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Poincaré Formula in Irreducible Hermitian Symmetric Spaces

Takashi SAKAI

University of Tsukuba

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Abstract. We describe the Poincaré formula for two complex submanifolds in irreducible Hermitian symmetric spaces. That can be expressed as a constant times the product of the volumes of two submanifolds under some conditions.

1. Introduction

Let G/K be a homogeneous space with an invariant Riemannian metric, and let M and N be submanifolds of G/K with $\dim M + \dim N \geq \dim(G/K)$. Then, for almost all g in G, $M \cap gN$ is again a manifold. So the function $g \mapsto \mathrm{vol}(M \cap gN)$ is measurable on G, and the integral

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g)$$

makes sense, where $d\mu_G$ denotes the invariant measure on G. It is called the Poincaré formula that the equality between the above integral and some geometric properties of M and N. For example, in the case where M and N are submanifolds of a real space form, this integral is equal to a constant times the product of the volumes of M and N. This was studied by Poincaré, Blaschke, Santaló and others (see [4] for reference). Furthermore Santaló [3] showed that if M and N are complex submanifolds of a complex projective space, then the Poincaré formula is expressed as a constant times the product of the volumes of two submanifolds.

Afterward Howard [1] obtained the generalized Poincaré formula on Riemannian homogeneous spaces G/K. Finally he asserted that if G is unimodular and acts transitively on the sets of tangent spaces to each of submanifolds M and N, then the integral is equal to a constant times the product of the volumes of two submanifolds.

In this paper, we attempt to express the Poincaré formula for two complex submanifolds in irreducible Hermitian symmetric spaces. And we show the following theorem:

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THEOREM 1.1. *Let G/K be an irreducible Hermitian symmetric space of complex dimension n. Assume that K acts irreducibly on the exterior algebra $\bigwedge^p(T_o(G/K))^{(1,0)}$. Then for any complex submanifolds M and N (each possibly with boundary) of G/K of complex dimensions $(n-p)$ and $(n-q)$ respectively with $p+q \leq n$, we have*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{(n-p)!(n-q)!\mathrm{vol}(K)}{n!(n-p-q)!} \mathrm{vol}(M)\mathrm{vol}(N) .$$

REMARK 1.2. In the case where $p+q=n$, this has been obtained by Kang, Takahashi, Tasaki and the author [2].

REMARK 1.3. If $G/K = \mathbf{C}P^n$ and $G = U(n+1)$, then K-action on $T_o(G/K)$ is orbit equivalent to the canonical action of a unitary group $U(n)$. So K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$ for any p. Thus Theorem 1.1 is actually an extension of the formula (2.2) by Santaló [3].

2. Preliminaries

In this section, we shall review the generalized Poincaré formula for Riemannian homogeneous spaces obtained by Howard [1].

We begin with a definition of the angle between subspaces. Let E be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. For two vector subspaces V and W of dimensions p and q in E with $p+q \leq n$, we take orthonormal bases $v_1, \dots, v_p$ and $w_1, \dots, w_q$ of V and W respectively. Then we define $\sigma(V, W)$, the angele between V and W, by

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\| ,$$

where

$$\|x_1 \wedge \cdots \wedge x_k\| \geq \det(\langle x_i, x_j \rangle) .$$

This definition is independent of the choice of orthonormal bases.

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of elements of K. This metric induces a G-invariant Riemannian metric on G/K. We denote by o the origin of G/K. For x and y in G/K and vector subspaces V in $T_x(G/K)$ and W in $T_y(G/K)$ we define $\sigma_K(V, W)$, the angle between V and W, by

$$\sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1} V , \quad dk_o^{-1}(dg_y)_o^{-1} W) d\mu_K(k) ,$$

where g_x and g_y are elements of G with $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y. Suppose that G is unimodular and let M and N be submanifolds of G/K with $\dim M + \dim N \geq \dim(G/K)$. With these notations, the generalized Poincaré formula for homogeneous spaces can be stated

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu_{M\times N}(x, y) . \tag{2.1}$$

Let V_o be a p dimensional subspace of $T_o(G/K)$. Then a p dimensional submanifold M of G/K is of type V_o if and only if for all x in M there exists g in G with $g_* V_o = T_x M$. Equality (2.1) implies that if M is a submanifold of G/K of type V_o and N of type W_o for some subspaces V_o and W_o of $T_o(G/K)$, then σ_K is a constant function on $M \times N$ and

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(M)\mathrm{vol}(N) .$$

In the case where G/K is a complex projective space $\mathbf{C}P^n$, any p dimensional complex submanifolds is a type V_o for any p dimensional complex subspace V_o in $T_o(G/K)$. Thus for any complex submanifolds M and N of complex dimensions p and q with $p + q \geq n$,

$$\begin{aligned} &\int_{U(n+1)} \mathrm{vol}(M \cap gN) d\mu_G(g) \\ &\quad = \frac{\mathrm{vol}(\mathbf{C}P^{p+q-n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbf{C}P^p)\mathrm{vol}(\mathbf{C}P^q)} \mathrm{vol}(M)\mathrm{vol}(N) . \end{aligned} \tag{2.2}$$

This was first obtained by Santaló [3]. For further details, see [1] as a reference.

3. Poincaré formula of complex submanifolds

Let (V, J) be a complex vector space with an inner product $\langle \cdot, \cdot \rangle$. We consider the exterior algebra $\bigwedge^p V^{(1,0)}$ of degree p on holomorphic vector space $V^{(1,0)}$. We extend $\langle \cdot, \cdot \rangle$ to a complex bilinear form on $V^{\mathbf{C}}$, and denote by the same symbol. Note that if X and Y are both in $V^{(1,0)}$ (or $V^{(0,1)}$) then $\langle X, Y \rangle = 0$. Expressing the norm on this exterior algebra by Gramian, we have following lemma immediately.

LEMMA 3.1. *For each* $\xi_1 \wedge \cdots \wedge \xi_p$ *in* $\wedge^p V^{(1,0)}$

$$\|\xi_1 \wedge \cdots \wedge \xi_p \wedge \overline{\xi_1 \wedge \cdots \wedge \xi_p}\| = \|\xi_1 \wedge \cdots \wedge \xi_p\|^2 .$$

PROPOSITION 3.2. *Let G be a unimodular Lie group and G/K an almost Hermitian homogeneous space of complex dimension n. Assume that K acts irreducibly on the exterior algebras $\bigwedge^p (T_o(G/K))^{(1,0)}$ and $\bigwedge^q (T_o(G/K))^{(1,0)}$ with $p + q \leq n$. Then there exists a positive constant C such that for any almost complex submanifolds M and N of G/K of complex dimensions $(n - p)$ and $(n - q)$ respectively*

$$\int_G \mathrm{vol}(M \cap gN) dg = C\mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

PROOF. From (2.1), it is sufficent if we show that $\sigma_K(T_x^{\perp}M, T_y^{\perp}N)$ is a positive constant function on $M \times N$.

Let $\{u_i, Ju_i\}_{1\leq i\leq p}$ and $\{v_i, Jv_i\}_{1\leq i\leq q}$ be orthonormal bases of $(dg_x)_o^{-1}(T_x^{\perp}M)$ and $(dg_y)_o^{-1}(T_y^{\perp}M)$ respectively. We put

$$\xi_i = \frac{1}{\sqrt{2}}\left(u_i - \sqrt{-1}Ju_i\right), \quad \eta_i = \frac{1}{\sqrt{2}}\left(v_i - \sqrt{-1}Jv_i\right).$$

Then $\{\xi_i\}_{1\leq i\leq p}$ and $\{\eta_i\}_{1\leq i\leq q}$ are unitary bases of $(dg_x)_o^{-1}(T_x^{\perp}M)^{(1,0)}$ and $(dg_y)_o^{-1}(T_y^{\perp}N)^{(1,0)}$ respectively. We note

$$u_i \wedge Ju_i = -\sqrt{-1}\xi_i \wedge \bar{\xi}_i, \quad v_i \wedge Jv_i = -\sqrt{-1}\eta_i \wedge \bar{\eta}_i.$$

So we have

$$\begin{aligned}&\|u_1 \wedge Ju_1 \wedge \cdots \wedge u_p \wedge Ju_p \wedge \mathrm{Ad}(k)(v_1 \wedge Jv_1 \wedge \cdots \wedge v_q \wedge Jv_q)\|\\ &= \|\xi_1 \wedge \bar{\xi}_1 \wedge \cdots \wedge \xi_p \wedge \bar{\xi}_p \wedge \mathrm{Ad}(k)(\eta_1 \wedge \bar{\eta}_1 \wedge \cdots \wedge \eta_q \wedge \bar{\eta}_q)\|.\end{aligned}$$

Now we put

$$\xi = \xi_1 \wedge \cdots \wedge \xi_p, \quad \eta = \eta_1 \wedge \cdots \wedge \eta_q.$$

From Lemma 3.1

$$\begin{aligned}\sigma_K(T_x^{\perp}M, T_y^{\perp}N) &= \int_K \|\xi \wedge \mathrm{Ad}(k)\eta \wedge \overline{\xi \wedge \mathrm{Ad}(k)\eta}\| d\mu_K(k)\\ &= \int_K \|\xi \wedge \mathrm{Ad}(k)\eta\|^2 d\mu_K(k).\end{aligned}$$

Fix η, and define Q_η by

$$Q_\eta(X, Y) = \int_K \langle X \wedge \mathrm{Ad}(k)\eta, \overline{Y \wedge \mathrm{Ad}(k)\eta}\rangle d\mu_K(k)$$

for each X, Y in $\wedge^p(T_o(G/K))^{(1,0)}$. Then Q_η is a K-invariant Hermitian form on $\bigwedge^p(T_o(G/K))^{(1,0)}$. From Schur's lemma, there is a positive constant C_η such that

$$Q_\eta(X, Y) = C_\eta\langle X, \bar{Y}\rangle,$$

since K acts irreducibly on $\bigwedge^p(T_o(G/K))^{(1,0)}$. So we have

$$\sigma_K(T_x^{\perp}M, T_y^{\perp}N) = C_\eta\|\xi\|^2 = C_\eta.$$

This implies that $\sigma_K(T_x^{\perp}M, T_y^{\perp}N)$ is independent of $T_x^{\perp}M$. In the same way, we can show that $\sigma_K(T_x^{\perp}M, T_y^{\perp}N)$ is also independent of $T_y^{\perp}N$ by the assumption which K acts irreducibly on $\bigwedge^q(T_o(G/K))^{(1,0)}$. This concludes the proof.

TABLE 1.

	compact type	
$A\ III$	$SU(l)/S(U(m) \times U(l-m))$	any p (if $m=1$) p=1 (if $m \geq 2$)
$D\ III$	$SO(2l)/U(l)$	$p = 1, 2$
$BD\ I$	$SO(2l)/SO(2) \times SO(2l-2)$ $SO(2l+1)/SO(2) \times SO(2l-1)$	$p \neq l-1$ any p
$C\ I$	$Sp(l)/U(l)$	$p = 1, 2$
$E\ III$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$p = 1, 2, 3$
$E\ VII$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$	$p = 1, 2, 3, 4$

In Table 1, we give p when K acts irreducibly on $\bigwedge^p (T_o(G/K))^{(1,0)}$ for irreducible Hermitian symmetric spaces. Although we show the case of compact type, it is clear that their non-compact duals also give the same results of Table 1.

PROOF OF THEOREM 1.1 From Table 1, if K acts irreducibly on $\bigwedge^p (T_o(G/K))^{(1,0)}$ for $p \leq n/2$, then K acts irreducibly on $\bigwedge^r (T_o(G/K))^{(1,0)}$ for any $r \leq p$. In addition, if K acts irreducibly on $\bigwedge^p (T_o(G/K))^{(1,0)}$, then K also acts irreducibly on $\bigwedge^{n-p} (T_o(G/K))^{(1,0)}$, since it is a dual representation of a unitary representation. From these facts, it is sufficient if we show the Theorem with $p \leq q \leq n-p$.

From the proof of Proposition 3.2, we have

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \int_K \|\xi_1 \wedge \cdots \wedge \xi_p \wedge \mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_q)\|^2 d\mu_K(k)\,.$$

We put a $p \times q$ matrix $A = (a_{ij}) = (\langle \xi_i, \overline{\mathrm{Ad}(k)\eta_j} \rangle)$, then

$$\|\xi_1 \wedge \cdots \wedge \xi_p \wedge \mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_q)\|^2 = \det \begin{bmatrix} I_p & A \\ A^* & I_q \end{bmatrix},$$

where I_p and I_q are unit matrixes of degree p and q respectively. Expanding with respect to the diagonal element 1, we can expand the right hand side to the sum of minor determinants as follows:

$$\det \begin{bmatrix} I_p & A \\ A^* & I_q \end{bmatrix} = 1 + \sum_{a=1}^{p} \sum_{b=1}^{q} \Bigg(\sum_{\substack{i_1 < \cdots < i_a \\ j_1 < \cdots < j_b}} \det \begin{bmatrix} O & A^{i_1 \cdots i_a}_{j_1 \cdots j_b} \\ (A^{i_1 \cdots i_a}_{j_1 \cdots j_b})^* & O \end{bmatrix} \Bigg),$$

where

$$A^{i_1 \cdots i_a}_{j_1 \cdots j_b} = \begin{bmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_b} \\ \vdots & \ddots & \vdots \\ a_{i_a j_1} & \cdots & a_{i_a j_b} \end{bmatrix}.$$

If $A^{i_1\cdots i_a}_{j_1\cdots j_b}$ is not a square matrix, then

$$\det\begin{bmatrix} O & A^{i_1\cdots i_a}_{j_1\cdots j_b} \\ (A^{i_1\cdots i_a}_{j_1\cdots j_b})^* & O \end{bmatrix} = 0\,.$$

Therefore we have

$$\begin{aligned}
&\det\begin{bmatrix} I_p & A \\ A^* & I_q \end{bmatrix} \\
&= 1 + \sum_{a=1}^{p}\Bigg(\sum_{\substack{i_1<\cdots<i_a\\ j_1<\cdots<j_a}} \det\begin{bmatrix} O & A^{i_1\cdots i_a}_{j_1\cdots j_a} \\ (A^{i_1\cdots i_a}_{j_1\cdots j_a})^* & O \end{bmatrix}\Bigg) \\
&= 1 + \sum_{a=1}^{p}(-1)^a\Bigg(\sum_{\substack{i_1<\cdots<i_a\\ j_1<\cdots<j_a}} \left|\det A^{i_1\cdots i_a}_{j_1\cdots j_a}\right|^2\Bigg) \\
&= 1 + \sum_{a=1}^{p}(-1)^a\Bigg(\sum_{\substack{i_1<\cdots<i_a\\ j_1<\cdots<j_a}} |\langle \xi_{i_1}\wedge\cdots\wedge\xi_{i_a}, \overline{\mathrm{Ad}(k)(\eta_{j_1}\wedge\cdots\wedge\eta_{j_a})}\rangle|^2\Bigg)\,.
\end{aligned}$$

Since K acts irreducibly on $\bigwedge^a(T_o(G/K))^{(1,0)}$ for any integer $a \le p$, in a similar way that in [2] we have

$$\int_K |\langle \xi_{i_1}\wedge\cdots\wedge\xi_{i_a}, \overline{\mathrm{Ad}(k)(\eta_{j_1}\wedge\cdots\wedge\eta_{j_a})}\rangle|^2 d\mu_K(k) = \frac{\mathrm{vol}(K)}{\binom{n}{a}}\,.$$

Thus

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \sum_{a=0}^{p}(-1)^a\frac{\binom{p}{a}\binom{q}{a}}{\binom{n}{a}}\mathrm{vol}(K)\,.$$

Here

$$\sum_{a=0}^{p}(-1)^a\frac{\binom{p}{a}\binom{q}{a}}{\binom{n}{a}}$$

is a constant only depending on n, p and q. In the case where G/K is a complex space form, for any n, p and q it satisfies the conditions of our Theorem. Hence comparing with (2.2), we have

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \frac{(n-p)!(n-q)!}{n!(n-p-q)!}\mathrm{vol}(K)\,.$$

This completes the proof.

COROLLARY 3.3. *If $p + q \leq n$, then*

$$\sum_{a=0}^{\min\{p,q\}} (-1)^a \frac{\binom{p}{a}\binom{q}{a}}{\binom{n}{a}} = \frac{(n-p)!(n-q)!}{n!(n-p-q)!} .$$

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Present Address:
DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE,
TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OHSAWA, HACHIOJI, TOKYO, 192–0397 JAPAN.
e-mail: tsakai@comp.metro-u.ac.jp