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Hausdorff Dimension of a Cantor set on R^1

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1. Introduction

It is well-known that the ergodic properties of one dimensional dynamical system are determined by the spectrum of the Perron-Frobenius operator associated with transformations (for example [2], [3]). In [4] and [5], constructing renewal equations on symbolic dynamics, Mori defined Fredholm matrix $\Phi(z)$. He proved that the determinant $\det(I - \Phi(z))$ plays similar role as the Fredholm determinant of nuclear operators, that is, the zeros of the determinant are the reciprocals of the eigenvalues of the Perron-Frobenius operator. So he also call $\det(I - \Phi(z))$ the Fredholm determinant. He also showed the reciprocal of the Fredholm determinant equals the dynamical zeta function.

Using this idea, constructing α-Fredholm matrix, Mori determined the Hausdorff dimension of Cantor sets generated by piecewise linear transformations on intervals, and studied the ergodic properties of the dynamical system on Cantor sets ([8]).

In [9], he also calculated the Hausdorff dimension of Cantor sets on a plane generated by Koch-like mappings or Sierpinskii-like mappings, using the spectra of the Perron-Frobenius operator associated with piecewise linear mappings on a plane (cf. [7]).

For Cantor sets generated by transformations which are not necessarily piecewise linear, one considers $\log|F'|$ as potential, and approximates a transformation F by piecewise linear transformations on symbolic dynamics. Then zeta functions corresponding to these piecewise linear transformations converges to that of F. Using this fact, Mori ([6]) estimated the Hausdorff dimension of Cantor sets generated by piecewise C^2 and Markov transformations. Jenkinson and Pollicott ([10]) also estimated the Hausdorff dimension of the Cantor set generated by continued fraction expansion in a similar way.

In this article, we will consider a Cantor set generated by a piecewise linear and Markov transformation on $\mathbf{R}^1$ with parameter p. Roughly speaking, our Cantor set is the set of points which returns infinitely many times. We will consider the Hausdorff dimension of this Cantor set as a function of p. If $0 < p < 1/4$, Markov chain is recurrent, hence the Hausdorff

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dimension of our Cantor set equals 1. On the contrary, if $1/4 < p < 1/2$, Markov chain is transient and the Hausdorff dimension of Cantor set is less than 1. Therefore at $p = 1/4$, Markov chain has drastic change from recurent to transient. We may call this a phase transition of Markov chain. But as we will see later, the Hausdorff dimension is a smooth function of p.

2. Rough sketch of the proof in [8]

To make our discussion clear, we will review how to calculate the Hausdorff dimension generated by piecewise linear Markov transformations on a bounded interval, and at the same time we will explain notations. In [8], we considered general piecewise linear transformations on a bounded interval. However, we will treat in this section only piecewise linear and Markov cases.

DEFINITION 1. 1. We call a transformation F from a bounded interval into itself piecewise linear, if there exists at most countable set $\mathcal{A}$, and for each $a \in \mathcal{A}$, there exists a subinterval $\langle a\rangle \subset I$ and $\{\langle a\rangle\}_{a\in\mathcal{A}}$ is a partition of I, and F' is constant on each $\langle a\rangle$.

2. A piecewise linear transformation F is called Markov if for each $a \in \mathcal{A}$, if $F(\langle a\rangle) \cap \langle b\rangle^o \neq \emptyset$, then $\overline{F(\langle a\rangle)} \supset \langle b\rangle$, where J^o and $\overline{J}$ are the interior and the closure of a set J, respectively.

A finite sequence of symbols $w = a_1 \cdots a_n$ ($a_i \in \mathcal{A}$) is called a word, and we define

$$|w| = n, \quad \text{(the length of a word)}$$

$$\langle w\rangle = \bigcap_{i=1}^{n} F^{-i+1}(\langle a_i\rangle)\,. \quad \text{(interval corresponding to a word)}$$

Put $\mathcal{A}_0 \subset \mathcal{A}$ and let $\mathcal{A}_0$ be a finite set. Then we define a Cantor set

$$C = \{x \in I : F^n(x) \in \cup_{a\in\mathcal{A}_0}\langle a\rangle,\ \text{for all } n \geq 0\}\,.$$

We will express the Hausdorff dimension of C by $\dim_H C$. To estimate $\dim_H C$, we cover C by words with same length. Let $\Phi(\alpha)$ be a $\mathcal{A}_0 \times \mathcal{A}_0$ matrix defined by

$$\Phi(\alpha)_{a,b} = \begin{cases} \left|(F|_{\langle a\rangle})'\right|^{-\alpha} & \text{if } F(\langle a\rangle) \supset \langle b\rangle^o, \\ 0 & \text{otherwise.} \end{cases}$$

We call this matrix the Fredholm matrix. Assume that the matrix $\Phi(\alpha)$ is aperiodic and irreducible. Then we get

$$\sum_{|w|=n} |\langle w\rangle|^\alpha = (1, \ldots, 1)\Phi(\alpha)^n \boldsymbol{x}\,, \tag{1}$$

where $\boldsymbol{x}$ is the vector such that $\boldsymbol{x}_a = |\langle a\rangle|^\alpha$ and $|J|$ is the Lebesgue measure of a set J. First note that because the matrix $\Phi(\alpha)$ is nonnegative and irreducible, from the Perron-Frobenius's theorem, the maximal eigenvalue of $\Phi(\alpha)$ is positive and simple. So, if $\Phi(\alpha)$ has no eigenvalues greater than 1, then the right hand term of (1) converges to 0. This shows $\alpha \geq \dim_H C$.

On the contrary, if there exists at least one eigenvalue which is greater than 1, then the right hand term of (1) diverges. To calculate the Hausdorff dimension, we need to estimate all the coverings by intervals, and here we only calculate covers by words with same length. But from the above discussion, we can expect $\alpha \leq \dim_H C$, and we can get the estimate from below rigorously using the Billingsley's theorem ([1]). This says the Hausdorff dimension $\dim_H C$ equals the maximal solution of $\det(I - \Phi(\alpha)) = 0$.

EXAMPLE 1. Let $I = [0, 1]$, $F = 3x \pmod 1$, $\mathcal{A} = \{0, 1, 2\}$, $\mathcal{A}_0 = \{0, 2\}$, and $\langle 0\rangle = [0, \frac{1}{3})$, $\langle 1\rangle = [\frac{1}{3}, \frac{2}{3})$, and $\langle 2\rangle = [\frac{2}{3}, 1]$. The C is the so called mid third Cantor set. For this case,

$$\Phi(\alpha) = \begin{pmatrix} (\frac{1}{3})^\alpha & (\frac{1}{3})^\alpha \\ (\frac{1}{3})^\alpha & (\frac{1}{3})^\alpha \end{pmatrix},$$

and $\det(I - \Phi(\alpha)) = 1 - 2(\frac{1}{3})^\alpha$. The Hausdorff dimension of mid third Cantor set equals $\frac{\log 2}{\log 3}$.

3. Definition

Choose $p, q \in (0, \frac{1}{2}]$ such that $2p + 2q = 1$. Let $\mathcal{A} = \{\pm 0, \pm 1, \dots\}$ be an alphabet, and $\mathcal{A}_N$ be a subset of $\mathcal{A}$ such that $\mathcal{A}_N = \{\pm 0, \pm 1, \dots, \pm(2N-1)\}$. For any symbol $k \in \mathcal{A}$, $\langle \pm k\rangle$ is a corresponding subinterval ($k = 0, 1, 2 \cdots$):

$$\begin{aligned} \langle +2k\rangle &= [k, k + 2p)\,, \\ \langle +(2k+1)\rangle &= [k + 2p, k + 1)\,, \end{aligned}$$

and $\langle -k\rangle$ is the mirror image of the positive part except $\langle -0\rangle = (-2p, 0)$. Then

$$\begin{aligned} &[k, k+1) = \langle +2k\rangle \cup \langle +(2k+1)\rangle\,, \\ &(-1, 0) = \langle -0\rangle \cup \langle -1\rangle\,, \\ &(-k-1, -k] = \langle -2k\rangle \cup \langle -(2k+1)\rangle \quad (k \geq 1)\,, \\ &\bigcup_{k=-\infty}^{\infty} \langle k\rangle = \mathbf{R}\,. \end{aligned}$$

Set a transformation F from $\mathbf{R}$ to $\mathbf{R}$ as follows. For $x \geq 0$, we define F_+ as follows;

$$F_+(x) = \begin{cases} \dfrac{1}{p}(x-k) + k & \text{if } x \in \langle +2k\rangle\,, \\ \dfrac{1}{q}(x-k-1) + k + 1 & \text{if } x \in \langle +(2k+1)\rangle\,, \end{cases}$$

and define F (see Figure 1):

$$F(x) = \begin{cases} F_+(x) & \text{if } x \geq 0\,, \\ -F_+(-x) & \text{if } x < 0\,. \end{cases}$$

Then F satisfies a Markov condition, and its Markov diagram is Figure 3. If $p > q$, for almost every $x \in \mathbf{R}$ the orbit $F^n(x)$ goes to infinity. The reason is the following: We will see this dynamical system as Markov chain. Then the probability from $[k, k+1)$ to go to $[k+1, k+2)$ is greater than the probability from $[k, k+1)$ to $[k-1, k)$. Hence this Markov chain is transient.

Now we are interested in a Cantor set generated by F :

$$C = \{x \in \mathbf{R} : F^n(x) \text{ returns } (-1, 1) \text{ infinitely often}\}\,.$$

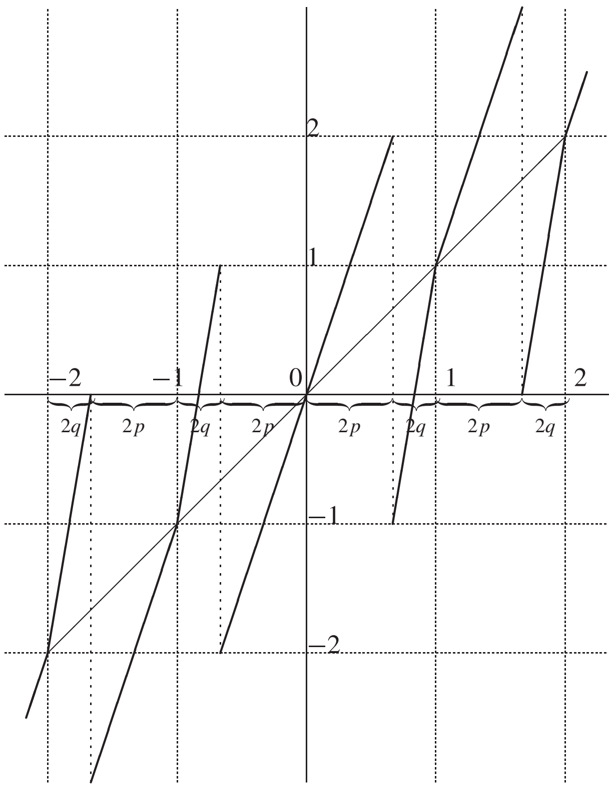

FIGURE 1. The graph of F.

To calculate $\dim_H C$, we consider following four kinds of Cantor sets. For $N \in \mathbf{N}$

$$C_N = \bigcap_{k=0}^{\infty} F_N^{-k}((-N, N)) ,$$

where F_N is a transformation F restricted to $(-N, N)$ (we give the graph of F_2 in Figure 2, for this case, when a point comes in $[1 + p, 1 + 2p)$ and $(-1 - 2p, -1 - p]$, then it will disappear). Induce the transformations F and F_N to $(-1, 1)$, and let $C(-1, 1)$ and $C_N(-1, 1)$ be the sets generated by them, respectively. Similarly we define F_N^+ which is a transformation F restricted to $[0, N)$, and $C_N^+[0, 1)$ is the Cantor set generated by the transformation which is induced F_N^+ to $[0, 1)$. These induced Cantor sets are equal to the restriction: i.e.

$$C(-1, 1) = C \cap (-1, 1) ,$$
$$C_N(-1, 1) = C_N \cap (-1, 1) .$$

Note that C_1^+ looks like the mid third Cantor set, that is, we remove the interval $[p, 2p]$ first from $[0, 1]$, then remove with same ratio repeatedly. But we can not choose p and q such that

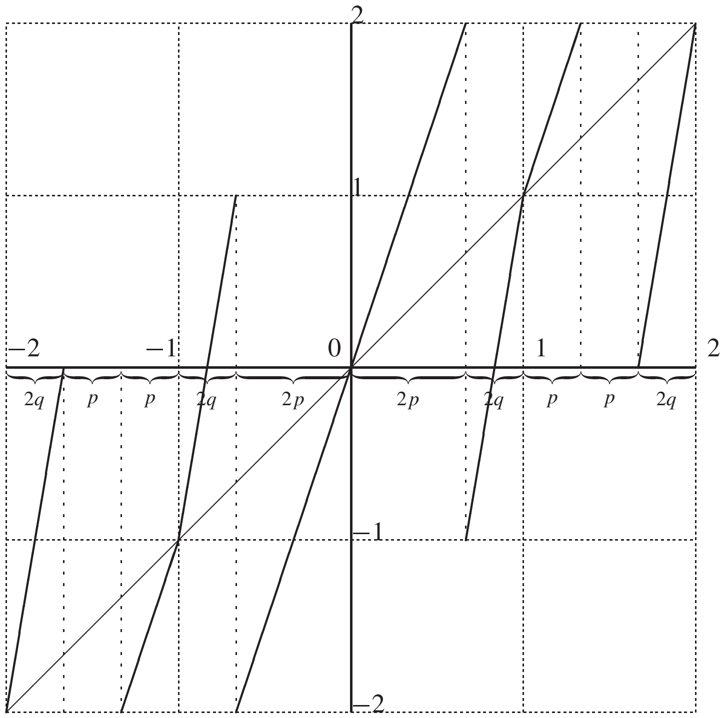

FIGURE 2. The graph of F_2.

C_1^+ becomes the mid third Cantor set. By the definition, we can see easily

$$\begin{aligned}&C_1 \subset C_2 \subset \cdots \subset C\,,\\&C_1(-1,1) \subset C_2(-1,1) \subset \cdots \subset C(-1,1)\,,\\&C_1^+[0,1) \subset C_2^+[0,1) \subset \cdots \subset C\,,\end{aligned}$$

therefore

$$\begin{aligned}&\dim_H C_1 \le \dim_H C_2 \le \cdots \le \dim_H C\,,\\&\dim_H C_1(-1,1) \le \dim_H C_2(-1,1) \le \cdots \le \dim_H C(-1,1)\,,\\&\dim_H C_1^+[0,1) \le \dim_H C_2^+[0,1) \le \cdots \le \dim_H C\,.\end{aligned}$$

At the same time, we can see

$$C_N^+[0,1) \subset C_N(-1,1) \subset C_N\,,$$

therefore we also have

$$\dim_H C_N^+[0,1) \le \dim_H C_N(-1,1) \le \dim_H C_N\,.$$

Let $\Phi(N;\alpha)$ be a $\mathcal{A}_N \times \mathcal{A}_N$ matrix whose components are

$$\Phi(N;\alpha)_{a,b} = \begin{cases} \left|(F|_{\langle a\rangle})'\right|^{-\alpha} & \text{if } F(\langle a\rangle) \cap \langle b\rangle^o \neq \phi\,,\\ 0 & \text{if } F(\langle a\rangle) \cap \langle b\rangle^o = \phi\,,\end{cases}$$

where $a, b \in \mathcal{A}_N$. Similarly $\Phi(\alpha)$, $\Phi^+(N;\alpha)$ are defined by F and F_N^+ respectively. By induced transformations F_N and F_N^+ to $(-1,1)$ and $[0,1)$, we can also define $\Phi_{(-1,1)}(N;\alpha)$ and $\Phi^+_{[0,1)}(N;\alpha)$, respectively. Note that $\Phi(\alpha)$ is an infinite dimensional matrix. We call them Fredholm α-matrices. By the definition of F, F_N and F_N^+, for $a, b \ge +0$,

$$\begin{aligned}\Phi(\alpha)_{a,b} &= \Phi(N;\alpha)_{a,b} = \Phi^+(N;\alpha)_{a,b}\\ &= \begin{cases} p^\alpha\,, & \text{if } a = 2k\,,\ a \le b \le a+3\,,\\ q^\alpha\,, & \text{if } a = 2k+1\,,\ a-3 \le b \le a\,,\\ 0\,, & \text{otherwise}\,.\end{cases}\end{aligned}$$

Let α_N be the maximal zero of $\det(I - \Phi(N;\alpha))$. In other word, $\Phi(N;\alpha_N)$ has an eigenvalue 1. Let also α_N^+, $\alpha_{(-1,1)}(N)$ and $\alpha_{[0,1)}(N)$ be the one for the matrices $\Phi^+(N;\alpha)$, $\Phi_{(-1,1)}(N;\alpha)$ and $\Phi^+_{[0,1)}(N;\alpha)$, respectively. By [8], we get $\dim_H C_N = \alpha_N$, and $\dim_H C_N^+ = \alpha_N^+$. Since $\dim_H C_N$ is monotone increasing with respect to N and $\dim_H C_N \le 1$ (because $C_N \subset \mathbf{R}$), there exists $\lim_{N\to\infty} \dim_H C_N$, and we denote it by α_0. ($\alpha_0 = \lim_{N\to\infty} \alpha_N$)

Using intervals $[0,1)$, $[1,2)\cdots$, we can express F by a Markov diagram as Figure 3. As we mentioned before, the Markov chain is recurrent when $p \le q$, and transient when $p > q$. Define

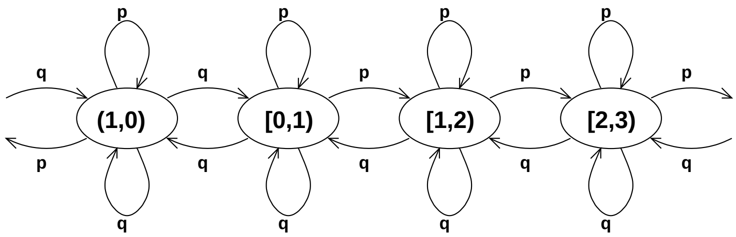

FIGURE 3. Markov diagram of F.

$$\psi_1(\alpha) = p^\alpha \sum_{n=0}^{\infty} (p^\alpha + q^\alpha)^n q^\alpha = \frac{p^\alpha q^\alpha}{1 - p^\alpha - q^\alpha} ,$$

which corresponds to the sum of the words which start from $[0, 1)$, then stay $[1, 2)$ and then return to $[0, 1)$, that is,

$$\psi_1(\alpha) = \sum |\langle w \rangle|^\alpha ,$$

where the sum is over all the words $w = a_1 \cdots a_n$ $(n \geq 2)$ such that $a_1 = +0, a_2, \cdots , a_{n-1} = +2$ or $+3$ and $a_n = +0$ or $+1$. Inductively, we define

$$\psi_N(\alpha) = p^\alpha \sum_{n=0}^{\infty} (p^\alpha + q^\alpha + \psi_{N-1}(\alpha))^n q^\alpha ,$$

which corresponds to the sum of the words which start from $[0, 1)$ then go to $[1, 2)$, going at most $[N, N + 1)$, then go back to $[0, 1)$ for the first time. If $p^\alpha + q^\alpha + \psi_{N-1}(\alpha) < 1$, then the sum converges and we get the recurrence formula:

$$\psi_N(\alpha) = \frac{p^\alpha q^\alpha}{1 - p^\alpha - q^\alpha - \psi_{N-1}(\alpha)} .$$

For convenience, we put $\psi_0(\alpha) = 0$. Using $\psi_N(\alpha)$, we get $\Phi^+_{[0,1)}(N; \alpha)$ and $\Phi_{(-1,1)}(N; \alpha)$

$$\Phi^+_{[0,1)}(N; \alpha) = \begin{pmatrix} p^\alpha + \psi_{N-1}(\alpha) & p^\alpha + \psi_{N-1}(\alpha) \\ q^\alpha & q^\alpha \end{pmatrix} ,$$

$$\Phi_{(-1,1)}(N; \alpha)$$
$$= \begin{pmatrix} q^\alpha & q^\alpha & q^\alpha & q^\alpha \\ p^\alpha + \psi_{N-1}(\alpha) & p^\alpha + \psi_{N-1}(\alpha) & 0 & 0 \\ 0 & 0 & p^\alpha + \psi_{N-1}(\alpha) & p^\alpha + \psi_{N-1}(\alpha) \\ q^\alpha & q^\alpha & q^\alpha & q^\alpha \end{pmatrix} .$$

Then

$$\det(\lambda I - \Phi^+_{[0,1)}(N; \alpha)) = \lambda(\lambda - (p^\alpha + q^\alpha + \psi_{N-1}(\alpha))) ,$$

$$\begin{aligned}&\det(\lambda I - \Phi_{(-1,1)}(N;\alpha))\\&\quad = \lambda^2(\lambda - p^\alpha - \psi_{N-1}(\alpha))(\lambda - (p^\alpha + \psi_{N-1}(\alpha) + 2q^\alpha)).\end{aligned}$$

Now let

$$D(\alpha) = (1 - p^\alpha - q^\alpha)^2 - 4p^\alpha q^\alpha = \left(1 - (p^{\frac{\alpha}{2}} + q^{\frac{\alpha}{2}})^2\right)\left(1 + (p^{\frac{\alpha}{2}} + q^{\frac{\alpha}{2}})^2\right). \tag{2}$$

We will express by α_D the unique solution of $D(\alpha) = 0$.

LEMMA 1. *If $D(\alpha) \geq 0$, that is, $\alpha \geq \alpha_D$, then*

$$\lim_{N\to\infty} \psi_N(\alpha) = \frac{1 - p^\alpha - q^\alpha - \sqrt{D(\alpha)}}{2}.$$

Otherwise, $\psi_N(\alpha)$ diverges.

PROOF. Note first $1 - p^\alpha - q^\alpha - \sqrt{D(\alpha)} > 0$. Let

$$f(x) = \frac{p^\alpha q^\alpha}{1 - p^\alpha - q^\alpha - x}$$

then $\psi_N(\alpha)$ is the N-th iteration of the function $f(x)$ starting from $\psi_0(\alpha)$ (i.e. $\psi_N(\alpha) = f^N(\psi_0(\alpha))$. The solutions of

$$x = \frac{p^\alpha q^\alpha}{1 - p^\alpha - q^\alpha - x}$$

equal $\frac{1-p^\alpha-q^\alpha\pm\sqrt{D(\alpha)}}{2}$. Since $\psi_0(\alpha) = 0$, if $D(\alpha) \geq 0$, then $\psi_N(\alpha)$ converges to the smaller one. □

If $\psi_N(\alpha)$ converges, then we define $\psi(\alpha) = \lim_N \psi_N(\alpha)$.

For convenience, we define $\Phi(0;\alpha) = O$.

LEMMA 2. *For $N = 1, 2, 3, \cdots$, we get*

$$\det(I - \Phi(N;\alpha)) = \left(\prod_{k=0}^{N-1} \det(I - \Phi^+_{[0,1)}(k;\alpha))\right)^2 \det(I - \Phi_{(-1,1)}(N;\alpha)). \tag{3}$$

PROOF. We get the proof by induction. When $N = 1$, the claim is trivial because $\Phi(1;\alpha) = \Phi_{(-1,1)}(1;\alpha)$. Now for $0 \leq k \leq N-1$ the component from $\Phi(\alpha)_{-2k-1,-2k-1}$ to $\Phi(\alpha)_{-2k+2,-2k+2}$, (whose size is 4×4) of $I - \Phi(N;\alpha)$ equals

$$\begin{pmatrix} 1-q^\alpha & -q^\alpha & -q^\alpha & -q^\alpha \\ -p^\alpha & 1-p^\alpha & 0 & 0 \\ 0 & 0 & 1-q^\alpha & -q^\alpha \\ -p^\alpha & -p^\alpha & -p^\alpha & 1-p^\alpha \end{pmatrix},$$

and the other components of rows and columns from $-2k-1$ to $2k+2$ are zero. From the inductive assumption, using the method described below, at the k-th step, we get a matrix of the form:

$$\begin{vmatrix} 1-q^\alpha & -q^\alpha & -q^\alpha & -q^\alpha \\ -p^\alpha-\psi_k(\alpha) & 1-p^\alpha-\psi_k(\alpha) & 0 & 0 \\ 0 & 0 & 1-q^\alpha & -q^\alpha \\ -p^\alpha & -p^\alpha & -p^\alpha & 1-p^\alpha \end{vmatrix}.$$

Now

1. Subtract $\frac{p^\alpha}{p^\alpha+\psi_k(\alpha)}\times$ the second row from the fourth row.
2. Subtract also $\frac{q^\alpha}{p^\alpha+\psi_k(\alpha)}\times$ the second row from the first row.

Then we get a new matrix:

$$\begin{vmatrix} 1 & -\frac{q^\alpha}{p^\alpha+\psi_k(\alpha)} & -q^\alpha & -q^\alpha \\ -p^\alpha-\psi_k(\alpha) & 1-p^\alpha-\psi_k(\alpha) & 0 & 0 \\ 0 & 0 & 1-q^\alpha & -q^\alpha \\ 0 & -\frac{p^\alpha}{p^\alpha+\psi_k(\alpha)} & -p^\alpha & 1-p^\alpha \end{vmatrix}.$$

Next by adding $(p^\alpha+\psi_k(\alpha))\times$ the first row to the second row, we get

$$\begin{vmatrix} 1 & -\frac{q^\alpha}{p^\alpha+\psi_k(\alpha)} & -q^\alpha & -q^\alpha \\ 0 & 1-p^\alpha-q^\alpha-\psi_k(\alpha) & -(p^\alpha+\psi_k(\alpha))q^\alpha & -(p^\alpha+\psi_k(\alpha))q^\alpha \\ 0 & 0 & 1-q^\alpha & -q^\alpha \\ 0 & -\frac{p^\alpha}{p^\alpha+\psi_k(\alpha)} & -p^\alpha & 1-p^\alpha \end{vmatrix}.$$

We can erase the first row and the first column and get the new matrix of the form:

$$\begin{vmatrix} 1-p^\alpha-q^\alpha-\psi_k(\alpha) & -(p^\alpha+\psi_k(\alpha))q^\alpha & -(p^\alpha+\psi_k(\alpha))q^\alpha \\ 0 & 1-q^\alpha & -q^\alpha \\ -\frac{p^\alpha}{p^\alpha+\psi_k(\alpha)} & -p^\alpha & 1-p^\alpha \end{vmatrix}.$$

Next, by adding $\frac{1}{1-p^\alpha-q^\alpha-\psi_k(\alpha)}\times\frac{p^\alpha}{p^\alpha+\psi_k(\alpha)}\times$ the first row to the third row, we get

$$\begin{vmatrix} 1-p^\alpha-q^\alpha-\psi_k(\alpha) & -(p^\alpha+\psi_k(\alpha))q^\alpha & -(p^\alpha+\psi_k(\alpha))q^\alpha \\ 0 & 1-q^\alpha & -q^\alpha \\ 0 & -p^\alpha-\psi_{k+1}(\alpha) & 1-p^\alpha-\psi_{k+1}(\alpha) \end{vmatrix}$$

$$= (1-p^\alpha-q^\alpha-\psi_k(\alpha))\begin{vmatrix} 1-q^\alpha & -q^\alpha \\ -p^\alpha-\psi_{k+1}(\alpha) & 1-p^\alpha-\psi_{k+1}(\alpha) \end{vmatrix},$$

where recall

$$\psi_{k+1}(\alpha) = \frac{p^\alpha q^\alpha}{1-p^\alpha-q^\alpha-\psi_k(\alpha)}.$$

Continuing this procedure $N-1$ times also from the bottom, we get the proof. $\square$

LEMMA 3. 1. $\dim_H C_N^+[0,1) = \alpha_{[0,1)}(N)$.
2. $\dim_H C_N(-1,1) = \alpha_{(-1,1)}(N)$.

PROOF. By the definition of Hausdorff dimension, $\dim_H C_N^+[0,1) \leq \alpha_{[0,1)}(N)$. We will show that $\dim_H C_N^+[0,1)$ is greater than or equal to $\alpha_{[0,1)}(N)$.

We showed following results in [8]: Let C be a Cantor set generated by a piecewise linear, expanding, topologically transitive and Markov transformation F with finite symbols. We construct α-Fredholm matrix $\Phi(\alpha)$, and α_0 be the maximal zero of $\det(I - \Phi(\alpha))$. Then by the eigenvector associated with the eigenvalue 1 of $\Phi(\alpha_0)$, we can construct a 1-dimensional transformation G. The dynamical system on the unit interval associated with G and the dynamical system on C associated with F have the same symbolic dynamics. Thus we can induce the Lebesgue measure on the unit interval where G acts to C, denote it by μ_1. We denote the Lebesgue measure on the interval where F acts by μ_2. Then comparing them, we have

$$C \subset \left\{ x : \lim_{n\to\infty} \frac{\log \mu_1 \langle a_1^x \cdots a_n^x \rangle}{\log \mu_2 \langle a_1^x \cdots a_n^x \rangle} = \alpha_0 \right\}.$$

Then we get by Billingsley's theorem ([1]), between the Hausdorff dimensions of measures $\dim_{\mu_1}$ and $\dim_{\mu_2}$, we have

$$\dim_{\mu_2} = \alpha_0 \dim_{\mu_1} = \alpha_0 .$$

We can apply this discusssion to our case. Namely, the dimension considered by the covering by words of the Lebesgue measure equals $\alpha_{[0,1)}(N)$ which is the maximal solution of $\det(I - \Phi^+_{[0,1)}(N;\alpha)) = 0$. In the case of [8], we get the Hausdorff dimension of C equals the Hausdorff dimension of the measure $\dim_{\mu_2}$ using Markov property. Different from transformations that we considered in [8], the induced transformation F_N to $[0,1)$ has countably many subintervals of monotonicity. So, what we need to show is that the Hausdorff dimension of the Lebesgue measure equals the Hausdorff dimension. What we need to show is: for any interval $J \subset [0,1)$, we need to find a constant $K^+ = K^+(\alpha, N)$ such that there exists a covering $\{\langle w_j \rangle\}$ of $J \cap C_N^+[0,1)$ by words with $\sum |\langle w_j \rangle|^\alpha < K^+ |J|^\alpha$ for $\alpha > \alpha_{[0,1)}(N)$. Here $|J|$ stands for the Lebesgue measure of J. If we can find such a constant K^+ for any covering $\{J_i\}$ of $C_N^+[0,1)$ by intervals, we can take $\{\langle w_{ij} \rangle\}$ such that

1. $\bigcup_j \langle w_{ij} \rangle \supset J_i \cap C_N^+[0,1)$,
2. $\sum_{i,j} |\langle w_{ij} \rangle|^\alpha < K^+ \sum_i |J_i|^\alpha$

This leads to the conclusion that $\dim_H C_N^+[0,1) \geq \alpha_{[0,1)}(N)$, and we get the proof of the lemma.

Now we will determine a constant K^+. For a fixed interval J, we take words with the minimal length which are completely contained in J. Let $a_1 \cdots a_n$ be one of them. Next if $J \not\subset \langle a_1 \cdots a_{n-1} \rangle$, we choose a word with minimal length which are completely contained in

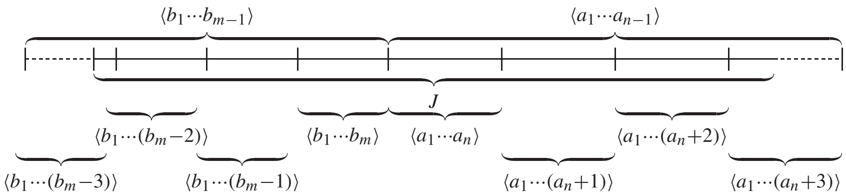

FIGURE 4. Covering of J.

$J\backslash\langle a_1\cdots a_{n-1}\rangle$. Let $b_1\cdots b_m$ be one of them. Each symbol connects to 4 symbols, for example, $+0$ connects to $+0$, 1, 2 and 3. So there exist at most 3 words (for example, $a_1\cdots a_{n-1}a_n$, $a_1\cdots a_{n-1}(a_n+1)$ and $a_1\cdots a_{n-1}(a_n+2)$), contained in J with length n and m, respectively. We can choose another at most 2 words which intersect with J, and those 8 words cover J (see Figure 4). For each one of these 8 words, if it contains a symbol with negative integers including -0, we need not use them to cover $J\cap C_N^+[0,1)$. If it ends with the symbol either $+0$ or $+1$, this word corresponds also to a word in the sense of $C_N^+[0,1)$. If it ends with the positive integer $+2k$ or $+(2k+1)$ greater than $+1$, we need to use them with words in the sense of $C_N^+[0,1)$.

We need another notation. Let $\psi_{[N,k)}(\alpha)$ be a sum corresponding to words starting from $\langle +2k\rangle$ or $\langle +(2k+1)\rangle$, going at most $2N+1$ and going back to $+0$ or 1. Then $\psi_N(\alpha)$ can be divided as follows:

1. words which do not reach $\langle +2k\rangle$ or $\langle +(2k+1)\rangle$,
2. words which reach $\langle +2k\rangle$ or $\langle +(2k+1)\rangle$, and go back to $\langle +0\rangle$ or $\langle +1\rangle$.

The sum of the first one equals $\psi_{k-1}(\alpha)$. To reach $\langle +2k\rangle$ or $\langle +(2k+1)\rangle$, first start from $\langle +0\rangle$ or $\langle +1\rangle$, then reach $\langle +2\rangle$ or $\langle +3\rangle$, stay for a while at $\langle +2\rangle$ or $\langle +3\rangle$, then go to $\langle +4\rangle$ or $\langle +5\rangle$, stay for a while $\langle +2\rangle\cup\langle +3\rangle\cup\langle +4\rangle\cup\langle +5\rangle$, then go to $\langle +6\rangle$ or $\langle +7\rangle$, and so on. Thus the second term must be

$$\begin{aligned}
&p^\alpha \frac{1}{1-p^\alpha-q^\alpha}\, p^\alpha \frac{1}{1-p^\alpha-q^\alpha-\psi_1(\alpha)}\cdots \\
&\qquad p^\alpha \frac{1}{1-p^\alpha-q^\alpha-\psi_{k-2}(\alpha)}\, p^\alpha \psi_{[N,k]}(\alpha) \\
&= \frac{p^\alpha}{(q^\alpha)^{k-1}}\psi_1(\alpha)\cdots\psi_{k-1}(\alpha)\psi_{[N,k]}(\alpha)\,.
\end{aligned}$$

Hence,

$$\psi_{[N,k]}(\alpha) = \frac{(q^\alpha)^{k-1}}{p^\alpha}\,\frac{\psi_N(\alpha)-\psi_{k-1}(\alpha)}{\psi_1(\alpha)\cdots\psi_{k-1}(\alpha)}\,. \tag{4}$$

Recall

$$\psi_m(\alpha) = \frac{p^\alpha q^\alpha}{1 - p^\alpha - q^\alpha - \psi_{m-1}(\alpha)}\,.$$

Thus

$$\psi_{m-1}(\alpha) = 1 - p^\alpha - q^\alpha - \frac{p^\alpha q^\alpha}{\psi_m(\alpha)}\,.$$

Therefore for $m > n$

$$\psi_m(\alpha) - \psi_n(\alpha) = \frac{\psi_m(\alpha)\psi_n(\alpha)(\psi_{m-1}(\alpha) - \psi_{n-1}(\alpha))}{p^\alpha q^\alpha}\,.$$

Substituting this to (4), we get

$$\psi_{[N,k]}(\alpha) = \frac{1}{(p^\alpha)^k}\psi_N(\alpha)\cdots\psi_{N-k+1}(\alpha)\,.$$

For fixed N, put

$$K' = \max_{k=1,\cdots,N}\left\{\frac{1}{(p^\alpha)^k}\psi_N(\alpha)\cdots\psi_{N-k+1}(\alpha)\right\} < \infty\,.$$

Thus, the sum of words which do not start from $[0, 1)$ are less than $K'|J|$. Summing up all these and taking $K^+ = 8K'^\alpha$, we get

$$\sum_{i,j}|\langle w_{ij}\rangle|^\alpha < K^+\sum_i |J_i|^\alpha\,.$$

This proves the lemma. The proof of the second assertion is just the same as above. □

LEMMA 4. $\dim_H C_N = \dim_H C_N(-1, 1)$.

PROOF. In the above equation (3) in Lemma 2, the left-hand side is a polynomial of p^α and q^α, and the right-hand side is the factorization into the rational expression. By the Perron-Frobenius' Theorem, the maximal zero of the left-hand side is simple, so it is not the zero of $\Phi^+_{[0,1)}(k;\alpha)$ $(0 \le k \le N-1)$. Therefore it is the zero of $\Phi_{(-1,1)}(N;\alpha)$. If this zero of $\Phi_{(-1,1)}(N;\alpha)$ is the diverging point of $\Phi^+_{[0,1)}(k;\alpha)$, by the same reason, this zero is not simple. This shows the maximal zero of the equation (3) is the maximal zero of the $\Phi_{(-1,1)}(N;\alpha)$. This proves the lemma. □

4. Eigenvectors

In this section, we consider the conditions that $\Phi(N;\alpha)$ has an eigenvalue equal to 1. From the equation

$$\Phi(N;\alpha)\boldsymbol{x} = \boldsymbol{x}, \quad \boldsymbol{x} = \begin{pmatrix} x_{-2N-1} \\ x_{-2N} \\ \vdots \\ x_{-1} \\ x_{-0} \\ x_{+0} \\ \vdots \\ x_{2N+1} \end{pmatrix},$$

we get the following equations.

$$\begin{aligned}
x_{2N+1} &= q^\alpha x_{2N-2} + q^\alpha x_{2N-1} + q^\alpha x_{2N} + q^\alpha x_{2N+1}\,, && (5)\\
x_{2N} &= p^\alpha x_{2N} + p^\alpha x_{2N+1}\,, && (6)\\
x_{2N-2} &= p^\alpha x_{2N-2} + p^\alpha x_{2N-1} + p^\alpha x_{2N} + p^\alpha x_{2N+1}\,, && (7)\\
&\vdots \\
x_1 &= q^\alpha x_{-1} + q^\alpha x_{-0} + q^\alpha x_{+0} + q^\alpha x_1\,, && (8)\\
x_{+0} &= p^\alpha x_{+0} + p^\alpha x_1\,,\\
x_{-0} &= p^\alpha x_{-0} + p^\alpha x_{-1}\,,\\
x_{-1} &= q^\alpha x_{-1} + q^\alpha x_{-0} + q^\alpha x_{+0} + q^\alpha x_1\,, && (9)\\
&\vdots \\
x_{-2N} &= p^\alpha x_{-2N} + p^\alpha x_{-2N-1}\,,\\
x_{-2N-1} &= q^\alpha x_{-2N+2} + q^\alpha x_{-2N+1} + q^\alpha x_{-2N} + q^\alpha x_{-2N-1}\,,
\end{aligned}$$

By (8) and (9), we get

$$x_1 = q^\alpha x_{-1} + q^\alpha x_{-0} + q^\alpha x_0 + q^\alpha x_1 = x_{-1}\,. \tag{10}$$

From the Perron-Frobenius' theorem, the maximal eigenvalue is simple, so we get $x_k = x_{-k}$ for $0 \le k \le 2N+1$. By (6), we get some constant K for which

$$\begin{pmatrix} x_{2N} \\ x_{2N+1} \end{pmatrix} = \begin{pmatrix} p^\alpha \\ 1 - p^\alpha \end{pmatrix} K\,,$$

and by (10)

$$\begin{pmatrix} 1 - 2q^\alpha \\ 2q^\alpha \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

By (5) and (7), we get

$$\begin{pmatrix} x_{2N-2} \\ x_{2N-1} \end{pmatrix} = A \begin{pmatrix} x_{2N} \\ x_{2N+1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & \frac{p^\alpha}{q^\alpha} \\ -1 & \frac{1-p^\alpha-q^\alpha}{q^\alpha} \end{pmatrix}.$$

Thus, we can inductively get

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = A^N \begin{pmatrix} x_{2N} \\ x_{2N+1} \end{pmatrix} = A^N \begin{pmatrix} p^\alpha \\ 1-p^\alpha \end{pmatrix} K. \tag{11}$$

Note that $\begin{pmatrix} p^\alpha \\ 1-p^\alpha \end{pmatrix}$ is not an eigenvector of A. It is because if $\begin{pmatrix} p^\alpha \\ 1-p^\alpha \end{pmatrix}$ is an eigenvector of A then

$$\begin{aligned} \frac{p^\alpha(1-p^\alpha)}{q^\alpha} &= \lambda p^\alpha, \\ -p^\alpha + \frac{(1-p^\alpha-q^\alpha)(1-p^\alpha)}{q^\alpha} &= \lambda(1-p^\alpha). \end{aligned}$$

So

$$\frac{(1-p^\alpha)}{q^\alpha} = \lambda = -\frac{p^\alpha}{1-p^\alpha} + \frac{(1-p^\alpha-q^\alpha)}{q^\alpha}.$$

Therefore, we get

$$(1-p^\alpha)^2 = -p^\alpha q^\alpha + (1-p^\alpha-q^\alpha)(1-p^\alpha).$$

This leads to the conclusion that $q = 0$, and this is the contradiction.

As in the above observation, the eigenvector associated with the eigenvalue 1 of $\Phi(N;\alpha)$ can be expressed by two eigenvalues of A. Note here

$$\det(\lambda I - A) = \lambda^2 - \frac{1-p^\alpha-q^\alpha}{q^\alpha}\lambda + \frac{p^\alpha}{q^\alpha}.$$

Then, the eigenvalues of A are

$$\lambda_\pm = \frac{1-p^\alpha-q^\alpha \pm \sqrt{D(\alpha)}}{2q^\alpha},$$

and the eigenvectors of A with respect to $\lambda_\pm$ are

$$v_\pm = \begin{pmatrix} 1-p^\alpha-q^\alpha \mp \sqrt{D(\alpha)} \\ 2q^\alpha \end{pmatrix},$$

respectively. Let us decompose $\begin{pmatrix} p^\alpha \\ 1-p^\alpha \end{pmatrix} K$ to eigenvectors of A:

$$\begin{pmatrix} p^\alpha \\ 1-p^\alpha \end{pmatrix} = a_+ \begin{pmatrix} 1-p^\alpha-q^\alpha-\sqrt{D(\alpha)} \\ 2q^\alpha \end{pmatrix} - a_- \begin{pmatrix} 1-p^\alpha-q^\alpha+\sqrt{D(\alpha)} \\ 2q^\alpha \end{pmatrix},$$

where

$$a_\pm = \frac{1-p^\alpha+q^\alpha \pm \sqrt{D(\alpha)}}{4\sqrt{D(\alpha)}} \cdot \frac{2p^\alpha}{1-p^\alpha-q^\alpha \mp \sqrt{D(\alpha)}},$$

Then we get an expression:

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = (a_+\lambda_+^N v_+ - a_-\lambda_-^N v_-)K.$$

LEMMA 5. *If* $p > q$ $(p+q=\frac{1}{2},\ 0<q<\frac{1}{2})$, *then* $\alpha_0 \le \alpha_D$.

PROOF. Note first $1-2q^\alpha > 0$. Otherwise, either x_0 or x_1 is negative. If $\alpha > \alpha_D$, $D(\alpha)$ is positive, therefore the eigenvalues $\lambda_\pm$ are positive, and $\lambda_+\lambda_- > 1$. Then, we get

$$\begin{aligned}\begin{pmatrix} 1-2q^\alpha \\ 2q^\alpha \end{pmatrix} &= \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \\ &= \lambda_+^N a_+ \begin{pmatrix} 1-p^\alpha-q^\alpha+\sqrt{D(\alpha)} \\ 2q^\alpha \end{pmatrix} K - \lambda_-^N a_- \begin{pmatrix} 1-p^\alpha-q^\alpha-\sqrt{D(\alpha)} \\ 2q^\alpha \end{pmatrix} K.\end{aligned}$$

Thus we get

$$\frac{1-2q^\alpha}{2q^\alpha} = \frac{\lambda_+^N a_+(1-p^\alpha-q^\alpha-\sqrt{D(\alpha)}) - \lambda_-^N a_-(1-p^\alpha-q^\alpha+\sqrt{D(\alpha)})}{(\lambda_+^N a_+ - \lambda_-^N a_-)2q^\alpha}.$$

Therefore,

$$\begin{aligned}1-2q^\alpha &= \frac{a_+}{a_+-(\lambda_-/\lambda_+)^N a_-}(1-p^\alpha-q^\alpha-\sqrt{D(\alpha)}) \\ &\quad - \left(\frac{\lambda_-}{\lambda_+}\right)^N \frac{a_-}{a_+-(\lambda_-/\lambda_+)^N a_-}(1-p^\alpha-q^\alpha+\sqrt{D(\alpha)}).\end{aligned}$$

Hence

$$p^\alpha - q^\alpha = -\sqrt{D(\alpha)} + \left(\frac{\lambda_-}{\lambda_+}\right)^N \frac{a_-}{a_+}(-p^\alpha+q^\alpha+\sqrt{D(\alpha)}). \tag{12}$$

If N is sufficiently large, this contradicts the assumption $p > q$. Thus if $\alpha > \alpha_D$, $\Phi(N;\alpha)$ cannot have an eigenvalue equal to 1. Assume that $\alpha_0 > \alpha_D$, then there exists sufficiently large N such that $\alpha_N > \alpha_D$, and for this N, $D(\alpha) > 0$. This is the contradiction. Therefore we get $\alpha_N \le \alpha_D$ for all N, that is, $\alpha_0 \le \alpha_D$. □

LEMMA 6. $\alpha_0 \geq \alpha_D$.

PROOF. Note that by Perron-Frobenius' theorem every $x_0, x_1, \dots, x_{2N+1}$ must be positive, and that $\lambda_\pm$ and $a_\pm$ are complex conjugates, respectively. Thus the argument of λ_+ must be in $[0, \frac{\pi}{2N}]$. When N goes to infinity, the argument converges to 0. This means that $D(\alpha) = 0$ at $\alpha = \alpha_0$. This proves $\alpha_D \leq \alpha_0$. □

Combining Lemma 5 and Lemma 6, we get $\alpha_D = \alpha_0$. This says $\dim_H C \geq \alpha_0 = \alpha_D$.

5. Covering of C

In the previous section, we see $\dim_H C \geq \alpha_0$. Our final purpose is to show that α_0 is equal to $\dim_H C$. To show this, we consider a certain δ-covering of C as follows. Let $f_\alpha(k; n)_{ab}$ be a sum corresponding to words with length n, which start from a, end with b, and there exists no symbol a in the subsequent letters from k to $n-1$, that is,

$$f_\alpha(k; n)_{a,b} = \sum_w |\langle w \rangle|^\alpha ,$$

where, the summation is all over the words w, which satisfy the above condition. $\{f_\alpha(k; n)_{a,b} ; a, b \in \mathcal{A}, n \geq k\}$ gives a cover of C by words. Since $\Phi(\alpha)^n$ corresponds to words with length n, the diagonal components are expressed by $f_\alpha(k; n)$ as follows;

$$\begin{aligned}
&(\Phi(\alpha)^n)_{a,a} \\
&\quad = f_\alpha(k; n)_{a,a} + f_\alpha(k; n-1)_{a,a}\Phi(\alpha)_{a,a} + \cdots + f_\alpha(k; k)_{a,a}(\Phi(\alpha)^{n-k})_{a,a} \\
&\quad = \sum_{i=k}^{n} f_\alpha(k; i)_{a,a}(\Phi(\alpha)^{n-i})_{a,a} .
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&(\Phi(\alpha)^n)_{a,b} \\
&\quad = f_\alpha(k; n)_{a,b} + f_\alpha(k; n-1)_{a,a}\Phi(\alpha)_{a,b} + \cdots + f_\alpha(k; k)_{a,a}(\Phi(\alpha)^{n-k})_{a,b} \\
&\quad = f_\alpha(k; n)_{a,b} + \sum_{i=k}^{n-1} f_\alpha(k; i)_{a,a}(\Phi(\alpha)^{n-i})_{a,b} .
\end{aligned}$$

Let

$$\phi(\alpha)^k_{a,b} = \sum_{n \geq k} (\Phi(\alpha)^n)_{a,b} , \tag{13}$$

that is, $\phi(\alpha)^k_{a,b}$ includes all words start from $\langle a \rangle$ and end with $\langle b \rangle$, and whose lengths are longer than k. Then

$$\begin{aligned}\phi(\alpha)^k_{a,a} &= \sum_{n\geq k}\sum_{i=k}^{n} f_\alpha(k;i)_{a,a}(\Phi(\alpha)^{n-i})_{a,a}\\ &= \sum_{i=k}^{\infty}\sum_{n\geq i} f_\alpha(k;i)_{a,a}(\Phi(\alpha)^{n-i})_{a,a}\\ &= \sum_{i=k}^{\infty} f_\alpha(k;i)_{a,a}\phi(\alpha)^0_{a,a}\,,\end{aligned}$$

and

$$\sum_{n\geq k} f_\alpha(k;n)_{a,a} = \frac{\phi(\alpha)^k_{a,a}}{\phi(\alpha)^0_{a,a}}\,.$$

For $b \neq a$, we get

$$\begin{aligned}\phi(\alpha)^k_{a,b} &= \sum_{n\geq k} f_\alpha(k;n)_{a,b} + \sum_{n\geq k}\sum_{i=k}^{n-1} f_\alpha(k;i)_{a,a}(\Phi(\alpha)^{n-i})_{a,b}\\ &= \sum_{n\geq k} f_\alpha(k;n)_{a,b} + \sum_{i=k}^{\infty}\sum_{n\geq i+1} f_\alpha(k;i)_{a,a}(\Phi(\alpha)^{n-i})_{a,b}\\ &= \sum_{n\geq k} f_\alpha(k;n)_{a,b} + \sum_{i=k}^{\infty} f_\alpha(k;i)_{a,a}\phi(\alpha)^0_{a,b}\,.\end{aligned}$$

Here note that $(\Phi(\alpha)^0)_{a,b} = 0$ if $a \neq b$. Then

$$\sum_{n\geq k} f_\alpha(k;n)_{a,b} = \phi(\alpha)^k_{a,b} - \frac{\phi(\alpha)^0_{a,b}}{\phi(\alpha)^0_{a,a}}\phi(\alpha)^k_{a,a}\,.$$

Thus, by (13) for fixed α, if $\phi(\alpha)^0_{a,b}$ converges, then $\phi(\alpha)^k_{a,b}$ converges to zero as $k \to \infty$. Let $\mathcal{A}_1 = \{-1, -0, +0, +1\}$. For fixed $\delta > 0$, there exists a sufficiently large k such that $\{f_\alpha(k;n)_{a,b};\, a, b \in \mathcal{A}_1, n \geq k\}$ gives a δ-covering of $C[0,1)$, and

$$\begin{aligned}\sum_{b\in\mathcal{A}_1}\sum_{n\geq k} f_\alpha(k;n)_{a,b} &= \frac{\phi(\alpha)^k_{a,a}}{\phi(\alpha)^0_{a,a}} + \sum_{\substack{b\neq a\\ b\mathcal{A}_1}}\left(\phi(\alpha)^k_{a,b} - \frac{\phi(\alpha)^k_{a,a}}{\phi(\alpha)^0_{a,a}}\phi(\alpha)^0_{a,b}\right)\\ &= \frac{\phi(\alpha)^k_{a,a} + \sum_{b\in\mathcal{A}_1}(\phi(\alpha)^k_{a,b}\phi(\alpha)^0_{a,a} - \phi(\alpha)^k_{a,a}\phi(\alpha)^0_{a,b})}{\phi(\alpha)^0_{a,a}} \qquad (14)\end{aligned}$$

covers $\langle a\rangle$.

THEOREM 1. *If $p > q$, then the Hausdorff dimension of C equals $\alpha_0 = \alpha_D$. On the contrary, if $p \leq q$, then the Hausdorff dimension of C equals* 1.

PROOF. As we showed before the Hausdorff dimension of C is greater than or equal to $\alpha_0 = \alpha_D$. Since $\dim_H C = \dim_H C(-1, 1)$ by Lemma 4, instead of C, we consider the Hausdorff dimension of $C(-1, 1)$. By (14), a covering of $C(-1, 1)$ is given as follows.

$$\begin{aligned}&\sum_{a,b\in\mathcal{A}_1}\sum_{n\geq k} f_\alpha(k;n)_{a,b}\\ &\quad=\sum_{a\in\mathcal{A}_1}\left(\frac{\phi(\alpha)^k_{a,a}+\sum_{b\in\mathcal{A}}(\phi(\alpha)^k_{a,b}\phi(\alpha)^0_{a,a}-\phi(\alpha)^k_{a,a}\phi(\alpha)^0_{a,b})}{\phi(\alpha)^0_{a,a}}\right). \qquad (15)\end{aligned}$$

Since the summation over a and b is finite, if $\phi(\alpha)^0_{a,b}$ converges, then (15) converges. By Lemma 1, if $\alpha > \alpha_D = \alpha_0$, then $\psi_N(\alpha)$ converges to $\psi(\alpha) = \frac{1-p^\alpha-q^\alpha-\sqrt{D(\alpha)}}{2}$. Note that

$$1-(p^\alpha+q^\alpha+2\psi(\alpha)) = \sqrt{D(\alpha)} > 0\,,$$

and

$$\sum_{a,b\in\{0,1\}}\phi(\alpha)^0_{a,b} = \frac{1}{1-p^\alpha-q^\alpha-(1+q^\alpha)\psi(\alpha)}\,.$$

This shows $\sum_{b\in\mathcal{A}_1}\phi(\alpha)^0_{a,b} < \infty$. Therefore, there exists a sequence of coverings of C, for which the above sum converges to 0. This proves α is greater than or equal to the Hausdorff dimension of C. Namely, the $\dim_H C \leq \alpha_0$. This proves the first part of the theorem.

On the contrary if $p \leq q$, we get by the equation (12)

$$2p^\alpha + 2q^\alpha = 1\,.$$

Note $2p + 2q = 1$. This shows $\alpha = 1$. □

Since the transformation F depends on p, we express $F = F_p$. So we consider the family of transformations $\{F_p \mid 0 < p < \frac{1}{2}\}$. According to F_p, we denote the corresponding Cantor sets by C_p.

If $0 < p \leq \frac{1}{4}$, then F_p is recurrent. In this case $\dim_H C = 1$. On the contrary, if $\frac{1}{4} < p < \frac{1}{2}$, then F_p is transient. In this case, by the expression (2), $\dim_H C$ is the solution α of

$$p^{\frac{\alpha}{2}} + \left(\frac{1}{2}-p\right)^{\frac{\alpha}{2}} = 1\,.$$

Therefore

$$\frac{d\alpha}{dp} = \frac{\alpha\{p^{\frac{\alpha}{2}-1}-(\frac{1}{2}-p)^{\frac{\alpha}{2}-1}\}}{p^{\frac{\alpha}{2}}\log p + (\frac{1}{2}-p)^{\frac{\alpha}{2}}\log(\frac{1}{2}-p)}\,.$$

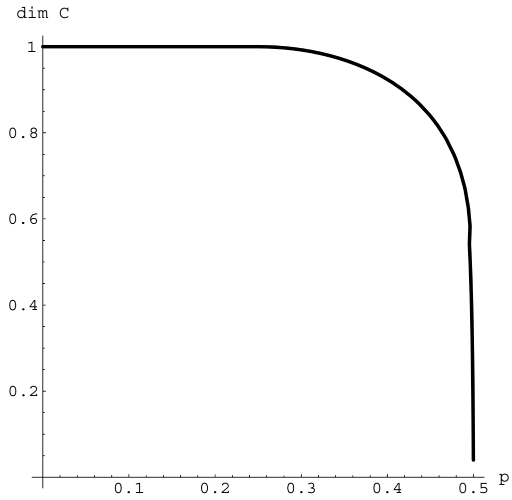

FIGURE 5. Relation between p and $\dim_H C$.

So if $p \downarrow \frac{1}{4}$ then $\frac{d\alpha}{dp} \to 0$. This implies the Hausdorff dimension of C_p is differentiable at $p = \frac{1}{4}$. But, as easily seen, the Hausdorff dimension of C does not belong to C^2 at $p = q = \frac{1}{4}$. Summarizing the results:

THEOREM 2. *The Hausdorff dimension* $\dim_H(C_p)$ *is* C^1 *but not* C^2.

The Hausdroff dimension $\dim_H C$ as a function of p is shown in Figure 5.

6. Example

Let $p = \frac{1}{3}$ and $q = \frac{1}{6}$. Then the Fredholm α-matrix associated with F_1 equals

$$\begin{pmatrix} \left(\frac{1}{6}\right)^\alpha & \left(\frac{1}{6}\right)^\alpha & \left(\frac{1}{6}\right)^\alpha & \left(\frac{1}{6}\right)^\alpha \\ \left(\frac{1}{3}\right)^\alpha & \left(\frac{1}{3}\right)^\alpha & 0 & 0 \\ 0 & 0 & \left(\frac{1}{3}\right)^\alpha & \left(\frac{1}{3}\right)^\alpha \\ \left(\frac{1}{6}\right)^\alpha & \left(\frac{1}{6}\right)^\alpha & \left(\frac{1}{6}\right)^\alpha & \left(\frac{1}{6}\right)^\alpha \end{pmatrix}.$$

In this case $D(\alpha) = 0$ has the solution $\alpha_D \sim 0.97907262$.

The next list shows the maximal solutions of

$$p^\alpha + \psi_N(\alpha) + 2q^\alpha = 1\,,$$

that is, the Hausdorff dimension of C_N calculated by computer. For these value $\Phi(N; \alpha)$ has an eigenvalue 1. and the graph is the relation between p and $\dim_H C$.

	$(p, q) =$	$(\frac{1}{3}, \frac{1}{6})$	$(\frac{499}{1000}, \frac{1}{1000})$	$(\frac{26}{100}, \frac{24}{100})$
$\dim_H C_N$	$N = 2$	0.8723	0.4386	0.9221
	$N = 3$	0.9197	0.4745	0.9592
	$N = 10$	0.9689	0.5144	0.9944
	$N = 30$	0.9775	0.5199	0.9989
	$N = 50$	0.9785	0.5204	0.9993
$\dim_H C$	α_D	0.979073	0.5271	0.99971

References

[1] P.BILLINGSLEY, *Ergodic Theory and Information*, John Wiley (1965).

[2] F. HOFBAUER and G. KELLER, Zeta functions and transfer–operators for piecewise linear transformations, J. reine angew. Math. **352** (1984), 100–113.

[3] A. LASOTA and J.A. YORKE, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[4] M. MORI, Fredholm determinant for piecewise linear transformations, Osaka J. Math. **27** (1990), 81–116.

[5] M. MORI, Fredholm determinant for piecewise monotonic transformations, Osaka J. Math. **29** (1992), 497–529.

[6] M. MORI, Dynamical system on Cantor set, Tokyo J. Math. **21** (1998), 217–232.

[7] M. MORI, Fredholm determinant for piecewise linear transformations on a plane, Tokyo J. Math. **21** (1998), 477–510.

[8] M. MORI, Cantor sets generated by Piecewise Linear Map, Proc. of the Institute of Natural Sciences Nihon University **35** (2000), 1–27.

[9] M. MORI, Dynamical systems on fractals in a Plane, Osaka J. Math. **38** (2001), 321–348.

[10] O. JENKINSON and M. POLLICOTT, Computing the dimension of dynamically defined sets: E_2 and bounded continued fraction, Ergod. Dynam. Systems **21** (2001), 1429–1445.

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