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Cone-Parameter Convolution Semigroups and Their Subordination

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Abstract. Convolution semigroups of probability measures with parameter in a cone in a Euclidean space generalize usual convolution semigroups with parameter in $[0,\infty)$. A characterization of such semigroups is given and examples are studied. Subordination of cone-parameter convolution semigroups by cone-valued cone-parameter convolution semigroups is introduced. Its general description is given and inheritance properties are shown. In the study the distinction between cones with and without strong bases is important.

1. Introduction

The structure of convolution semigroups of probability measures on $\mathbf{R}^d$ with parameter in $[0,\infty)$ is well-known: (i) $\{\mu_t : t \geqslant 0\}$ is a convolution semigroup if and only if μ_1 is infinitely divisible and $\mu_t = {\mu_1}^{t*}$ (the convolution power); (ii) a probability measure μ on $\mathbf{R}^d$ is infinitely divisible if and only if the characteristic function (Fourier transform) $\hat{\mu}(z)$ of μ is expressed as

$$(1.1)\qquad \hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \int_{\mathbf{R}^d} g(z,x)\,\nu(dx) + i\langle z,\gamma\rangle\right], \quad z \in \mathbf{R}^d\,,$$

where $g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\leqslant 1\}}(x)$, A is a nonnegative-definite symmetric $d\times d$ matrix, ν is a measure on $\mathbf{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int (1\wedge |x|^2)\,\nu(dx) < \infty$, and $\gamma \in \mathbf{R}^d$. The expression is unique and called the Lévy–Khintchine representation of μ; (A,ν,γ) is called the (generating) triplet of μ; A is the Gaussian covariance matrix, ν is the Lévy measure, and γ is a location parameter. See [3], [4], and [11] for general d and many textbooks in probability theory for $d = 1$. A natural generalization of the parameter set $[0,\infty)$ is a cone in the Euclidean space $\mathbf{R}^M$. Bochner [4], pp. 106–108, made a heuristic study of this generalization but, after that, there have been no works in this direction. Recently, Barndorff-Nielsen, Pedersen, and Sato [1] studied the case of the parameter set $\mathbf{R}_+^N$ in connection with multiparameter subordination of multiparameter Lévy processes, where subordinators are Lévy processes (with usual time parameter) taking values in $\mathbf{R}_+^N$. Many examples are discussed

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in [1]. As the set $\mathbf{R}_+^N$ is a typical cone, it is natural to consider subordinators which take values in a cone K in $\mathbf{R}^M$ and subordinands which are Lévy processes with parameter in K. Thus we have renewed interest in convolution semigroups with parameter in a cone. Another background fact is that the class $\mathbf{S}_d^+$ of nonnegative-definite symmetric $d \times d$ matrices is a $d(d+1)/2$-dimensional cone not isomorphic to $\mathbf{R}_+^{d(d+1)/2}$ and that there is a remarkable convolution semigroup $\{\mu_s\colon s \in \mathbf{S}_d^+\}$ defined by $\mu_s = N_d(0, s)$, Gaussian distribution on $\mathbf{R}^d$ with mean 0 and covariance matrix s. It is tempting to study properties and seek applications of this convolution semigroup, as it is a natural object.

In this paper we give, in Section 2, a characterization of cone-parameter convolution semigroups, which is connected with the representation (1.1), and some applications of it. Then, in Section 3, we discuss examples which illustrate the characterization. Given two cones K_1 and K_2 in $\mathbf{R}^{M_1}$ and $\mathbf{R}^{M_2}$, respectively, we study in Section 4 the composition of a K_2-parameter convolution semigroup (subordinand) with a K_2-valued K_1-parameter convolution semigroup (subordinator). This yields a new K_1-parameter convolution semigroup (subordinated). This is an extension of Bochner's subordination [4].

A usual convolution semigroup $\{\mu_t\colon t \geqslant 0\}$ of probability measures on $\mathbf{R}^d$ induces, uniquely in law, a Lévy process $\{X_t\colon t \geqslant 0\}$ with $\mathcal{L}(X_t) = \mu_t$. Here $\mathcal{L}(X_t)$ stands for the law (distribution) of X_t. In a companion paper [7] we discuss whether this fact generalizes to cone-parameter case under appropriate definition of cone-parameter Lévy processes. It turns out that neither existence nor uniqueness in law holds for the induced cone-parameter Lévy process in general. This implies that, in the cone-parameter case, subordination of convolution semigroups is of importance independently of subordination of Lévy processes.

2. Characterization of cone-parameter convolution semigroups

We consider elements of $\mathbf{R}^d$ as column vectors. We denote the coordinates of $x \in \mathbf{R}^d$ by x_j, and use either the notation $x = (x_j)_{1\leqslant j\leqslant d}$ or $x = (x_1, \cdots, x_d)^\top$. The inner product on $\mathbf{R}^d$ is $\langle x, y\rangle$ and the norm is $|x|$. For a measure μ on $\mathbf{R}^d$, $\mathrm{Supp}(\mu)$ denotes the support of μ, that is, the smallest closed set whose complement has μ-measure 0. Let δ_c denote the distribution (= probability measure) concentrated at a point c. Such a distribution is called trivial. For $a, b \in \mathbf{R}$, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For a distribution μ on $\mathbf{R}^d$, the characteristic function $\hat{\mu}(z)$ of μ is

$$\hat{\mu}(z) = \int_{\mathbf{R}^d} e^{i\langle z,x\rangle}\mu(dx)\,, \quad z \in \mathbf{R}^d\,.$$

For distributions μ_n $(n = 1, 2, \cdots)$ and μ on $\mathbf{R}^d$, $\mu_n \to \mu$ means weak convergence of μ_n to μ, that is, $\lim_{n\to\infty} \int f(x)\mu_n(dx) = \int f(x)\mu(dx)$ for all bounded continuous functions f on $\mathbf{R}^d$.

We call a subset K of $\mathbf{R}^M$ a *cone* if it is a non-empty closed convex set closed under multiplication by nonnegative reals and containing no straight line through 0 and if $K \neq \{0\}$.

DEFINITION 2.1. Given a cone K, we call $\{\mu_s : s \in K\}$ a *K-parameter convolution semigroup on* $\mathbf{R}^d$ if it is a family of probability measures on $\mathbf{R}^d$ satisfying

$$\mu_{s^1} * \mu_{s^2} = \mu_{s^1+s^2} \quad \text{for } s^1, s^2 \in K, \tag{2.1}$$

$$\mu_{t_n s} \to \delta_0 \quad \text{for } s \in K \tag{2.2}$$

whenever $\{t_n\}$ is a sequence of reals strictly decreasing to 0.

It is clear that, for the cone $\mathbf{S}_d^+$ defined in Section 1, the system $\{\mu_s : s \in \mathbf{S}_d^+\}$ with $\mu_s = N_d(0, s)$ forms an $\mathbf{S}_d^+$-parameter convolution semigroup on $\mathbf{R}^d$. We call it the *canonical* $\mathbf{S}_d^+$*-parameter convolution semigroup*.

If $\{e^1, \cdots, e^N\}$ is a linearly independent system in $\mathbf{R}^M$, then the set of $s = s_1 e^1 + \cdots + s_N e^N$ with nonnegative $s_1, \cdots, s_N$ is the smallest cone that contains $e^1, \cdots, e^N$. It is called the cone generated by $\{e^1, \cdots, e^N\}$.

DEFINITION 2.2. Let K be a cone in $\mathbf{R}^M$. If $\{e^1, \cdots, e^N\}$ is a linearly independent system such that K is the cone generated by it, then $\{e^1, \cdots, e^N\}$ is called a *strong basis* of K. If $\{e^1, \cdots, e^N\}$ is a basis of the linear subspace L generated by K and if $e^1, \cdots, e^N$ are in K, then $\{e^1, \cdots, e^N\}$ is called a *weak basis* of K. In this case K is called an N-dimensional cone. A cone in $\mathbf{R}^M$ is called *nondegenerate* if it is M-dimensional.

Any cone has a weak basis. A cone in $\mathbf{R}$ is either $[0, \infty)$ or $(-\infty, 0]$, and has a strong basis. Any nondegenerate cone in $\mathbf{R}^2$ is a closed sector with angle $< \pi$ and has a strong basis. A nondegenerate cone in $\mathbf{R}^3$ has a strong basis if and only if it is a triangular cone. For any N, the nonnegative orthant $\mathbf{R}_+^N$ is a cone with a strong basis. Conversely, if a cone K has a strong basis $\{e^1, \cdots, e^N\}$, then it is isomorphic to $\mathbf{R}_+^N$, that is, there is a linear transformation T from the linear subspace L generated by K onto $\mathbf{R}^N$ such that $TK = \mathbf{R}_+^N$.

Given a cone K in $\mathbf{R}^M$, write $s^1 \leqslant_K s^2$ if $s^2 - s^1 \in K$. This defines a partial order in $\mathbf{R}^M$. A sequence $\{s^n\}$ in $\mathbf{R}^M$ is said to be *K-increasing* if $s^n \leqslant_K s^{n+1}$ for each n; *K-decreasing* if $s^{n+1} \leqslant_K s^n$ for each n.

The following proposition is basic. A proof is given in the appendix. We call H a *strictly supporting hyperplane* of a cone K in $\mathbf{R}^M$, if H is an $(M-1)$-dimensional linear subspace such that $H \cap K = \{0\}$.

PROPOSITION 2.3. *Any cone K in $\mathbf{R}^M$ has the following properties.*

(i) *There exists a strictly supporting hyperplane H of K.*

(ii) *Let H be a strictly supporting hyperplane of K and let $s^0 \in K \setminus \{0\}$. Then the hyperplane $s^0 + H$ does not contain* 0. *Let D be the closed half space containing* 0 *with boundary $s^0 + H$. Then $K \cap D$ is a bounded set.*

(iii) *If $\{s^n\}_{n=1,2,\cdots}$ is a K-decreasing sequence in K, then it is convergent.*

A weak basis of K is not unique. But, a strong basis of K is essentially unique, if it exists.

PROPOSITION 2.4. *If* $\{e^1, \cdots, e^N\}$ *and* $\{f^1, \cdots, f^N\}$ *are both strong bases of* K, *then these systems are identical up to scaling and permutation.*

PROOF. Since the two systems are strong bases, we have

$$e^j = e^j_1 f^1 + \cdots + e^j_N f^N \quad \text{for } j = 1, \cdots, N,$$
$$f^k = f^k_1 e^1 + \cdots + f^k_N e^N \quad \text{for } k = 1, \cdots, N,$$

where $f^k_j \geqslant 0$ and $e^j_l \geqslant 0$ for all k, j, l. Since $f^k = \sum_{j,l} f^k_j e^j_l f^l$, we get

$$\sum_{j=1}^{N} f^k_j e^j_l = 0 \quad \text{or 1 according as } k \neq l \text{ or } k = l.$$

Fix k. Since $f^k \neq 0$, we can find k' such that $f^k_{k'} > 0$. If $l \neq k$, then $f^k_j e^j_l = 0$ for all j and thus $e^{k'}_l = 0$. That is, $e^{k'} = e^{k'}_k f^k$. Hence $e^{k'}_k > 0$ and $f^k = (e^{k'}_k)^{-1} e^{k'}$. The mapping from k to k' is onto, since $f^1, \cdots, f^N$ are linearly independent. This finishes the proof. □

REMARK 2.5. Given s^1, s^2 in a cone K, we call $u \in K$ the greatest lower bound of s^1 and s^2 and write $u = s^1 \wedge_K s^2$, if

$$\{v \in K : v \leqslant_K s^1\} \cap \{v \in K : v \leqslant_K s^2\} = \{v \in K : v \leqslant_K u\}. \tag{2.3}$$

Similarly, u is called the least upper bound, written $u = s^1 \vee_K s^2$, if

$$\{v \in K : s^1 \leqslant_K v\} \cap \{v \in K : s^2 \leqslant_K v\} = \{v \in K : u \leqslant_K v\}. \tag{2.4}$$

If K has a strong basis $\{e^1, \cdots, e^N\}$, then for any $s^1, s^2 \in K$, $s^1 \wedge_K s^2$ and $s^1 \vee_K s^2$ exist (in other words, K is a lattice). Indeed, if $s^j = s^j_1 e^1 + \cdots + s^j_N e^N$ for $j = 1, 2$, then $s^1 \wedge_K s^2 = (s^1_1 \wedge s^2_1)e^1 + \cdots + (s^1_N \wedge s^2_N)e^N$ and $s^1 \vee_K s^2 = (s^1_1 \vee s^2_1)e^1 + \cdots + (s^1_N \vee s^2_N)e^N$. But, in a general cone K, $s^1 \wedge_K s^2$ and $s^1 \vee_K s^2$ do not necessarily exist. For example, let K be a circular cone in $\mathbf{R}^3$. Then, for some s^1 and s^2 in K, $s^1 \wedge_K s^2$ does not exist. This is seen in the following way. Denote $x = (x_j)_{1 \leqslant j \leqslant 3} \in \mathbf{R}^3$ and let K have the x_3-axis as the axis of rotation. We have $\{v \in K : v \leqslant_K s\} = (s - K) \cap K$ for $s \in K$. The section of the left-hand side of (2.3) by some plane $x_3 =$ constant is not a connected set, if $s^1 - s^2 \notin K \cup (-K)$. Thus, the relation (2.3) is not always possible. Similarly, the relation (2.4) is not always possible.

Let $ID(\mathbf{R}^d)$ be the class of infinitely divisible distributions on $\mathbf{R}^d$. Let $\mathcal{B}_0(\mathbf{R}^d)$ be the class of Borel sets B in $\mathbf{R}^d$ such that $\inf_{x \in B} |x| > 0$. Any $\mu \in ID(\mathbf{R}^d)$ has the representation (1.1) by the triplet (A, ν, γ). If ν satisfies $\int_{|x| \leqslant 1} |x| \nu(dx) < \infty$, then let $\gamma^0 = \gamma - \int_{|x| \leqslant 1} x \nu(dx)$ and call γ^0 the drift of μ. For $\mu \in ID(\mathbf{R}^d)$ and $r \in \mathbf{R}$, we define $\hat{\mu}(z)^r$,

$z \in \mathbf{R}^d$, as $\hat{\mu}(z)^r = e^{r \log \hat{\mu}(z)}$, where $\log \hat{\mu}(z)$ is the distinguished logarithm of $\hat{\mu}(z)$ in [11], p. 33. In other words,

$$\hat{\mu}(z)^r = \exp\left[r\left(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbf{R}^d} g(z,x)\nu(dx)\right)\right].$$

If $\mu \in ID(\mathbf{R}^d)$ and $r \geqslant 0$, then $\hat{\mu}(z)^r$ is the characteristic function of a distribution in $ID(\mathbf{R}^d)$, denoted by μ^{r*} or μ^r. However, if $r < 0$, then $\hat{\mu}(z)^r$ is not a characteristic function for any nontrivial μ in $ID(\mathbf{R}^d)$.

PROPOSITION 2.6. *Let K_1 and K_2 be cones in $\mathbf{R}^M$ such that $K_1 \subseteq K_2$. If $\{\mu_s : s \in K_2\}$ is a K_2-parameter convolution semigroup then its restriction $\{\mu_s : s \in K_1\}$ is a K_1-parameter convolution semigroup.*

PROOF. Evident from Definition 2.1. □

PROPOSITION 2.7. *Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbf{R}^d$. Then, $\mu_0 = \delta_0$ and $\mu_s \in ID(\mathbf{R}^d)$ for $s \in K$. We have $\mu_{ts} = {\mu_s}^t$ for $t \geqslant 0$. The triplet (A_s, ν_s, γ_s) of μ_s satisfies*

(2.5) $$A_{s^1+s^2} = A_{s^1} + A_{s^2}, \quad \nu_{s^1+s^2} = \nu_{s^1} + \nu_{s^2}, \quad \gamma_{s^1+s^2} = \gamma_{s^1} + \gamma_{s^2},$$

(2.6) $$A_{ts} = tA_s, \quad \nu_{ts} = t\nu_s, \quad \gamma_{ts} = t\gamma_s.$$

If, moreover, $\int_{|x|\leqslant 1} |x|\nu_s(dx) < \infty$ for all $s \in K$, then, for the drift γ_s^0 of μ_s, we have

(2.7) $$\gamma^0_{s^1+s^2} = \gamma^0_{s^1} + \gamma^0_{s^2}, \quad \gamma^0_{ts} = t\gamma^0_s.$$

PROOF. Since $\mu_0 = \mu_0 * \mu_0$ by (2.1), we have $\hat{\mu}_0(z) = \hat{\mu}_0(z)^2$ and hence $\hat{\mu}_0(z) = 1$ if $\hat{\mu}_0(z) \neq 0$. This shows that $\hat{\mu}_0(z) = 1$ for all z, as $\hat{\mu}_0(0) = 1$ and $\hat{\mu}_0(z)$ is continuous. Hence $\mu_0 = \delta_0$. Since $\{\mu_{ts} : t \geqslant 0\}$ is an $\mathbf{R}_+$-parameter convolution semigroup by Proposition 2.6, we have $\mu_s \in ID(\mathbf{R}^d)$ and $\mu_{ts} = {\mu_s}^t$. Equations (2.5)–(2.7) are obvious consequences. □

THEOREM 2.8. *Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbf{R}^d$ with triplets (A_s, ν_s, γ_s). Let $\{e^1, \cdots, e^N\}$ be a weak basis of K. Then, for all $s \in K$, μ_s is determined by $\mu_{e^1}, \cdots, \mu_{e^N}$. More precisely, for $s = s_1e^1 + \cdots + s_Ne^N \in K$ we have*

(2.8) $$\hat{\mu}_s(z) = \hat{\mu}_{e^1}(z)^{s_1} \cdots \hat{\mu}_{e^N}(z)^{s_N}, \quad z \in \mathbf{R}^d,$$

(2.9) $$A_s = s_1A_{e^1} + \cdots + s_NA_{e^N},$$

(2.10) $$\nu_s(B) = s_1\nu_{e^1}(B) + \cdots + s_N\nu_{e^N}(B) \quad \text{for } B \in \mathcal{B}_0(\mathbf{R}^d),$$

(2.11) $$\gamma_s = s_1\gamma_{e^1} + \cdots + s_N\gamma_{e^N}.$$

Keep in mind that some of $s_1, \cdots, s_N$ may be negative.

PROOF OF THEOREM. Any $s \in K$ is represented uniquely as $s = s_1e^1 + \cdots + s_Ne^N$, with $s_1, \cdots, s_N \in \mathbf{R}$. Let $s_j^+ = s_j \vee 0$ and $s_j^- = -(s_j \wedge 0)$. Then $s_j = s_j^+ - s_j^-$. We have

$s = u - v$ with $u = s_1^+ e^1 + \cdots + s_N^+ e^N \in K$ and $v = s_1^- e^1 + \cdots + s_N^- e^N \in K$. Hence $\mu_s * \mu_v = \mu_u$. Using Proposition 2.7, we can express $\hat{\mu}_u(z)$ and $\hat{\mu}_v(z)$ by $\hat{\mu}_{e^1}(z), \cdots, \hat{\mu}_{e^N}(z)$. Noting that $\hat{\mu}_v(z) \neq 0$ by infinite divisibility, we have

$$\hat{\mu}_s(z) = \frac{\hat{\mu}_u(z)}{\hat{\mu}_v(z)} = \frac{\hat{\mu}_{e^1}(z)^{s_1^+} \cdots \hat{\mu}_{e^N}(z)^{s_N^+}}{\hat{\mu}_{e^1}(z)^{s_1^-} \cdots \hat{\mu}_{e^N}(z)^{s_N^-}},$$

which is (2.8). Now (2.9)–(2.11) follow from (2.8) by the uniqueness of the expression as formulated in [11], E 12.2. □

COROLLARY 2.9. *Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbf{R}^d$. If $\{s^n\}_{n=1,2,\dots}$ is a sequence in K with $|s^n - s^0| \to 0$, then $\mu_{s^n} \to \mu_{s^0}$.*

PROOF. Let $|s^n - s^0| \to 0$. Decompose s^n as $s^n = s_1^n e^1 + \cdots + s_N^n e^N$ for $n = 0, 1, \cdots$. Then $s_j^n \to s_j^0$ for $j = 1, \cdots, N$ and (2.8) shows that $\hat{\mu}_{s^n}(z) \to \hat{\mu}_{s^0}(z)$ for all z. □

If $K = [0, \infty)$, then for any $\rho \in ID(\mathbf{R}^d)$ there exists a convolution semigroup $\{\mu_t \colon t \geqslant 0\}$ satisfying $\mu_1 = \rho$. We ask the question whether this fact generalizes to the case of a general cone K. The answer follows from Theorem 2.8.

DEFINITION 2.10. Let $\{e^1, \cdots, e^N\}$ be a weak basis of K and let $\rho_1, \cdots, \rho_N \in ID(\mathbf{R}^d)$. We call $\{\rho_1, \cdots, \rho_N\}$ *admissible with respect to* $\{e^1, \cdots, e^N\}$, if there exists (uniquely, by Theorem 2.8) a K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$ such that $\mu_{e^j} = \rho_j$ for $j = 1, \cdots, N$.

THEOREM 2.11. *Let $\{e^1, \cdots, e^N\}$ be a weak basis of K. Let $\rho_1, \cdots, \rho_N \in ID(\mathbf{R}^d)$ and let (A_j, ν_j, γ_j) be the generating triplet of ρ_j. Then the following three statements are equivalent.*

(a) *$\{\rho_1, \cdots, \rho_N\}$ is admissible with respect to $\{e^1, \cdots, e^N\}$.*

(b) *If $s_1, \cdots, s_N \in \mathbf{R}$ are such that $s_1 e^1 + \cdots + s_N e^N \in K$, then $\hat{\rho}_1(z)^{s_1} \cdots \hat{\rho}_N(z)^{s_N}$ is an infinitely divisible characteristic function.*

(c) *If $s_1, \cdots, s_N \in \mathbf{R}$ are such that $s_1 e^1 + \cdots + s_N e^N \in K$, then $s_1 A_1 + \cdots + s_N A_N \in \mathbf{S}_d^+$ and $s_1 \nu_1(B) + \cdots + s_N \nu_N(B) \geqslant 0$ for $B \in \mathcal{B}_0(\mathbf{R}^d)$.*

PROOF. By Theorem 2.8, (a) implies (b). Conversely, suppose that (b) is true. For each $s \in K$, define $\mu_s \in ID(\mathbf{R}^d)$ by (2.8) with $\mu_{e^j} = \rho_j$. Since $s_1, \cdots, s_N$ are determined by s, this is well-defined by virtue of (b). The property $\mu_{s^1 + s^2} = \mu_{s^1} * \mu_{s^2}$ is obvious. If t_n strictly decreases to 0, then $t_n s \to 0$ and hence $\mu_{t_n s} \to \delta_0$. This shows (a). The equivalence of (b) and (c) is a consequence of E 12.3 of [11]. □

A characterization of strong bases follows from this theorem.

COROLLARY 2.12. *Let $\{e^1, \cdots, e^N\}$ be a weak basis of K. Then, every choice of $\{\rho_1, \cdots, \rho_N\}$ in $ID(\mathbf{R}^d)$ is admissible with respect to $\{e^1, \cdots, e^N\}$ if and only if $\{e^1, \cdots, e^N\}$ is a strong basis of K.*

PROOF. If $\{e^1, \cdots, e^N\}$ is a strong basis, then the condition (b) of the theorem above is automatically satisfied for any $\{\rho_1, \cdots, \rho_N\}$ in $ID(\mathbf{R}^d)$, since $s_j \geqslant 0$ for $j = 1, \cdots, N$. Conversely, suppose that $\{e^1, \cdots, e^N\}$ is not a strong basis. Then, we can choose j_0 such that there exists $s = s_1 e^1 + \cdots + s_N e^N \in K$ with $s_{j_0} < 0$. Let $\rho \in ID(\mathbf{R}^d)$ be nontrivial and $\rho_j = \rho$ for $j \neq j_0$ and $\rho_{j_0} = \rho^c$ with c so large that $(1-c)s_{j_0} > s_1 + \cdots + s_N$. By the theorem above, $\{\rho_1, \cdots, \rho_N\}$ is then not admissible with respect to $\{e^1, \cdots, e^N\}$. □

When we are given a cone K and its weak basis $\{e^1, \cdots, e^N\}$, we can sometimes rewrite the condition (c) in Theorem 2.11 as more tractable properties of $A_1, \cdots, A_N$ and $\nu_1, \cdots, \nu_N$. This will be shown in Section 3.

Let us give some other applications of Theorem 2.11. For a $d \times d$ matrix A, $A(\mathbf{R}^d) = \{Ax : x \in \mathbf{R}^d\}$ denotes the range of A. For linear subspaces $L, L_1, \cdots, L_N$ of $\mathbf{R}^d$, L is said to be the direct sum of $L_1, \cdots, L_N$ if $L = L_1 + \cdots + L_N$ and if the expression of $x \in L$ in the form $x = x^1 + \cdots + x^N$ with $x^j \in L_j$, $j = 1, \cdots, N$, is unique.

PROPOSITION 2.13. *Let $\{e^1, \cdots, e^N\}$ be a weak basis of K and suppose that there is $s \in K$ satisfying $s = s_1 e^1 + \cdots + s_N e^N$ with $s_{j_0} < 0$. Let $L_1, \cdots, L_N$ be linear subspaces of $\mathbf{R}^d$ such that $L_1 + \cdots + L_N$ is the direct sum of $L_1, \cdots, L_N$. If $\{\rho_1, \cdots, \rho_N\}$ in $ID(\mathbf{R}^d)$ is admissible with respect to $\{e^1, \cdots, e^N\}$ and if* $\mathrm{Supp}(\rho_j) \subseteq L_j$ *for $j = 1, \cdots, N$, then ρ_{j_0} is trivial.*

PROOF. *Step 1.* Let us prove the assertion under the assumption that L_j, $j = 1, \cdots, N$, are orthogonal. Let (A_j, ν_j, γ_j) be the generating triplet of ρ_j. It follows from $\mathrm{Supp}(\rho_j) \subseteq L_j$ that $A_j(\mathbf{R}^d) \subseteq L_j$, $\mathrm{Supp}(\nu_j) \subseteq L_j$ and $\gamma_j \in L_j$ (cf. Proposition 24.17 of [11]). Now choose s such that $s_{j_0} < 0$. Let $z \in L_{j_0}$. Then, by (c) of Theorem 2.11, $0 \leqslant \langle z, (s_1 A_1 + \cdots + s_N A_N) z\rangle = s_{j_0}\langle z, A_{j_0} z\rangle$. Hence $\langle z, A_{j_0} z\rangle = 0$. It follows that $A_{j_0} z = 0$. Since $A_j(\mathbf{R}^d) = \{A_j z : z \in A_j(\mathbf{R}^d)\}$ and $A_j(\mathbf{R}^d) \subseteq L_j$, we see that $A_j(\mathbf{R}^d) = \{A_j z : z \in L_j\}$. Therefore, $A_{j_0}(\mathbf{R}^d) = \{0\}$, that is, $A_{j_0} = 0$. Let B be a Borel set in L_{j_0}. Then $\nu_j(B) \leqslant \nu_j(L_{j_0} \cap L_j) = 0$ for $j \neq j_0$. Hence $s_{j_0}\nu_{j_0}(B) \geqslant 0$. Since $s_{j_0} < 0$, this means that $\nu_{j_0}(B) = 0$. That is, $\nu_{j_0} = 0$. Thus, ρ_{j_0} is trivial.

Step 2. General case. There exists a linear transformation T from $\mathbf{R}^d$ onto $\mathbf{R}^d$ such that the images $L_j^\sharp$ of L_j by T, $j = 1, \cdots, N$, are orthogonal. Denote $\rho_j^\sharp(B) = \rho_j(T^{-1}B)$. It is readily seen that $\{\rho_1^\sharp, \cdots, \rho_N^\sharp\}$ is admissible. Since $\rho_j^\sharp(L_j^\sharp) = \rho_j(T^{-1}L_j^\sharp) = \rho_j(L_j) = 1$, we have $\mathrm{Supp}(\rho_j^\sharp) \subseteq L_j^\sharp$. Hence, by Step 1, $\rho_{j_0}^\sharp$ is trivial, that is, ρ_{j_0} is trivial. □

Let K and $\tilde{K}$ be cones satisfying $K \subseteq \tilde{K}$. Let $\{\mu_s : s \in K\}$ and $\{\tilde{\mu}_s : s \in \tilde{K}\}$ be, respectively, K- and $\tilde{K}$-parameter convolution semigroups on $\mathbf{R}^d$. We say that $\{\tilde{\mu}_s : s \in \tilde{K}\}$ is an extension of $\{\mu_s : s \in K\}$ if $\tilde{\mu}_s = \mu_s$ for all $s \in K$.

PROPOSITION 2.14. *Let K be an N-dimensional cone with strong basis $\{e^1, \cdots, e^N\}$. Then there exists a K-parameter convolution semigroup $\{\mu_s : s \in K\}$ on $\mathbf{R}$ such that, for any*

N-dimensional cone $\tilde{K}$ satisfying $\tilde{K} \supseteq K$ and $\tilde{K} \neq K$, $\{\mu_s : s \in K\}$ is not extendable to a $\tilde{K}$-parameter convolution semigroup. In particular if, for the Lévy measures ν_j of μ_{e^j}, there are $B_j \in \mathcal{B}_0(\mathbf{R})$, $j = 1, \cdots, N$, such that $\nu_j(B_j) > 0$ and $\nu_k(B_j) = 0$ for $k \neq j$, then $\{\mu_s : s \in K\}$ is not extendable.

PROOF. Let $\{\mu_s : s \in K\}$ be as above and let $\tilde{K}$ be an N-dimensional cone satisfying $\tilde{K} \supseteq K$ and $\tilde{K} \neq K$. Suppose that $\{\mu_s : s \in K\}$ is extendable to $\{\tilde{\mu}_s : s \in \tilde{K}\}$. Since $\{e^1, \cdots, e^N\}$ is a weak basis of $\tilde{K}$ but not a strong basis, there is $s \in \tilde{K}$ such that $s = s_1 e^1 + \cdots + s_N e^N$ with $s_j < 0$ for some j. The Lévy measure $\tilde{\nu}_s$ of $\tilde{\mu}_s$ satisfies $\tilde{\nu}_s = s_1\nu_1 + \cdots + s_N\nu_N$ by Theorem 2.8. Hence $\tilde{\nu}_s(B_j) = s_j\nu_j(B_j) < 0$, which is absurd. □

3. Examples

In this section, the first example concerns the structure of the cone $\mathbf{S}_d^+$. Then we seek admissibility conditions for some cones in $\mathbf{R}^3$. We will use the notion of dual cones. The last example is a polyhedral cone in $\mathbf{R}^M$.

EXAMPLE 3.1. Consider the class $\mathbf{S}_d^+$ of nonnegative-definite symmetric $d \times d$ matrices $s = (s_{jk})_{j,k=1}^d$. The lower triangle $(s_{jk})_{k \leqslant j}$ with $d(d+1)/2$ entries determines s. We identify $\mathbf{S}_d^+$ with a subset of $\mathbf{R}^{d(d+1)/2}$, considering $(s_{jk})_{k \leqslant j}$ as a column vector. Then $\mathbf{S}_d^+$ is a nondegenerate cone in $\mathbf{R}^{d(d+1)/2}$.

Let us show that $\mathbf{S}_2^+$ *is isomorphic to a circular cone in* $\mathbf{R}^3$. Indeed, let $K = \mathbf{S}_2^+$. Then $s = (s_{jk})_{j,k=1}^2 \in K$ is identified with $(x_1, x_2, x_3)^\top$, where $x_1 = s_{11}$, $x_2 = s_{22}$, $x_3 = s_{21}$, and hence

$$K = \{(x_1, x_2, x_3)^\top : x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 x_2 - x_3^2 \geqslant 0\}.$$

Consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined by $T(x_1, x_2, x_3)^\top = (y_1, y_2, y_3)^\top$ with

$$x_1 = y_1 + y_3, \quad x_2 = -y_1 + y_3, \quad x_3 = y_2.$$

Then $u \in \tilde{K} = TK$ is expressed as

$$y_1 + y_3 \geqslant 0, \quad -y_1 + y_3 \geqslant 0, \quad (y_1 + y_3)(-y_1 + y_3) - y_2^2 \geqslant 0.$$

This is written as $y_3 \geqslant 0$, $y_3^2 - y_1^2 - y_2^2 \geqslant 0$, which describes a circular cone. This expression of $\mathbf{S}_d^+$ by a quadratic inequality seems to exist only for $d = 2$, because the boundary of $\mathbf{S}_d^+$ is expressed by $\det(s) = 0$, which is an equation of degree d.

For any $d \geqslant 2$, the cone $\mathbf{S}_d^+$ does not have a strong basis. The proof is as follows. If $\mathbf{S}_d^+$ has a strong basis, then, for any choice of $s^1, s^2 \in \mathbf{S}_d^+$, the greatest lower bound $s^1 \wedge_{\mathbf{S}_d^+} s^2$ exists by Remark 2.5. But, in a circular cone in $\mathbf{R}^3$, two elements do not always have a greatest lower bound by Remark 2.5. By the isomorphism of $\mathbf{S}_2^+$ to a circular cone, $\mathbf{S}_2^+$ does not have a strong basis. For $d \geqslant 3$, consider $s^p = (s^p_{jk})^d_{j,k=1}$ in $\mathbf{S}_d^+$ for $p = 1, 2$ such that $s^p_{jk} = 0$ whenever $j \geqslant 3$ or $k \geqslant 3$. For $s = (s_{jk})^d_{j,k=1} \in \mathbf{S}_d^+$, we have $s \leqslant_{\mathbf{S}_d^+} s^p$ if and only if $s_{jk} = 0$ for $j \geqslant 3$ or $k \geqslant 3$ and $u \leqslant_{\mathbf{S}_2^+} u^p$ where $u = (s_{jk})^2_{j,k=1}$ and $u^p = (s^p_{jk})^2_{j,k=1}$. That is, finding the greatest lower bound of s^1 and s^2 in $\mathbf{S}_d^+$ is equivalent to finding the greatest lower bound of u^1 and u^2 in $\mathbf{S}_2^+$. Since u^1 and u^2 do not always have a greatest lower bound in $\mathbf{S}_2^+$, s^1 and s^2 do not always have a greatest lower bound in $\mathbf{S}_d^+$. Thus, $\mathbf{S}_d^+$ does not have a strong basis.

Let K be a cone in $\mathbf{R}^M$. Let $K' = \{u \in \mathbf{R}^M : \langle u, s\rangle \geqslant 0 \text{ for all } s \in K\}$. Then K' is again a cone in $\mathbf{R}^M$. It is called the *dual cone* of K. We have $(K')' = K$. If $K = \mathbf{R}^M_+$, then $K = K'$. For two cones K_1, K_2 in $\mathbf{R}^M$, we have $K_1 \subseteq K_2$ if and only if $K_1' \supseteq K_2'$.

EXAMPLE 3.2. Let

$$e^1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right)^\top, \quad e^2 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right)^\top, \quad e^3 = (0, -1, 1)^\top \tag{3.1}$$

in $\mathbf{R}^3$. These points are on the circle $x_1^2 + x_2^2 = 1$, $x_3 = 1$, and form an equilateral triangle. Let Γ_1 and Γ_2 be the line segments e^3e^1 and e^2e^3, respectively. Let C be the arc e^1e^2 of the circle. Let D be the closed convex set on the plane $x_3 = 1$, surrounded by Γ_1, C and Γ_2. Let $K = \{s = tu \in \mathbf{R}^3 : u \in D \text{ and } t \geqslant 0\}$. Then $\{e^1, e^2, e^3\}$ is a weak basis of this cone K. For s and u in $\mathbf{R}^3$, denote $s = s_1e^1 + s_2e^2 + s_3e^3$ and $u_j = \langle u, e^j\rangle$ for $j = 1, 2, 3$. We have

$$\langle u, s\rangle = u_1s_1 + u_2s_2 + u_3s_3 \,. \tag{3.2}$$

Then, $u \in K'$ if and only if

$$\begin{cases} u_j \geqslant 0 \quad \text{for } j = 1, 2, 3 \\ au_1 + (1-a)u_2 - a(1-a)u_3 \geqslant 0 \quad \text{for } 0 \leqslant a \leqslant 1 \,. \end{cases} \tag{3.3}$$

An alternative characterization is that $u \in K'$ if and only if

$$\begin{cases} u_j \geqslant 0 \quad \text{for } j = 1, 2, 3 \\ \sqrt{u_3} \leqslant \sqrt{u_1} + \sqrt{u_2} \,. \end{cases} \tag{3.4}$$

The proof is as follows. A few calculations show that $s \in C$ if and only if

$$s = (1 - a(1-a))^{-1}(ae^1 + (1-a)e^2 - a(1-a)e^3) \quad \text{with } 0 \leqslant a \leqslant 1 \,. \tag{3.5}$$

If $u \in K'$, then (3.3) holds, since $\langle u, e^j\rangle \geqslant 0$ and $\langle u, s\rangle \geqslant 0$ for s of (3.5). If u satisfies (3.3), then we can show that $u \in K'$. Indeed, for $s \in C$ we have $\langle u, s\rangle \geqslant 0$ by (3.5); for s in the triangle with vertices e^1, e^2, e^3 we have $\langle u, s\rangle \geqslant 0$, since $u_j \geqslant 0$ for $j = 1, 2, 3$; finally for s in D but not in the triangle there is a number $0 \leqslant \gamma \leqslant 1$ and $\tilde{s} \in C$ such that $s = \gamma e^3 + (1-\gamma)\tilde{s}$ and hence $\langle u, s\rangle \geqslant 0$. To see the equivalence of (3.3) and (3.4), notice that, if $u_1 \geqslant 0$ and $u_2 \geqslant 0$, then the infimum of $u_1/(1-a) + u_2/a$ for $0 < a < 1$ equals $(\sqrt{u_1} + \sqrt{u_2})^2$.

Let us consider admissibility for K and $\{e^1, e^2, e^3\}$. A system $\{\rho_1, \rho_2, \rho_3\}$ in $ID(\mathbf{R}^d)$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if the triplets (A_j, ν_j, γ_j) of ρ_j, $j = 1, 2, 3$, satisfy

$$\begin{cases} aA_1 + (1-a)A_2 - a(1-a)A_3 \in \mathbf{S}_d^+ \quad \text{for } 0 < a < 1, \\ a\nu_1 + (1-a)\nu_2 - a(1-a)\nu_3 \geqslant 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d) \text{ for } 0 < a < 1 \end{cases} \tag{3.6}$$

or, equivalently,

$$\begin{cases} \sqrt{\langle A_3 z, z\rangle} \leqslant \sqrt{\langle A_1 z, z\rangle} + \sqrt{\langle A_2 z, z\rangle} \quad \text{for } z \in \mathbf{R}^d, \\ \sqrt{\nu_3(B)} \leqslant \sqrt{\nu_1(B)} + \sqrt{\nu_2(B)} \quad \text{for } B \in \mathcal{B}_0(\mathbf{R}^d)\,. \end{cases} \tag{3.7}$$

Indeed, for $u_1, u_2, u_3 \geqslant 0$, the condition that $u_1 s_1 + u_2 s_2 + u_3 s_3 \geqslant 0$ for all $s = s_1 e^1 + s_2 e^2 + s_3 e^3 \in K$ is expressed as above. Hence, by Theorem 2.11 we get the result.

For example, if $\rho_1 = \rho_2 = \rho$ with triplet (A, ν, γ), then the admissibility condition for $\{\rho, \rho, \rho_3\}$ is that $4A - A_3 \in \mathbf{S}_d^+$ and $4\nu - \nu_3 \geqslant 0$ on $\mathcal{B}_0(\mathbf{R}^d)$.

EXAMPLE 3.3. Let K be the circular cone in $\mathbf{R}^3$ defined by $x_1^2 + x_2^2 \leqslant x_3^2$ and $x_3 \geqslant 0$. Let e^1, e^2, e^3 be as in (3.1). These form a weak basis of K. Notice that the points e^1, e^2, e^3 are located on the circle C defined by $x_1^2 + x_2^2 = 1, x_3 = 1$ and that the triangle $e^1e^2e^3$ is equilateral. Thus K is the union of three cones, each of which is isomorphic to the cone of Example 3.2. Hence we conclude the following. Let $\rho_j \in ID(\mathbf{R}^d)$ with triplet (A_j, ν_j, γ_j) for $j = 1, 2, 3$. Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if, for $(k, l, m) = (1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$,

$$\begin{cases} aA_k + (1-a)A_l - a(1-a)A_m \in \mathbf{S}_d^+ \quad \text{for } 0 < a < 1\,, \\ a\nu_k + (1-a)\nu_l - a(1-a)\nu_m \geqslant 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d) \text{ for } 0 < a < 1 \end{cases} \tag{3.8}$$

or, equivalently,

$$\begin{cases} \sqrt{\langle A_m z, z\rangle} \leqslant \sqrt{\langle A_k z, z\rangle} + \sqrt{\langle A_l z, z\rangle} \quad \text{for } z \in \mathbf{R}^d\,, \\ \sqrt{\nu_m(B)} \leqslant \sqrt{\nu_k(B)} + \sqrt{\nu_l(B)} \quad \text{for } B \in \mathcal{B}_0(\mathbf{R}^d)\,. \end{cases} \tag{3.9}$$

For example, for any $\rho \in ID(\mathbf{R}^d)$, $\{\rho, \rho, \rho\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ and the associated semigroup $\{\mu_s : s \in K\}$ satisfies $\mu_s = \rho$ for any $s \in C$, which is proved from (3.5). As another example, let $\rho_1 = \rho_2 = \rho \in ID(\mathbf{R}^d)$ with triplet (A, ν, γ). Then, like in Example 3.2, $\{\rho, \rho, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if $4A - A_3 \in \mathbf{S}_d^+$ and $4\nu - \nu_3 \geqslant 0$ on $\mathcal{B}_0(\mathbf{R}^d)$.

Suppose that $\mathrm{Supp}(\rho_j) \subseteq L_j$ for $j = 1, 2, 3$, where L_1, L_2, L_3 are linear subspaces of $\mathbf{R}^d$ such that $L_1 + L_2 + L_3$ is the direct sum. Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ only if each ρ_j is trivial, as is seen in Proposition 2.13.

EXAMPLE 3.4. Let K be the least cone in $\mathbf{R}^3$ containing $e^1, \cdots, e^4$, where

$$e^1 = (0, 0, 1)^\top, \quad e^2 = (1, 1, 1)^\top, \quad e^3 = (1, 0, 1)^\top, \quad e^4 = (0, 1, 1)^\top.$$

That is, K is the set of s such that

$$s = \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3 + \alpha_4 e^4 \quad \text{with } \alpha_1, \cdots, \alpha_4 \geqslant 0, \tag{3.10}$$

but this expression of s is not unique. Note that the section $K \cap \{(x_1, x_2, x_3)^\top : x_1, x_2 \in \mathbf{R}\}$ for $x_3 > 0$ is the square with vertices $(0, 0, x_3)^\top$, $(x_3, 0, x_3)^\top$, $(x_3, x_3, x_3)^\top$ and $(0, x_3, x_3)^\top$. Let us use $\{e^1, e^2, e^3\}$ as a weak basis of K. As in Example 3.2, for s and u in $\mathbf{R}^3$, denote $s = s_1 e^1 + s_2 e^2 + s_3 e^3$ and $u_j = \langle u, e^j \rangle$ for $j = 1, 2, 3$. Then we have (3.2). It follows from $e^4 = e^1 + e^2 - e^3$ that $u \in K'$ if and only if

$$\begin{cases} u_j \geqslant 0 \quad \text{for } j = 1, 2, 3, \\ u_1 + u_2 - u_3 \geqslant 0. \end{cases} \tag{3.11}$$

Indeed, if $u \in K'$, then we get (3.11) by letting $s = e^j$, $j = 1, \cdots, 4$; conversely, if (3.11) holds, then $\langle u, s \rangle \geqslant 0$ for all $s \in K$ by (3.10). In particular, there are vectors $u^1, \cdots, u^4 \in K'$ such that $\langle u^1, s \rangle = s_1$, $\langle u^2, s \rangle = s_2$, $\langle u^3, s \rangle = s_1 + s_3$, $\langle u^4, s \rangle = s_2 + s_3$. Let us show that any $u \in K'$ is written as

$$u = \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^3 + \beta_4 u^4 \quad \text{with } \beta_1, \cdots, \beta_4 \geqslant 0. \tag{3.12}$$

Let $u \in K'$. Then, using (3.11), we can find $\beta_1, \cdots, \beta_4 \geqslant 0$ such that

$$\langle u, s \rangle = (\beta_1 + \beta_3) s_1 + (\beta_2 + \beta_4) s_2 + (\beta_3 + \beta_4) s_3.$$

For instance, if $u_1 \leqslant u_3$, let $\beta_1 = 0$, $\beta_2 = u_1 + u_2 - u_3$, $\beta_3 = u_1$, $\beta_4 = u_3 - u_1$, and if $u_1 > u_3$, let $\beta_1 = u_1 - u_3$, $\beta_2 = u_2$, $\beta_3 = u_3$, $\beta_4 = 0$. By rearranging terms we see $\langle u, s \rangle = \langle \beta_1 u^1 + \cdots + \beta_4 u^4, s \rangle$ for $s \in \mathbf{R}^3$ and hence (3.12) holds.

The admissibility condition for K and $\{e^1, e^2, e^3\}$ is as follows. Let $\rho_j \in ID(\mathbf{R}^d)$ with triplet (A_j, ν_j, γ_j). Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if

$$\begin{cases} A_1 + A_2 - A_3 \in \mathbf{S}_d^+ \\ \nu_1 + \nu_2 - \nu_3 \geqslant 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d). \end{cases} \tag{3.13}$$

This is an immediate consequence of Theorem 2.11 and the characterization (3.11).

EXAMPLE 3.5. Example 3.4 is partly generalized as follows. Let K be a cone in $\mathbf{R}^M$. Suppose that there are $e^1, \cdots, e^L$ with $L > M$ such that $\{e^1, \cdots, e^M\}$ is linearly independent and K is the smallest cone that contains $e^1, \cdots, e^L$. This means that K is the set of s such that

$$s = \alpha_1 e^1 + \cdots + \alpha_L e^L \quad \text{with } \alpha_1, \cdots, \alpha_L \geqslant 0.$$

Such a cone is called a polyhedral cone (cf. Rockafellar [9]). We use $\{e^1, \cdots, e^M\}$ as our weak basis of K. For s and u in $\mathbf{R}^M$, we use s_j and u_j in the meaning that $s = s_1 e^1 + \cdots + s_M e^M$ and $\langle u, e^j \rangle = u_j$ for $j = 1, \cdots, M$. Then,

$$\langle u, s \rangle = u_1 s_1 + \cdots + u_M s_M .$$

It follows from the linear independence of $\{e^1, \cdots, e^M\}$ that there are unique expressions

$$e^j = a_1^j e^1 + \cdots + a_M^j e^M \quad \text{for } j = M+1, \cdots, L .$$

Then we can prove the following. The proof is similar to Example 3.4.

(i) $u \in K'$ if and only if

$$\begin{cases} u_j \geqslant 0 & \text{for } j = 1, \cdots, M , \\ a_1^j u_1 + \cdots + a_M^j u_M \geqslant 0 & \text{for } j = M+1, \cdots, L . \end{cases}$$

(ii) Let $\{\rho_1, \cdots, \rho_M\} \subset ID(\mathbf{R}^d)$ and let (A_j, ν_j, γ_j) be the triplet of ρ_j. Then, $\{\rho_1, \cdots, \rho_M\}$ is admissible with respect to $\{e^1, \cdots, e^M\}$ if and only if, for $j = M+1, \cdots, L$,

$$\begin{cases} a_1^j A_1 + \cdots + a_M^j A_M \in \mathbf{S}_d^+ , \\ a_1^j \nu_1 + \cdots + a_M^j \nu_M \geqslant 0 \quad \text{on } \mathcal{B}_0(\mathbf{R}^d). \end{cases}$$

4. Subordination of cone-parameter convolution semigroups

In this section K_1 is an N_1-dimensional cone in $\mathbf{R}^{M_1}$ and K_2 is an N_2-dimensional cone in $\mathbf{R}^{M_2}$. We extend the concept of subordination to the case where subordinators and subordinands have parameters in K_1 and K_2, respectively. Then we discuss inheritance of self-decomposability, the L_m property and stability from subordinator to subordinated. As the subordinators have to be supported on K_2, we begin with the following lemma.

LEMMA 4.1. *Let $\rho \in ID(\mathbf{R}^{M_2})$ with triplet (A, ν, γ). Then* Supp$(\rho) \subseteq K_2$ *if and only if*

$$A = 0 , \quad \nu(\mathbf{R}^{M_2} \setminus K_2) = 0 , \quad \int_{K_2 \cap \{|s| \leqslant 1\}} |s| \nu(ds) < \infty , \quad \gamma^0 \in K_2 . \tag{4.1}$$

Here we recall that $\gamma^0 = \gamma - \int_{K_2 \cap \{|s| \leqslant 1\}} s \nu(ds)$, the drift of ρ. This lemma is found in Skorohod [25], Chapter 3, Theorem 21. A proof can be given by extending the proof of Theorem 21.5 of [11]. Here we have to use Proposition 2.3 as in [8], p. 70–72.

THEOREM 4.2. *Let $\{e^1, \cdots, e^{N_1}\}$ be a weak basis of K_1. Let $\{\rho_s : s \in K_1\}$ be a K_1-parameter convolution semigroup on $\mathbf{R}^{M_2}$. Let (A_s, ν_s, γ_s) be the triplet of ρ_s. Then* Supp$(\rho_s) \subseteq K_2$ *for all $s \in K_1$ if and only if the following conditions* (a) *and* (b) *are satisfied:*

(a) $A_{e^j} = 0$, $\nu_{e^j}(\mathbf{R}^{M_2} \setminus K_2) = 0$, *and* $\int_{K_2 \cap \{|s| \leqslant 1\}} |s| \nu_{e^j}(ds) < \infty$ *for* $j = 1, \cdots, N_1$;

(b) *if* $s_1, \cdots, s_{N_1} \in \mathbf{R}$ *are such that* $s_1 e^1 + \cdots + s_{N_1} e^{N_1} \in K_1$, *then* $s_1 \gamma^0_{e^1} + \cdots + s_{N_1} \gamma^0_{e^{N_1}} \in K_2$, *where* $\gamma^0_{e^j}$ *is the drift of* ρ_{e^j}.

If $\{e^1, \cdots, e^{N_1}\}$ is a strong basis, then condition (b) is simply written as $\gamma^0_{e^j} \in K_2$ for $j = 1, \cdots, N_1$. If $\{\rho_s : s \in K_1\}$ satisfies $\mathrm{Supp}(\rho_s) \subseteq K_2$ for all $s \in K_1$, then we say that it is *supported* on K_2.

PROOF OF THEOREM. Suppose that $\mathrm{Supp}(\rho_s) \subseteq K_2$ for all $s \in K_1$. Then the triplet (A_s, ν_s, γ_s) satisfies (4.1). By Theorem 2.8 we see that $\gamma^0_s = s_1 \gamma^0_{e^1} + \cdots + s_{N_1} \gamma^0_{N_1}$ for $s = s_1 e^1 + \cdots + s_{N_1} e^{N_1} \in K_1$. Hence (a) and (b) hold. The converse is similarly proved. □

Now we introduce subordination of convolution semigroups. For any measure μ and μ-integrable function f, we write $\mu(f) = \int f(x)\mu(dx)$.

THEOREM 4.3. *Let* $\{\mu_u : u \in K_2\}$ *be a* K_2*-parameter convolution semigroup on* $\mathbf{R}^d$ *and* $\{\rho_s : s \in K_1\}$ *a* K_1*-parameter convolution semigroup supported on* K_2. *Define a probability measure* σ_s *on* $\mathbf{R}^d$ *by*

$$\sigma_s(f) = \int_{K_2} \mu_u(f)\rho_s(du) \tag{4.2}$$

for bounded continuous functions f *on* $\mathbf{R}^d$. *Then* $\{\sigma_s : s \in K_1\}$ *is a* K_1*-parameter convolution semigroup on* $\mathbf{R}^d$.

We call this procedure to get $\{\sigma_s : s \in K_1\}$ *subordination of* $\{\mu_u : u \in K_2\}$ *by* $\{\rho_s : s \in K_1\}$. The new convolution semigroup is said to be *subordinate to* $\{\mu_u : u \in K_2\}$ *by* $\{\rho_s : s \in K_1\}$. Sometimes $\{\mu_u : u \in K_2\}$, $\{\rho_s : s \in K_1\}$ and $\{\sigma_s : s \in K_1\}$ are respectively called subordinand, subordinating (or subordinator), and subordinated.

PROOF OF THEOREM. If f is bounded and continuous, then $\mu_u(f)$ is continuous in u by Corollary 2.9, and hence the integral in (4.2) exists. It is linear in f, nonnegative for $f \geqslant 0$, and 1 for $f = 1$. It decreases to 0 whenever $f = f_n(x)$ decreases to 0 on $\mathbf{R}^d$ as $n \to \infty$. Thus there is a unique probability measure σ_s satisfying (4.2) (Dudley [5], Theorem 4.5.2). Moreover, $\{\sigma_s : s \in K_1\}$ is a convolution semigroup. Indeed, we have

$$\hat{\sigma}_s(z) = \int_{K_2} \hat{\mu}_u(z)\rho_s(du) , \quad z \in \mathbf{R}^d . \tag{4.3}$$

Since

$$\begin{aligned} \hat{\sigma}_{s^1+s^2}(z) &= \int_{K_2} \hat{\mu}_u(z)\rho_{s^1+s^2}(du) = \iint_{K_2 \times K_2} \hat{\mu}_{u^1+u^2}(z)\rho_{s^1}(du^1)\rho_{s^2}(du^2) \\ &= \iint_{K_2 \times K_2} \hat{\mu}_{u^1}(z)\hat{\mu}_{u^2}(z)\rho_{s^1}(du^1)\rho_{s^2}(du^2) = \hat{\sigma}_{s^1}(z)\hat{\sigma}_{s^2}(z) , \end{aligned}$$

we have $\sigma_{s^1+s^2} = \sigma_{s^1} * \sigma_{s^2}$. As $\{t_n\}$ strictly decreases to 0, $\rho_{t_n s}$ tends to δ_0, and hence $\hat{\sigma}_{t_n s}(z) \to 1$, that is, $\sigma_{t_n s} \to \delta_0$. □

Let us give the characteristic functions and the triplets of subordinated semigroups. Let $\mathbf{C}$ be the set of complex numbers. For $v = (v_1, \cdots, v_{N_2})^\top$ and $w = (w_1, \cdots, w_{N_2})^\top$ in $\mathbf{C}^{N_2}$, we write $\langle v, w\rangle = \sum_{k=1}^{N_2} v_k w_k$. In the case of ordinary subordination (that is, $K_1 = K_2 = \mathbf{R}_+$) the following theorem reduces to Theorem 30.1 of [11]. In the case where $K_1 = \mathbf{R}_+$ and $K_2 = \mathbf{R}_+^{N_2}$, it is in Theorems 3.3 and 4.7 of [1].

THEOREM 4.4. *Let $\{\mu_u : u \in K_2\}$, $\{\rho_s : s \in K_1\}$, and $\{\sigma_s : s \in K_1\}$ be the subordinand, subordinating, and subordinated convolution semigroups in Theorem* 4.3. *Let $\{h^1, \cdots, h^{N_2}\}$ be a weak basis of K_2. Let $(A_k^\mu, \nu_k^\mu, \gamma_k^\mu)$ be the triplet of μ_{h^k} for $k = 1, \cdots, N_2$. Let ν_s^ρ and $\gamma_s^{0\rho}$ be the Lévy measure and the drift of ρ_s for $s \in K_1$ and decompose $\gamma_s^{0\rho}$ as*

$$\gamma_s^{0\rho} = (\gamma_s^{0\rho})_1 h^1 + \cdots + (\gamma_s^{0\rho})_{N_2} h^{N_2} . \tag{4.4}$$

Let R be the orthogonal projection from $\mathbf{R}^{M_2}$ to the linear subspace L_2 generated by K_2 and let T be the linear transformation from $\mathbf{R}^{M_2}$ onto $\mathbf{R}^{N_2}$ defined by

$$Tu = (u_1, \cdots, u_{N_2})^\top \quad \text{where } Ru = u_1 h^1 + \cdots + u_{N_2} h^{N_2}.$$

Then we have the following.

(i) *For any $s \in K_1$,*

$$\hat{\sigma}_s(z) = \exp \Psi_s^\rho(w) , \quad z \in \mathbf{R}^d , \tag{4.5}$$

where

$$\Psi_s^\rho(w) = \int_{K_2} (e^{\langle w, Tu\rangle} - 1)\nu_s^\rho(du) + \langle T\gamma_s^{0\rho}, w\rangle \tag{4.6}$$

with $w = (w_1, \cdots, w_{N_2})^\top$ given by

$$w_k = -\frac{1}{2}\langle z, A_k^\mu z\rangle + \int_{\mathbf{R}^d} g(z, x)\nu_k^\mu(dx) + i\langle \gamma_k^\mu, z\rangle. \tag{4.7}$$

Here $g(z, x)$ is the function appearing in (1.1).

(ii) *For any $s \in K_1$ the triplet $(A_s^\sigma, \nu_s^\sigma, \gamma_s^\sigma)$ of σ_s is represented as follows*:

$$A_s^\sigma = \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k A_k^\mu , \tag{4.8}$$

$$\nu_s^\sigma(B) = \int_{K_2} \mu_u(B)\nu_s^\rho(du) + \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \nu_k^\mu(B), \quad B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}) , \tag{4.9}$$

$$\gamma_s^\sigma = \int_{K_2} \nu_s^\rho(du) \int_{|x|\leqslant 1} x\mu_u(dx) + \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \gamma_k^\mu . \tag{4.10}$$

(iii) *Fix* $s \in K_1$. *If* $\int_{K_2 \cap \{|u| \leqslant 1\}} |u|^{1/2} \nu_s^\rho(du) < \infty$ *and* $\gamma_s^{0\rho} = 0$, *then* $A_s^\sigma = 0$, $\int_{|x|\leqslant 1} |x| \nu_s^\sigma(dx) < \infty$, *and the drift* $\gamma_s^{0\sigma}$ *is zero.*

(iv) *Let* K_3 *be a cone in* $\mathbf{R}^d$. *If* $\mathrm{Supp}(\mu_u) \subseteq K_3$ *for all* $u \in K_2$, *then* $\mathrm{Supp}(\sigma_s) \subseteq K_3$ *for all* $s \in K_1$ *and*

$$\gamma_s^{0\sigma} = \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \, \gamma_k^{0\mu} . \tag{4.11}$$

PROOF OF THEOREM 4.4 (i). We start from the identity (4.3). For $u = u_1 h^1 + \cdots + u_{N_2} h^{N_2} \in K_2$ we have

$$\begin{aligned} \hat{\mu}_u(z) &= \hat{\mu}_{h^1}(z)^{u_1} \cdots \hat{\mu}_{h^{N_2}}(z)^{u_{N_2}} \\ &= \exp\Big[\sum_{k=1}^{N_2} u_k \Big(-\frac{1}{2}\langle z, A_k^\mu z\rangle + \int_{\mathbf{R}^d} g(z,x) \nu_k^\mu(dx) + i\langle \gamma_k^\mu, z\rangle \Big) \Big] \end{aligned} \tag{4.12}$$

by Theorem 2.8. Define $T\rho_s$ as $(T\rho_s)(B) = \rho_s(T^{-1}(B))$ for $B \in \mathcal{B}(\mathbf{R}^{N_2})$. Let $K_2^\sharp$ be the set of $w = (w_1, \cdots, w_{N_2})^\top \in \mathbf{C}^{N_2}$ such that $\mathrm{Re}\,(u_1 w_1 + \cdots + u_{N_2} w_{N_2}) \leqslant 0$ for all $u_1, \cdots, u_{N_2} \in \mathbf{R}$ satisfying $u_1 h^1 + \cdots + u_{N_2} h^{N_2} \in K_2$. We claim that

$$\int_{\mathbf{R}^{N_2}} e^{\langle w, \tilde{u}\rangle} (T\rho_s)(d\tilde{u}) = \int_{K_2} e^{\langle w, Tu\rangle} \rho_s(du) = \exp \Psi_s^\rho(w) \quad \text{for } w \in K_2^\sharp . \tag{4.13}$$

By [11], Proposition 11.10, the triplet $(A_s^{T\rho}, \nu_s^{T\rho}, \gamma_s^{T\rho})$ of $T\rho_s$ is given by the triplet $(A_s^\rho, \nu_s^\rho, \gamma_s^\rho)$ of ρ_s as

$$A_s^{T\rho} = T A_s^\rho T' , \quad \nu_s^{T\rho} = [\nu_s^\rho T^{-1}]_{\mathbf{R}^{N_2}\setminus\{0\}} ,$$

$$\gamma_s^{T\rho} = T\gamma_s^\rho + \int Tu (1_{\{|\tilde{u}|\leqslant 1\}}(Tu) - 1_{\{|u|\leqslant 1\}}(u)) \nu_s^\rho(du) ,$$

where T' is the transpose of T. Hence, $A_s^{T\rho} = 0$ and

$$\int_{|\tilde{u}|\leqslant 1} |\tilde{u}| \nu_s^{T\rho}(d\tilde{u}) = \int_{|Tu|\leqslant 1} |Tu| \nu_s^\rho(du) \leqslant \mathrm{const} \int_{|u|\leqslant 1} |u| \nu_s^\rho(du) + \int_{|u|>1} \nu_s^\rho(du) < \infty .$$

The drift $\gamma_s^{0T\rho}$ of $T\rho_s$ is represented as $\gamma_s^{0T\rho} = T\gamma_s^{0\rho}$, since

$$\begin{aligned} \gamma_s^{0T\rho} &= \gamma_s^{T\rho} - \int_{|\tilde{u}|\leqslant 1} \tilde{u} \nu_s^{T\rho}(d\tilde{u}) \\ &= T\gamma_s^\rho + \int Tu (1_{\{|\tilde{u}|\leqslant 1\}}(Tu) - 1_{\{|u|\leqslant 1\}}(u)) \nu_s^\rho(du) - \int_{|Tu|\leqslant 1} Tu \nu_s^\rho(du) \\ &= T\gamma_s^\rho - \int_{|u|\leqslant 1} Tu \nu_s^\rho(du) = T\gamma_s^{0\rho} . \end{aligned}$$

Hence, by (4.6), $\int e^{i\langle z,Tu\rangle}\rho_s(du) = \exp \Psi_s^{\rho}(iz)$ for $z \in \mathbf{R}^{N_2}$. If $w \in K_2^{\sharp}$, then Re $\langle w, Tu\rangle \leqslant 0$ for ρ_s-almost every u and hence $\int e^{\langle w,Tu\rangle}\rho_s(du)$ is finite. Now we can apply Theorem 25.17 of [11]. Thus, if $w \in K_2^{\sharp}$, then (4.6) is definable and (4.13) holds.

Now (4.5) follows from (4.3), (4.12), and (4.13), because w of (4.7) belongs to $K_2^{\sharp}$ by Theorem 2.11. This proves (i). □

We prepare lemmas to prove (ii)–(iv). We say that a subclass Λ of $ID(\mathbf{R}^d)$ is bounded if $\sup_{|z|\leqslant 1}\langle z, A_\mu z\rangle$, $\int_{\mathbf{R}^d}(|x|^2 \wedge 1)\nu_\mu(dx)$, and $|\gamma_\mu|$ are bounded with respect to $\mu \in \Lambda$.[1] Here $(A_\mu, \nu_\mu, \gamma_\mu)$ is the triplet of μ.

LEMMA 4.5. *Let Λ be a bounded subclass of $ID(\mathbf{R}^d)$. Then there are constants $C(\varepsilon)$, C_1, C_2, C_3 such that, for all $t \geqslant 0$,*

$$\sup_{\mu\in\Lambda}\int_{|x|>\varepsilon}\mu^t(dx) \leqslant C(\varepsilon)t \quad \textit{for } \varepsilon > 0\,, \tag{4.14}$$

$$\sup_{\mu\in\Lambda}\int_{|x|\leqslant 1}|x|^2\mu^t(dx) \leqslant C_1 t\,, \tag{4.15}$$

$$\sup_{\mu\in\Lambda}\left|\int_{|x|\leqslant 1}x\mu^t(dx)\right| \leqslant C_2 t\,, \tag{4.16}$$

$$\sup_{\mu\in\Lambda}\int_{|x|\leqslant 1}|x|\mu^t(dx) \leqslant C_3 t^{1/2}\,. \tag{4.17}$$

PROOF. Using Example 25.12 of [11], we can extend the proof of Lemma 30.3 of [11]. Details are omitted. □

LEMMA 4.6. *Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbf{R}^d$. Then there are constants $C(\varepsilon)$, C_1, C_2, C_3 such that, for all $s \in K$,*

$$\int_{|x|>\varepsilon}\mu_s(dx) \leqslant C(\varepsilon)|s| \quad \textit{for } \varepsilon > 0\,, \tag{4.18}$$

$$\int_{|x|\leqslant 1}|x|^2\mu_s(dx) \leqslant C_1|s|\,, \tag{4.19}$$

$$\left|\int_{|x|\leqslant 1}x\mu_s(dx)\right| \leqslant C_2|s|\,, \tag{4.20}$$

$$\int_{|x|\leqslant 1}|x|\mu_s(dx) \leqslant C_3|s|^{1/2}\,. \tag{4.21}$$

PROOF. Fix a strictly supporting hyperplane H of K and $s^0 \in K \setminus \{0\}$. Let $K_0 = K \cap (s^0 + H)$. Then, by Proposition 2.3 (ii), K_0 is a compact set. Now $\{\mu_s : s \in K_0\}$ is a

[1] That is, conditions (1)–(3) in E 12.5 of [11] are satisfied. The statement in E 12.5 contains an error; the condition that $\lim_{l\to\infty}\sup_{\mu\in M}\int_{|x|>l}\nu_\mu(dx) = 0$ should be added. Thus boundedness and precompactness are not equivalent.

bounded subclass of $ID(\mathbf{R}^d)$. Indeed, let $\{e^1, \cdots, e^N\}$ be a weak basis of K. Then $s \in K$ is uniquely expressed as $s = s_1 e^1 + \cdots + s_N e^N$, and $s_1, \cdots, s_N$ are continuous functions of s. Hence $\sup_{s \in K_0}(|s_1| + \cdots + |s_N|) < \infty$. This shows boundedness of $\{\mu_s : s \in K_0\}$, in view of (2.9)–(2.11) of Theorem 2.8. Since every $s \in K$ is written as $s = tr$ with some $t \geqslant 0$ and $r \in K_0$, Lemma 4.5 shows that there is $C(\varepsilon)$ such that

$$\int_{|x|>\varepsilon} \mu_s(dx) = \int_{|x|>\varepsilon} {\mu_r}^t(dx) \leqslant C(\varepsilon)t\,.$$

Let $c = \inf_{r \in K_0} |r|$. We have $c > 0$, since $0 \notin K_0$. Hence $t \leqslant c^{-1}|s|$, and we get (4.18) by changing a constant. The other assertions are proved similarly. □

PROOF OF THEOREM 4.4 (ii)–(iv). First let us prove (ii). We rewrite (4.5). For $w = (w_1, \cdots, w_{N_2})^\top$ of (4.7),

$$\begin{aligned}\langle T\gamma_s^{0\rho}, w\rangle = &-\frac{1}{2}\Big\langle z, \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k\, A_k^\mu z\Big\rangle \\ &+ \int_{\mathbf{R}^d} g(z,x)\Big(\sum_{k=1}^{N_2}(\gamma_s^{0\rho})_k\, \nu_k^\mu\Big)(dx) + i\Big\langle \sum_{k=1}^{N_2}(\gamma_s^{0\rho})_k\, \gamma_k^\mu, z\Big\rangle.\end{aligned}$$

This gives the summation terms in (4.8)–(4.10). Further, for w of (4.7),

$$\begin{aligned}\int_{K_2} (e^{\langle w, Tu\rangle} - 1)\nu_s^\rho(du) &= \int_{K_2}\Big(\prod_{k=1}^{N_2} \hat\mu_{h^k}(z)^{u_k} - 1\Big)\nu_s^\rho(du) \\ &= \int_{K_2}(\hat\mu_u(z) - 1)\nu_s^\rho(du) = \int_{K_2}\nu_s^\rho(du)\int_{\mathbf{R}^d}(e^{i\langle z,x\rangle} - 1)\mu_u(dx) \\ &= \int_{K_2}\nu_s^\rho(du)\int_{\mathbf{R}^d} g(z,x)\mu_u(dx) + i\int_{K_2}\nu_s^\rho(du)\Big\langle z, \int_{|x|\leqslant 1} x\mu_u(dx)\Big\rangle.\end{aligned}$$

Here the last equality is valid by Lemma 4.6. Define τ_s by $\tau_s(B) = \int_{K_2}\mu_u(B)\nu_s^\rho(du)$ for $B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\})$. Then, using Lemma 4.6, we can prove that $\int_{\mathbf{R}^d}(1 \wedge |x|^2)\tau_s(dx) < \infty$. Thus we get (4.8)–(4.10), where τ_s gives the first term in the expression (4.9).

To show (iii), let $\int_{K_2 \cap \{|u| \leqslant 1\}} |u|^{1/2}\nu_s^\rho(du) < \infty$ and $\gamma_s^{0\rho} = 0$. Then $A_s^\sigma = 0$ by (4.8). Use (4.9), (4.10) and (4.21) and notice that

$$\begin{aligned}\int_{|x|\leqslant 1}|x|\nu_s^\sigma(dx) &= \int_{K_2}\nu_s^\rho(du)\int_{|x|\leqslant 1}|x|\mu_u(dx) \\ &\leqslant C_3\int_{|u|\leqslant 1}|u|^{1/2}\nu_s^\rho(du) + \int_{|u|>1}\nu_s^\rho(du) < \infty\end{aligned}$$

and that

$$\gamma_s^{0\sigma} = \gamma_s^{\sigma} - \int_{|x|\leqslant 1} x\nu_s^{\sigma}(dx) = \gamma_s^{\sigma} - \int_{K_2} \nu_s^{\rho}(du) \int_{|x|\leqslant 1} x\mu_u(dx) = 0 .$$

Thus (iii) is true.

Let us show (iv). Assume that $\mathrm{Supp}(\mu_u) \subseteq K_3$ for $u \in K_2$. Since $\mathrm{Supp}(\rho_s) \subseteq K_2$ for all $s \in K_1$, we have $\mathrm{Supp}(\sigma_s) \subseteq K_3$ for all $s \in K_1$. Hence, by Lemma 4.1, $\int_{|x|\leqslant 1} |x|\nu_s^{\sigma}(dx) < \infty$. Thus the drift $\gamma_s^{0\sigma}$ of σ_s exists and $\gamma_s^{0\sigma} = \gamma_s^{\sigma} - \int_{|x|\leqslant 1} x\nu_s^{\sigma}(dx)$. The drift $\gamma_u^{0\mu}$ of μ_u also exists and has a similar expression. Now using (4.9) and (4.10), we get (4.11). □

A random variable Y on $\mathbf{R}$ (or its distribution) is said to be of type G if $Y \stackrel{\mathrm{d}}{=} Z^{1/2}X$, where $\stackrel{\mathrm{d}}{=}$ stands for the equality in distribution, X is a standard Gaussian, Z is nonnegative and infinitely divisible, and X and Z are independent (see [10]). Equivalently, Y is of type G if $\mathcal{L}(Y)$ is the same as the distribution at a fixed time of a Lévy process on $\mathbf{R}$ subordinate to Brownian motion. Barndorff-Nielsen and Pérez-Abreu [2] say that an $\mathbf{R}^d$-valued random variable Y (or its distribution) is of type ext$G$ if, for any $c \in \mathbf{R}^d$, $\langle c, Y\rangle$ is of type G. They say that an $\mathbf{R}^d$-valued random variable Y (or its distribution) is of type mult$G$ if

$$Y \stackrel{\mathrm{d}}{=} Z^{1/2}X , \tag{4.22}$$

where X is standard Gaussian on $\mathbf{R}^d$, Z is an $\mathbf{S}_d^+$-valued infinitely divisible random variable, $Z^{1/2}$ is the nonnegative-definite symmetric square root of Z, and X and Z are independent. If Y is of type mult$G$, then Y is of type ext$G$. Maejima and Rosiński [6] say that a probability measure μ on $\mathbf{R}^d$ is of type G (we call it type G in the MR sense) if μ is symmetric, infinitely divisible with arbitrary Gaussian covariance matrix and Lévy measure ν represented as $\nu(B) = E[\nu_0(X^{-1}B)]$ for $B \in \mathcal{B}(\mathbf{R}^d)$ where ν_0 is a measure on $\mathbf{R}^d$ and X is standard Gaussian on $\mathbf{R}$. They show that μ is of type mult$G$ if it is of type G in the MR sense, and that type ext$G$ distributions are not always of type G in the MR sense. Type mult$G$ is related to subordination of cone-parameter convolution semigroups.

THEOREM 4.7. *If $\{\sigma_t : t \geqslant 0\}$ is an $\mathbf{R}_+$-parameter convolution semigroup on $\mathbf{R}^d$ subordinate to the canonical $\mathbf{S}_d^+$-parameter convolution semigroup $\{\mu_u : u \in \mathbf{S}_d^+\}$ by an $\mathbf{R}_+$-parameter convolution semigroup $\{\rho_t : t \geqslant 0\}$ supported on $\mathbf{S}_d^+$, then, for any $t \geqslant 0$, σ_t is of type* mult$G$. *Conversely, any distribution on $\mathbf{R}^d$ of type* mult$G$ *is expressible as σ_1 of such an $\mathbf{R}_+$-parameter convolution semigroup $\{\sigma_t : t \geqslant 0\}$.*

PROOF. Let $\{\sigma_t : t \geqslant 0\}$ be as stated above. Then, by (4.3) and by the definition of the canonical $\mathbf{S}_d^+$-parameter convolution semigroup,

$$\hat{\sigma}_t(z) = \int_{\mathbf{S}_d^+} e^{-\langle z, uz\rangle/2} \rho_t(du) , \quad z \in \mathbf{R}^d . \tag{4.23}$$

Let Z_t be a random variable on $\mathbf{S}_d^+$ with distribution ρ_t, X a standard Gaussian on $\mathbf{R}^d$, where X and Z_t are independent. Then

$$Ee^{i\langle z, Z_t^{1/2}X\rangle} = Ee^{-\langle z, Z_t z\rangle/2} = \int_{\mathbf{S}_d^+} e^{-\langle z, uz\rangle/2}\rho_t(du)\,.$$

Therefore $\sigma_t = \mathcal{L}({Z_t}^{1/2}X)$, that is, σ_t is of type mult$G$.

The converse is obvious, since we can construct, from a given $\mathbf{S}_d^+$-valued infinitely divisible random variable Z, a convolution semigroup $\{\rho_t : t \geqslant 0\}$ supported on $\mathbf{S}_d^+$ with $\rho_1 = \mathcal{L}(Z)$. □

REMARK 4.8. Let $\sigma = \mathcal{L}(Y)$ be a distribution on $\mathbf{R}^d$ of type mult$G$ which satisfies (4.22) using Z and X and let ν^ρ and $\gamma^{0\rho}$ be the Lévy measure and the drift of $\rho = \mathcal{L}(Z)$. Note that ν^ρ is a measure on $\mathbf{S}_d^+$ and $\gamma^{0\rho} \in \mathbf{S}_d^+$. Then, by Theorem 4.7, σ is infinitely divisible and we can apply Theorem 4.4 to find the triplet $(A^\sigma, \nu^\sigma, \gamma^\sigma)$ of σ. Thus, we obtain that

$$\hat{\sigma}(z) = \exp\left[\int_{\mathbf{S}_d^+} (e^{-\langle z, uz\rangle/2} - 1)\nu^\rho(du) - \frac{1}{2}\langle z, \gamma^{0\rho} z\rangle\right],$$

and $A^\sigma = \gamma^{0\rho}$, $\gamma^\sigma = 0$ and $\nu^\sigma(B) = \int_{\mathbf{S}_d^+} \mu_u(B)\nu^\rho(du)$ with $\mu_u = N_d(0, u)$. These results are noticed in [2] without using subordination.

Inheritance of selfdecomposability and the L_m-property from subordinator to subordinated in subordination of an $\mathbf{R}_+^{N_2}$-parameter Lévy process was studied in [1]. In the rest of this section we extend their results to the cone-parameter case. Our method of proof is simpler than that of [1]. However, since we do not consider operator selfdecomposability and operator stability, the results here do not cover those in [1].

A distribution μ on $\mathbf{R}^d$ is said to be *selfdecomposable* if, for every $b > 1$, there is a distribution μ' on $\mathbf{R}^d$ such that

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\mu'}(z)\,, \quad z \in \mathbf{R}^d\,. \tag{4.24}$$

The class of selfdecomposable distributions on $\mathbf{R}^d$ is denoted by $L_0 = L_0(\mathbf{R}^d)$. Thus we also call them *of class* L_0. If $\mu \in L_0$, then μ is infinitely divisible, μ' is uniquely determined by μ and b, and μ' is also infinitely divisible.

For $m = 1, 2, \cdots$, $L_m = L_m(\mathbf{R}^d)$ is inductively defined as follows: $\mu \in L_m(\mathbf{R}^d)$ if and only if $\mu \in L_0(\mathbf{R}^d)$ and, for every $b > 1$, $\mu' \in L_{m-1}(\mathbf{R}^d)$. The class $L_\infty = L_\infty(\mathbf{R}^d)$ is defined to be the intersection of $L_m(\mathbf{R}^d)$ for $m = 0, 1, 2, \cdots$. We have

$$ID \supset L_0 \supset L_1 \supset \cdots \supset L_\infty \supset \mathfrak{S}\,, \tag{4.25}$$

where $\mathfrak{S} = \mathfrak{S}(\mathbf{R}^d)$ is the class of stable distributions on $\mathbf{R}^d$.

DEFINITION 4.9. Let K be a cone in $\mathbf{R}^M$. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbf{R}^d$. It is called *of class* L_m if $\mu_s \in L_m(\mathbf{R}^d)$ for every $s \in K$. Here

$m \in \{0, 1, \cdots, \infty\}$. Let $0 < \alpha \leqslant 2$. We call $\{\mu_s : s \in K\}$ *strictly α-stable* if, for every $s \in K$,

$$\mu_{as}(B) = \mu_s(a^{-1/\alpha}B) \quad \text{for all } a > 0 \text{ and } B \in \mathcal{B}(\mathbf{R}^d)\,. \tag{4.26}$$

If $\mu_{as} = \delta_0$ for all $a > 0$, then it satisfies (4.26) for every α. Our terminology is different from [11] in this respect. In [11] this case is excluded from the definition of strict α-stability. If $\{\mu_s\}$ is supported on a cone and $\mu_s \neq \delta_0$ for some s, then it cannot be strictly α-stable for $\alpha \in (1, 2]$. If $\{\mu_s\}$ is supported on a cone and strictly 1-stable, then μ_s is trivial for all s. These follow from Lemma 4.1.

THEOREM 4.10. *Let $\{\sigma_s : s \in K_1\}$ be a K_1-parameter convolution semigroup on $\mathbf{R}^d$ subordinate to a K_2-parameter convolution semigroup $\{\mu_u : u \in K_2\}$ by a K_1-parameter convolution semigroup $\{\rho_s : s \in K_1\}$ supported on K_2. Let $0 < \alpha \leqslant 2$. Suppose that $\{\mu_u : u \in K_2\}$ is strictly α-stable. Then the following are true.*

(i) *Let $m \in \{0, 1, \cdots, \infty\}$. If $\{\rho_s : s \in K_1\}$ is of class L_m, then $\{\sigma_s : s \in K_1\}$ is of class L_m.*

(ii) *Let $0 < \alpha' \leqslant 1$. If $\{\rho_s : s \in K_1\}$ is strictly α'-stable, then $\{\sigma_s : s \in K_1\}$ is strictly $\alpha\alpha'$-stable.*

We need two lemmas.

LEMMA 4.11. *Let K be a cone in $\mathbf{R}^M$. Let μ be in $L_0(\mathbf{R}^M)$ and satisfy* $\mathrm{Supp}(\mu) \subseteq K$. *Then, for any $b > 1$, the probability measure μ' defined by* (4.24) *satisfies* $\mathrm{Supp}(\mu') \subseteq K$.

PROOF. We fix $b > 1$ and denote by μ'' the probability measure defined by $\hat{\mu}''(z) = \hat{\mu}(b^{-1}z)$. Thus (4.24) means that $\mu = \mu' * \mu''$. Let (A, ν, γ), (A', ν', γ'), and (A'', ν'', γ'') be the triplets of μ, μ', and μ'', respectively. Then, $A = A' + A''$, $\nu = \nu' + \nu''$, and $\gamma = \gamma' + \gamma''$. Applying Lemma 4.1, we have

$$A = 0\,, \quad \nu(\mathbf{R}^M \setminus K) = 0\,, \quad \int_{|s| \leqslant 1} |s|\nu(ds) < \infty\,, \quad \gamma^0 \in K\,,$$

where γ^0 is the drift of μ. Therefore, we have $A' = 0$, $\nu'(\mathbf{R}^M \setminus K) = 0$, $\int_{|s|\leqslant 1} |s|\nu'(ds) < \infty$, and similarly for A'' and ν''. Thus μ' and μ'' have drifts $\gamma^{0'}$ and $\gamma^{0''}$, and $\gamma^0 = \gamma^{0'} + \gamma^{0''}$. Since $\gamma^{0''} = b^{-1}\gamma^0$, we have $\gamma^{0'} = (1 - b^{-1})\gamma^0 \in K$. Now we can conclude that μ' is supported on K, using Lemma 4.1 again. □

LEMMA 4.12. *Let K be a cone in $\mathbf{R}^M$. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup of class L_0 on $\mathbf{R}^d$. Fix $b > 1$ and define μ'_s by*

$$\hat{\mu}_s(z) = \hat{\mu}_s(b^{-1}z)\hat{\mu'_s}(z)\,. \tag{4.27}$$

Then $\{\mu'_s : s \in K\}$ is a K-parameter convolution semigroup.

PROOF. We have $\hat{\mu}_{s^1+s^2}(z) = \hat{\mu}_{s^1}(z)\hat{\mu}_{s^2}(z) = \hat{\mu}_{s^1+s^2}(b^{-1}z)\hat{\mu'}_{s^1}(z)\hat{\mu'}_{s^2}(z)$. On the other hand, $\hat{\mu}_{s^1+s^2}(z) = \hat{\mu}_{s^1+s^2}(b^{-1}z)\hat{\mu'}_{s^1+s^2}(z)$. Since $\hat{\mu}_s(z) \neq 0$, we have $\hat{\mu'}_{s^1+s^2}(z) =$

$\hat{\mu'}_{s^1}(z)\hat{\mu'}_{s^2}(z)$. As t_n strictly decreases to 0, $\hat{\mu}_{t_n s}(z) \to 1$ and hence, by (4.27), $\hat{\mu'}_{t_n s}(z) \to 1$. Therefore, $\{\mu'_s : s \in K\}$ is a K-parameter convolution semigroup. □

PROOF OF THEOREM 4.10. (i) Suppose that $\{\rho_s : s \in K\}$ is of class L_0. Fix $b > 1$. There are ρ'_s and ρ''_s such that $\rho_s = \rho'_s * \rho''_s$ and $\hat{\rho''}_s(z) = \hat{\rho}_s(b^{-1}z)$. Since $\mathrm{Supp}(\rho_s) \subseteq K_2$, we have $\mathrm{Supp}(\rho'_s) \subseteq K_2$ by Lemma 4.11. It is evident that $\mathrm{Supp}(\rho''_s) \subseteq K_2$. Therefore, by (4.3),

$$\hat{\sigma}_s(z) = \int_{K_2} \hat{\mu}_u(z)\rho_s(du) = \iint_{K_2 \times K_2} \hat{\mu}_{u^1}(z)\hat{\mu}_{u^2}(z)\rho'_s(du^1)\rho''_s(du^2)$$
$$= \int_{K_2} \hat{\mu}_{u^1}(z)\rho'_s(du^1) \int_{K_2} \hat{\mu}_{b^{-1}u^2}(z)\rho_s(du^2)\,.$$

Now we utilize the assumption that $\hat{\mu}_{au}(z) = \hat{\mu}_u(a^{1/\alpha}z)$ for $a > 0$. Then

$$\hat{\sigma}_s(z) = \hat{\sigma}_s(b^{-1/\alpha}z) \int_{K_2} \hat{\mu}_{u^1}(z)\rho'_s(du^1)\,. \tag{4.28}$$

By Lemma 4.12, $\int_{K_2} \hat{\mu}_{u^1}(z)\rho'_s(du^1)$ is the characteristic function of a subordinated convolution semigroup. Since $b^{1/\alpha}$ can be an arbitrary real larger than 1, (4.28) shows that $\sigma_s \in L_0$, that is, $\{\sigma_s : s \in K_1\}$ is of class L_0.

If $\{\rho_s : s \in K_1\}$ is of class L_1, then $\{\rho'_s : s \in K_1\}$ is of class L_0 by the definition of the class L_1 and $\int_{K_2} \hat{\mu}_{u^1}(z)\rho'_s(du^1)$ is the characteristic function of a convolution semigroup of class L_0, which, combined with (4.28), shows that $\{\sigma_s : s \in K_1\}$ is of class L_1. Repeating this argument, we see that, if $\{\rho_s : s \in K_1\}$ is of class L_m for some $m < \infty$, then $\{\sigma_s : s \in K_1\}$ is of class L_m. Finally, if $\{\rho_s : s \in K_1\}$ is of class L_∞, then $\{\sigma_s : s \in K_1\}$ is of class L_m for all $m < \infty$, that is, it is of class L_∞.

(ii) Assume that $\{\rho_s : s \in K_1\}$ is strictly α'-stable. Then

$$\hat{\sigma}_{as}(z) = \int_{K_2} \hat{\mu}_u(z)\rho_{as}(du) = \int_{K_2} \hat{\mu}_{a^{1/\alpha'}u}(z)\rho_s(du)$$
$$= \int_{K_2} \hat{\mu}_u(a^{1/(\alpha\alpha')}z)\rho_s(du) = \hat{\sigma}_s(a^{1/(\alpha\alpha')}z)\,.$$

This shows that $\{\sigma_s : s \in K_1\}$ is strictly $\alpha\alpha'$-stable. □

REMARK 4.13. Let Y be a random variable of type mult$G$ on $\mathbf{R}^d$. Then $\mathcal{L}(Y)$ can be embedded into an $\mathbf{R}_+$-parameter convolution semigroup subordinate to the canonical $\mathbf{S}_d^+$-parameter convolution semigroup, which is strictly 2-stable. Hence we can apply Theorem 4.10. Thus, if the $\mathbf{S}_d^+$-valued random variable Z in (4.22) is of class L_m, then Y is of class L_m.

REMARK 4.14. The problem how much we can weaken the assumption of strict α-stability of $\{\mu_u : u \in K_2\}$ in Theorem 4.10 is open even in the case of the ordinary subordination. In the subordination of Brownian motion with drift on $\mathbf{R}^d$ (2-stable but not strictly

2-stable), the selfdecomposability is inherited from subordinator to subordinated if $d = 1$ (Sato [12]), but it is not always inherited if $d \geqslant 2$ (Takano [14]).

Appendix

Proposition 2.3 is obvious in two or three dimensions. Here we present a general proof.

PROOF OF PROPOSITION 2.3. (i) Suppose that L is a linear subspace of $\mathbf{R}^M$ such that $L \cap K = \{0\}$. We will prove that there is an $(M-1)$-dimensional linear subspace H containing L such that $H \cap K = \{0\}$. This will entail the assertion (i) by taking $L = \{0\}$. Let $\dim L = l$. If $l = M - 1$, then there is nothing to prove. Suppose that $0 \leqslant l \leqslant M - 2$. It is enough to show that, under this assumption, there is an $(l+1)$-dimensional linear subspace $\tilde{L}$ of $\mathbf{R}^M$ such that $\tilde{L} \supseteq L$ and $\tilde{L} \cap K = \{0\}$. There is a 2-dimensional linear subspace D such that $D \cap L = \{0\}$. Denote $\tilde{K} = K - L = \{s - y \colon s \in K,\ y \in L\}$ and $K^\sharp = D \cap \tilde{K}$. Then we see that both $\tilde{K}$ and $K^\sharp$ are convex and closed under multiplication by nonnegative reals. Moreover $\tilde{K}$ is a closed set. Indeed, suppose that $x^n \in \tilde{K}$, $n = 1, 2, \cdots$, and $x^n \to x$. Then $x^n = s^n - y^n$ with $s^n \in K$ and $y^n \in L$. If there is a subsequence $\{s^{n_i}\}_{i=1,2,\dots}$ of $\{s^n\}$ such that $|s^{n_i}| \to \infty$, then $|s^{n_i}|^{-1} s^{n_i}$ tends to some $s \in K$ with $|s| = 1$ via a further subsequence while $|s^{n_i}|^{-1} x^{n_i} \to 0$, and hence $|s^{n_i}|^{-1} y^{n_i} \to s \in L$ via this subsequence, which contradicts $L \cap K = \{0\}$. Therefore $\{s^n\}_{n=1,2,\dots}$ is bounded. It also follows that $\{y^n\}_{n=1,2,\dots}$ is bounded. Choosing a convergent subsequence, we see that $x \in K - L = \tilde{K}$. Thus $\tilde{K}$ is closed. It follows that $K^\sharp$ is closed. If x and $-x$ are in $K^\sharp$, then $x = 0$. Indeed, let $x = s - y$ and $-x = s' - y'$ with $s, s' \in K$ and $y, y' \in L$. Then $s + s' = y + y' \in K \cap L = \{0\}$, and hence $s = s' = 0$, showing $x \in D \cap L = \{0\}$. It follows that $K^\sharp$ is a cone or a singleton $\{0\}$. If $K^\sharp$ is a cone, then it is a half line with endpoint 0 or a closed sector in D with angle $< \pi$. In any case there is a straight line $L^\sharp$ in D through 0 such that $L^\sharp \cap K^\sharp = \{0\}$. Now let $\tilde{L} = L + L^\sharp$. If $x \in \tilde{L} \cap K$, then $x - y \in L^\sharp \cap (K - L) = L^\sharp \cap K^\sharp = \{0\}$ for some $y \in L$, and hence $x \in K \cap L = \{0\}$. Hence $\tilde{L} \cap K = \{0\}$ and $\dim \tilde{L} = l + 1$.

(ii) If $0 \in s^0 + H$, then $-s^0 \in H$ and hence $s^0 \in H$, contradicting $H \cap K = \{0\}$. Therefore $0 \notin s^0 + H$. The set H has a representation $H = \{x \colon \langle x, \gamma \rangle = 0\}$ with $\gamma \neq 0$ such that $\langle s, \gamma \rangle > 0$ for all $s \in K \setminus \{0\}$. Thus we have $D = \{x \colon \langle x - s^0, \gamma \rangle \leqslant 0\}$. Let us show that $K \cap D$ is bounded. Suppose, on the contrary, that there is $\{x^n\}_{n=1,2,\dots}$ in $K \cap D$ with $|x^n| \to \infty$. Then $\langle |x^n|^{-1}(x^n - s^0), \gamma \rangle \leqslant 0$ and the limit s of a convergent subsequence of $\{|x^n|^{-1} x^n\}$ satisfies $|s| = 1$, $s \in K$, and $\langle s, \gamma \rangle \leq 0$, which is absurd.

(iii) Let $\{s^n\}$ be a K-decreasing sequence in K. Then $s^1 - s^n = (s^1 - s^2) + \cdots + (s^{n-1} - s^n) \in K$. If $s^1 = 0$, then we have $-s^n \in K$ and hence $s^n = 0$ for all n. Assume that $s^1 \neq 0$. Let $K \cap D$ be the bounded set in the assertion (ii) with s^1 in place of s^0. Using the representation of H in the proof of (ii), we have $\langle s^1 - s^n, \gamma \rangle \geqslant 0$. Hence $s^n \in K \cap D$. It follows that $\{s^n\}_{n=1,2,\dots}$ is bounded. Let $\{s^{n_i}\}$ and $\{s^{m_j}\}$ be subsequences of $\{s^n\}$ convergent to x and y, respectively. If $n_i > m_j$, then $s^{m_j} - s^{n_i} \in K$ and thus $s^{m_j} - x \in K$. Hence $y - x \in K$. Similarly, $x - y \in K$. Hence $x - y = 0$. Thus $\{s^n\}$ is convergent. □

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