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Some Remarks on the Coarse Index Theorem for Complete Riemannian Manifolds

Shingo KAMIMURA

Keio University

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Abstract. In this paper, we state two results on the coarse index for simple models of the coarse geometry. The first one is a coarse index theorem on trivial bundles over even dimensional closed Riemannian manifolds. It comes from the Poincaré duals for these base manifold in the trivial bundles. We next show some correspondence between the generator of the coarse cohomology of $\mathbf{R}^n$ and the Riesz transformation on $\mathbf{R}^n$.

1. Introduction

In [7] J. Roe has extended the Atiyah-Singer index theorem for closed Riemannian manifolds to bounded geometry, and afterward it has been extended to arbitrary complete Riemannian manifolds in [9], where J. Roe has proposed a new category, called *coarse category*, whose objects consist of metric spaces. In this category any compact metric spaces are identified with the space which consists of a point. So the coarse category can be said to give a method to classify noncompact spaces.

The coarse index theorem states that the coarse indecies of the Dirac operators on complete Riemannian manifolds are invariant on the coarse category. The new category plays role of parameter space in the coarse index theorem as if the topological or homotopical category plays role in the Atiyah-Singer index theorem.

Once we get the coarse index theorem, we may have some questions, for example, when the coarse indecies are computable for given complete noncompact Riemannian manifolds. The purpose of this paper is to attempt the computation of index in the following two cases: Dirac operators on trivial bundles over even dimensional closed Riemannian manifolds, and for Euclidean spaces. These spaces can be regarded as toy models of the coarse geometry, and especially for trivial bundles, various applications are expected from now on. For example, family index theorem, G-equivariant index theorem and the gauge theory.

Let (M, g) be a complete Riemannian manifold. The basic idea to construct invariants on the coarse category is to bound the support of functions on or kernel functions on M

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by using the Riemannian metric g. On this idea, we can obtain the two invariants in this category. The first one arising from the functions is the *coarse cohomology* of M, denoted by $HX^*(M)$. Roughly speaking, it is a *delocalized version* of the Alexander-Spanier cohomology $\bar{H}X^*(M)$. Recall that ([11]) the Alexander-Spanier cochains are functions $\varphi : M^{q+1} \longrightarrow \mathbf{R}$ with support in a bounded neighborhood of the multi-diagonal set $\mathrm{diag}(M^{q+1})$. On the other hand, the coarse cochains have supports *bounded in the direction of* $\mathrm{diag}(M^{q+1})$. So they can extend in the transeversal direction of $\mathrm{diag}(M^{q+1})$. It is easy to find that $HX^*(M)$ has various information of the infinity of M. In fact, for example, the 0-th dimensional cohomology group detects compactness of M and the 1-dimensional cohomology group detects the number of ends of M. In addition to the above property, the coarse cohomology has some interesting singularity. For example, the generator of $HX^1(\mathbf{R})$, denoted by φ_g, is the coboundary of Heaviside function. Strictly speaking, because Heaviside function is not a 0-th coarse cochain , so we should say that Heaviside function is a vartual potential of φ_g. Similarly, the generator of $HX^n(\mathbf{R}^n)$ is given as the n-times cup product of φ_g itself and it is deeply related to the Riesz transformation on $\mathbf{R}^n$.

From the way of its construction, we can define a canonical map from the q-dimensional coarse cohomology group $HX^q(M)$ to the q-dimensional compactly supported Alexander-Spanier cohomology $\bar{H}_c^q(M)$ by restricting the support of the coarse cochain to a bounded neighborhood of $\mathrm{diag}(M^{q+1})$. Composing to the isomorphism between $\bar{H}_c^q(M)$ and $H_{dR,c}^q(M)$ the q-dimensional compactly supported de Rham cohomology, we can get the *topological character* $\chi^t : HX^q(M) \longrightarrow H_{dR,c}^q(M)$, which contributes to the right-hand side of coarse index theorem, i.e. the geometric part of the index thoerem.

The second one, which arises from the kernel functions, is the *Roe algebra*, denoted by $C^*(M)$, which consists of bounded integral smoothing operators A, whose support $\mathrm{supp}(k_A)$ are localized around $\mathrm{diag}(M \times M)$ (we call this condition *bounded propagation* property of A). Strictly speaking, it is not $C^*(M)$ itself but the K_*-group of $C^*(M)$, $K_*(C^*(M))$, that would be the invariant in the coarse category. In the case of arbitrary closed Riemannian manifold M and a Dirac bundle S on M, $C^*(M)$ is nothing but $\mathcal{K}(L^2(S))$, all of compact operators on $L^2(S)$. Recall that the graded Dirac heat operator εe^{-tD^2} on even dimensional closed Riemannian manifolds is a compact operator; in paticular it is a trace-class operator. So εe^{-tD^2} can be a representative of an element of $K_0(\mathcal{K}(L^2(S))$, the K_0-functor of C^*-algebra $\mathcal{K}(L^2(S))$. Well known as the McKean-Singer formula ([6]), $\mathrm{tr}(\varepsilon e^{-tD^2})$, the trace of εe^{-tD^2} coinsides with the Fredholm index of D, $\mathrm{ind}(D)$.

But on arbitrary even dimensional complete Riemannian manifold M, D is not Fredholm and εe^{-tD^2} is not compact, in generally, of course not trace-class. Nevertheless we can show that εe^{-tD^2} is still an element of $C^*(M)$ using finite propagation property of the wave Dirac operator e^{itD} on M ([3]). Moreover it holds the condition of a representative of an element of $K_0(C^*(M))$. From the above argument, the correspondence of D with εe^{-tD^2} could be thought as a canonical *renormalization* of D associated to εe^{-tx^2}. In fact, εe^{-tx^2} is a representative of the generator of $K_0(C_0(\mathbf{R})\times\mathbf{Z}_2)$ and J. Roe has shown that such correspondence is nothing but a functional calculus map $\rho : C_0(\mathbf{R})\times\mathbf{Z}_2 \to C^*(M)$ ([9]). Here, $C_0(\mathbf{R})$ is all of continuous functions on $\mathbf{R}$ vanishing at infinity and $\mathbf{Z}_2$ is the cyclic group of degree 2 including ε as the nontrivial elememt and $\times$ means the C^*-crossed product. Similarly, we can define c-ind(D) for odd dimensional complete Riemannian manifolds by plugging D in the generator of $K_1(C_0(\mathbf{R}))$. Then c-ind(D) can be obtained as a representative of an element $K_1(C^*(M))$.

In formulating the coarse index theorem, it is a serious problem for us how to get the analytic index from $\rho(D)$, because $\rho(D)$ is no longer a trace-class operator. For such difficulty, the Connes' cyclic theory would give a strong prescription to us ([4]). He has proposed the cyclic cocycle as a generalization of the ordinary trace for operators. His basic idea is that we should regard the trace of operators simply as a multi-linear functional with some cyclic condition.

On complete noncompact Riemannian manifolds, c-ind(D) can be thought to have some information of infinity of M. So we must take the cyclic cocycle associated to the coarse cochains of M. In fact, we will define *the cyclic character* χ^c from $HX^q(M)$ to $HC^q(C^*(M))$ the q-dimensional cyclic cohomology of $C^*(M)$. And then, for arbitrary q-dimensional coarse cochain $\varphi \in HX^q(M)$ we define *the analytic index of D associated to φ* putting c-ind$_\varphi(D) := \langle c\text{-ind}(D), \chi^c(\varphi)\rangle$, where $\langle , \rangle$ means Connes' pairing.

The coarse index seems a natural generalization of the usual index, in the sense that these coinside for the case of closed Riemannian manifolds.

This paper is organized as follows. In the section 1, we describe the main results by giving some comments. The section 2 is devoted to the brief introduction to the coarse geometry. In the section 3, We give the proof of the index theorem for the trivial bundles over even dimensional compact manifolds. In the section 4, we compute the coarse index for the Euclidean spaces using a correspondence between the generator of $HX^n(\mathbf{R}^n)$ and the Riesz transformation on $\mathbf{R}^n$ through a Fredholm module on $C^*(\mathbf{R}^n)$.

1.1. Statement of results

THEOREM 1.1.1 (coarse index theorem for trivial bundles). *Let N be an even dimensional closed Riemannian manifold, $M = N \times \mathbf{R}^r$ be a trivial bundle over N, D_M and D_N be Dirac operators on M and N respectively. Then we have*

$$c\text{-ind}_{\varphi_{pd(N)}}(D_M) = c_r \mathrm{ind}(D_N)\,,$$

where $\varphi_{pd(N)}$ is the r-dimensional coarse cohomology class of M determined by the Poincaré dual pd(N) of N in M. The right-hand side of the above equation is the ordinary Fredholm index of D_N and c_r is the constant depending only on r given as follows:

$$c_r = \begin{cases} \dfrac{(r/2)!}{r!(2\pi i)^{r/2}}\,, & \textit{for even-dimensional } M\,, \\ \\ \dfrac{\{(r+1)/2\}!}{r!(2\pi i)^{(r+1)/2}}\,, & \textit{for odd-dimensional } M\,. \end{cases}$$

Theorem 1.1.1 states that by measuring $c\text{-ind}(D_M)$ by the cyclic cocycle arising from the Poincaré dual of N in M, we can take the ordinary Fredholm index of D_N out of $c\text{-ind}(D_M)$. By looking into the contribution of the rank of the trivial bundle to the constant in the above theorem, we have got a little more interesting result as follows.

THEOREM 1.1.2 (coarse index for $\mathbf{R}^n$). *Put* $F := \sum c_j R_j$, *where* R_j *are the Riesz transformations on* $L^2(\mathbf{R}^n)$ *and* c_j *the canonical Clifford actions on* $\mathbf{R}^n$ *and let* φ_g *be the generator of* $HX^n(\mathbf{R}^n)$, *then*

$$c\text{-ind}_{\varphi_g}(D_{\mathbf{R}^n}) = \langle \tau_{\sigma(F)}, \rho(D_{\mathbf{R}^n}) \rangle = C_n$$

where $\tau_{\sigma(F)}$ *is a canonical cyclic cocycle for Fredholm module* $(C^*(\mathbf{R}^n), F, L^2(\mathbf{R}^n))$ *and* C_n *is the constant appeared in the previous theorem.*

This theorem indicates that there is a natural correspondence between the generator of $HX^n(\mathbf{R}^n)$ and the total Riesz transformation on $L^2(\mathbf{R}^n)$.

2. Coarse geometry

In this section, we give a brief review of the coarse geometry given by J. Roe [9, 10]. We, however, give a slightly different definition of coarse index from Roe's one, though they are equivarent. We first introduce coarse category, in which the most remarkable fact is that every compact space has the same coarse type to a point, in other words, any compact spaces are trivial, and noncompact complete Riemannian manifolds have a nontrivial coarse structure in general. We next give two invariants on coarse spaces. The first is an algebra on complete Riemannian manifolds, called coarse C^*-algebra, consist of all bounded integral operators with finite propagation speed (strictly speaking, we must take the C^*-crosure of the above algebra and farthermore take the K-groups of it.), and the second is a cohomology on complete Riemannian manifolds, called coarse cohomology. Roughly speaking, it consists of Alexander-Spanier cochains which become non-acyclic because of the metric structure of complete Riemannian manifolds and we can calcurate it taking coarser cover of complete Riemannian manifolds as its Čech homology. The last subsection, we discribe a index theorem on complete Riemannian manifolds, called coarse index theorem, which is a natural generalization of the Atiyah-Singer index theorem for compact Riemannian manifolds.

2.1. Coarse structure

DEFINITION 2.1.1 (coarse map). Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$, not necessarily continuous, will be called a *coarse map* if
(a) (Uniform expansiveness) For every $r > 0$, there exists $s > 0$ such that

$$\forall x, x' \in X, d_X(x, x') < r \Rightarrow d_Y(f(x), f(x')) < s\,.$$

(b) (Metric properness) For each bounded set $B \subseteq Y$, the inverse image $f^{-1}(B)$ is bounded in X.

REMARK 2.1.2. For any continuous maps between proper metric spaces, where any bounded closed subsets are compact, metric properness is equivalent to ordinary properness. In paticuler, any complete Riemannian manifolds are proper metric spaces.

DEFINITION 2.1.3 (coarse equivalence). Two coarse maps $f_0, f_1 : X \to Y$ are *coarsely equivalent* ($f_0 \sim f_1$), if there exists a constant K such that

$$d(f_0(x), f_1(x)) \leq K$$

for all $x \in X$.

DEFINITION 2.1.4 (coarse type). Two metric spaces have the same *coarse type* ($X \cong Y$), if there exist coarse maps $f : X \to Y$, $g : Y \to X$ such that

$$g \circ f \sim id_X \quad and \quad f \circ g \sim id_Y\,.$$

EXAMPLE 2.1.5. Every compact metric spaces has the same coarse type to a point.

2.2. Roe algebra and functional calculus map. To define the index of Dirac operators on any complete Riemannian manifolds, which are not Fredholm in general, we first introduce the *Roe algebra* as a reciever of the index of Dirac operators, whose K_*-groups are invariant on coarse category.

DEFINITION 2.2.1 (Roe algebra). Let M be a complete Riemannian manifold, S be a Hermitian vector bundle over M, and $L^2(M, S)$ be all of L^2-sections of S. We define the *Roe algebra* of M, denoted by $C^*(M)$, as follows: $C^*(M)$ is the C^*-closure of $Cont(M)$, which consists of all bounded integral operators A on $L^2(M, S)$ whose kernel is smooth and the support of whose kernel is within a R-neighborhood of the diagonal set of $M \times M$, $N_R(diag(M^2))$. We call such operators A *a controlled operator*.

REMARK 2.2.2 (compact case). If M is a compact Riemannian manifold, $C^*(M)$ is nothing but $\mathcal{K}(L^2(M, S))$, the C^*-algebra consisting of all compact operators on $L^2(M, S)$.

To define some functional calculus map, we use the following fact, which is well-known as *finite propagation speed* of the wave operator on complete Riemannian manifolds:

LEMMA 2.2.3 (finite propagation speed [3]). *Let M be a complete Riemannian manifold, S be a Dirac bundle over M, and D be a Dirac operator on S. Then there exists a* *constant c such that for all* $s \in C_c^\infty(S)$,

$$supp(e^{itD}s) \subset N_{c|t|}(supp(s))\,,$$

where $C_c^\infty(S)$ *is all of compactly supported sections in* S *and* $supp(s)$ *is the support of* s.

Using Lemma 2.2.3 , we can get the following functional calculus map.

PROPOSITION 2.2.4 (odd case ([9])). *There exists a unique* C^**-homomorphism* ρ_{odd},

$$\rho_{odd} : C_0(\mathbf{R}) \to C^*(M)$$

such that for all $f \in \mathcal{S}(\mathbf{R})$,

$$\rho_{odd}(f) := f(D) = \int_{\mathbf{R}} \hat{f}(t) e^{itD} dt\,,$$

where $C_0(\mathbf{R})$ *is all of continuous functions on* $\mathbf{R}$ *vanishing at infinity and* $\mathcal{S}(\mathbf{R})$ *is functions of Schwartz class. We also denote the lift of the above* C^**-homomorphism to its* K*-theory by* ρ_{odd}.

OBSERVATION 2.2.5. *If the dimension of* M *is even, there naturaly exists a* $\mathbf{Z}_2$*-grading operator* ε *on* $L^2(M, S)$ *such that* $D\varepsilon + \varepsilon D = 0$. *On the other hand, we can consider that the cyclic group* $\mathbf{Z}_2 = \{e, \varepsilon\}$ *of order* 2 *acts on* $\mathbf{R}$ *such that for all* $x \in \mathbf{R}$, $\alpha_e(x) = x$, $\alpha_\varepsilon(x) = -x$. *Moreover, this action naturaly lift up to* $C_0(\mathbf{R})$, *which is a function algebra on* $\mathbf{R}$. *So replacing* $C_0(\mathbf{R})$ *to* C^**- crossed product* $C_0(\mathbf{R}) \times_\alpha \mathbf{Z}_2$, *we define the coarse index homomorphism in even case as follows, which is slightly different from the original one* [10, 11].

PROPOSITION 2.2.6 (even case ([7])). *There exists a unique* C^**-homomorphism* ρ_{even},

$$\rho_{even} : C_0(\mathbf{R}) \times_\alpha \mathbf{Z}_2 \to C^*(M)$$

such that for all $f_e, f_\varepsilon \in \mathcal{S}(\mathbf{R})$,

$$\begin{aligned} \rho_{even}(f_e U_e + f_\varepsilon U_\varepsilon) &:= f_e(D)\rho_{even}(U_e) + f_\varepsilon(D)\rho_{even}(U_\varepsilon) \\ &= f_e(D)e + f_\varepsilon(D)\varepsilon\,, \end{aligned}$$

where $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* U_e *and* U_ε *are corresponding unitary operators for* e *and* ε, *respectively.*

Using the above functional calculus map, we now define the coarse index of D.

DEFINITION 2.2.7 (coarse index). Let P_g be a generator of $K_0(C_0(\mathbf{R}) \times_\alpha \mathbf{Z}_2)$ and U_g be a generator of $K_1(C_0(\mathbf{R}))$. Then we define the coarse index of D as follows.

$$c\text{-ind}(D) := \begin{cases} \rho_{even}(P_g) \in K_0(C^*(M)) \\ \rho_{odd}(U_g) \in K_1(C^*(M))\,. \end{cases}$$

REMARK 2.2.8 (generator). The above generators, for example, can be concretely described as follows: For a projection $\frac{1}{1+x^2}\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $C_0(\mathbf{R}) \times_\alpha \mathbf{Z}_2$,

$$P_g := \left[\frac{1}{1+x^2}\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \in K_0(C_0(\mathbf{R}) \times_\alpha \mathbf{Z}_2)\,,$$

and for a unitary $\frac{x+i}{x-i} - 1$ in $C_0(\mathbf{R})$,

$$U_g := \left[\frac{x+i}{x-i} - 1\right] \in K_1(C_0(\mathbf{R}))\,.$$

Geometrically, if we regard x as a linear map on $\mathbf{R}$, $\frac{1}{1+x^2}\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix}$ is just the *graph projection* of x on $\mathbf{R} \times \mathbf{R}$. And $\frac{x+i}{x-i}$ is just the *Cayley map* from $\mathbf{R}$ to $\mathbf{C}$, whose mapping degree is $+1$.

2.3. Coarse cohomology and character map

DEFINITION 2.3.1 (coarse cohomology). Let M be a metric spaces. The *coarse complex* $(CX^q(M), \delta)$ is defined as follows: $CX^q(M)$ is the space of locally bounded functions $\varphi : M^{q+1} \to \mathbf{R}$ which satisfy the following support conditions: for each $R > 0$, the set

$$supp(\varphi) \cap N_R(diag(M^{q+1}))$$

is relatively compact in M^{q+1}. And δ is defined as the usual coboundary map of Alexander-Spanier cohomology [11], that is

$$(\delta\varphi)(x_0, \cdots, x_{q+1}) = \sum_{j=0}^{q+1}(-1)^j \varphi(x_0, \cdots, \hat{x_j}, \cdots, x_{q+1})\,.$$

Then we define the *coarse cohomology* $HX^*(M)$ to be the cohomology of the above complex.

REMARK 2.3.2. It is known that the coarse complex is replaced by the subcomplex consisting of all continuous functions, or all of smooth functions, if M is a smooth manifold. We can see in [5, 9] that these replacements dose not change the cohomology with coefficients in the constant sheaf $\mathbf{R}$.

DEFINITION 2.3.3 (cup product [9]). For any $\varphi \in CX^p(M)$ and $\psi \in CX^q(N)$, we define a cup product of them $\varphi \cup \psi \in CX^{p+q}(M \times N)$ as follows:

$$(\varphi \cup \psi)((x_0, y_0), \cdots, (x_{p+q}, y_{p+q})) := \varphi(x_0, \cdots, x_p)\psi(x_p, \cdots, x_{p+q})\,.$$

This definition is indused from a natural cup product of Alexander-Spanier cochains ([11]) and it passes to a product on cohomology.

It is a well-known fact [11] in algebraic topology that the compactly supported de Rham cohomology group $H^*_{dR,c}(M)$ is naturally isomorphic to the compactly supported Alexander-Spanier cohomology group $\bar{H}^*_c(M)$. Using this fact, we can transrate the delocalized information $HX^*(M)$ into the local one $H^*_{dR,c}(M)$ as follows:

DEFINITION 2.3.4 (topological character). We define the homomorphism χ^t,

$$\chi^t : HX^q(M) \longrightarrow H^q_{dR,c}(M)\,,$$

by sending

$$\varphi \in CX^q(M) \longmapsto \chi^t(\varphi) := \varphi|_{N_R(diag(M^{q+1}))} \in \bar{C}^q_c(M)\,,$$

where $R > 0$ is an arbitrary real number and $\bar{C}^q_c(M)$ is compactly supported Alexander-Spanier cochain group. We call χ^t the *topological character*.

Now we define the character valued in the cyclic cohomology (see [4]) of the coarse C^*-algebra $HC^*(C^*(M))$. Using this character, we can measure the coarse index.

DEFINITION 2.3.5 (cyclic character). We define the homomorphism χ^c,

$$\chi^c : HX^q(M) \longrightarrow HC^q(C^*(M))\,,$$

by sending

$$\varphi \in CX^q(M) \longmapsto \chi^c(\varphi) \in CC^q(C^*(M))$$

such that

$$\chi^c(\varphi)(A_0, \cdots, A_q) := \int_{M^{q+1}} k_{A_0}(x_0, x_1) \cdots k_{A_q}(x_q, x_0)\varphi(x_0, \cdots, x_q)dx_0 \cdots dx_q\,,$$

where $CC^q(C^*(M))$ is the cyclic cochain group. We call χ^c *cyclic character*. Considering the support conditions of k_{A_j} and φ, we can easily understand the well-defindness of the above integration.

REMARK 2.3.6 (φ-coarse index). Evaluating $c - \text{ind}(D)$ with $\chi^c(\varphi)$, we can get the coarse index of Dirac operators on complete Riemannian manifolds associated φ, i.e. as [4] we define the pairing

$$c\text{-ind}_\varphi(D) := \langle c\text{-ind}(D), \chi^c(\varphi)\rangle.$$

2.4. Coarse index theorem. Now we discribe the coarse index theorem by J. Roe. The key of this theorem is the localization of index theorem by Connes and Moscovici [5]:

THEOREM 2.4.1 (coarse index theorem). *Let M be a complete Riemannian manifold, D a Dirac operator on M and φ be a q-th coarse cohomology class of M. Then,*

$$c\text{-ind}_\varphi(D) = c_q\langle \chi^t(\varphi) \cup ch(\sigma(D)) \cup td(M), [M]\rangle\,,$$

where $ch(\sigma(D))$ *is the Chern character of the principal symbol of* D, $td(M)$ *and* $[M]$ *are the Todd class and the fundamental homology class of* M, *respectively. Here*, c_q *is the constant depending only on* q *giving as follows*,

$$c_q = \begin{cases} \dfrac{(q/2)!}{q!(2\pi i)^{q/2}}, & \textit{for } \dim M \textit{ even}, \\ \dfrac{\{(q+1)/2\}!}{q!(2\pi i)^{(q+1)/2}}, & \textit{for } \dim M \textit{ odd}. \end{cases}$$

3. Coarse index theorem for trivial bundles

In this section, we prove the coarse index theorem for trivial bundles over even dimensional closed Riemannian manifolds. The result can be seen very natural as follows.

3.1. Coarse cocycle determined by Poincaré dual. In Theorem 1.1.1, $\varphi_{pd(N)}$ is the r-dimensional coarse cohomology class in $HX^r(M)$ determined by the Poincaré dual of N in M. In this subsection, we see how $\varphi_{pd(N)}$ is determined by $pd(N)$ by showing the following lemma.

LEMMA 3.1.1. *Let M be a complete Riemannian manifold as in Theorem* 1.1.1. *Then the topological character*,

$$\chi^t : HX^r(M) \longrightarrow H^r_{dR,c}(M)$$

is an isomorphism. In particular, $\chi^t(\varphi_{pd(N)}) = pd(N)$.

PROOF. Computation for $HX^r(M)$: By Example 2.1.5,

$$N \cong \{pt\}.$$

So,

$$\begin{aligned} HX^r(M) &= HX^r(N \times \mathbf{R}^r) \\ &\cong HX^r(\{pt\} \times \mathbf{R}^r) \\ &\cong HX^r(\mathbf{R}^r) \\ &= \mathbf{R}. \end{aligned}$$

The last equation can be refered to [9].
Computaion for $H^r_{dR,c}(M)$: By Poincaré lemma for trivial bundles (for example, see [2]),

$$H^q_{dR,c}(N \times \mathbf{R}^r) \cong H^{q-r}_{dR,c}(N).$$

Using this fact,

$$H^r_{dR,c}(M) = H^r_{dR,c}(N \times \mathbf{R}^r)$$

$$\cong H^0_{dR,c}(N)$$
$$= \mathbf{R}\,.$$

Thus

$$\dim(HX^r(M)) = \dim(H^r_{dR,c}(M))\,.$$

On the other hand, χ^t is clearly surjective by the definition. So the result follows.

3.2. Proof of Theorem 1.1.1. By Theorem 2.4.1 and Lemma 3.1.1

$$\begin{aligned} c\text{-ind}_{\varphi_{pd(N)}}(D_M) &:= \\ \langle \rho(D_M), \chi^c(\varphi_{pd(N)})\rangle &= c_q\langle \chi^t(\varphi_{pd(N)}) \cup ch(\sigma(D_M)) \cup td(M), [M]\rangle \\ &= c_q\langle pd(N) \cup ch(\sigma(D_M)) \cup td(M), [M]\rangle \end{aligned}$$

Using the fact $\varphi_{pd(N)}$ is Poincaré dual to N,

$$c_q\langle \chi^t(\varphi_{pd(N)}) \cup ch(\sigma(D_M)) \cup td(M), [M]\rangle = c_q\langle ch(\sigma(D_M)) \cup td(M), [N]\rangle\,.$$

However, the restriction of $ch(\sigma(D_M)) \cup td(M)$ to N is just $ch(\sigma(D_N)) \cup td(N)$, because of the fact that the normal bundle to N in M is trivial, so that $td(M)|_N$ is the same as $td(N)$. Thus,

$$\begin{aligned} c_q\langle ch(\sigma(D_M)) \cup td(M), [N]\rangle &= c_q\langle ch(D_N) \cup td(N), [N]\rangle \\ &= c_q \operatorname{ind}(D_N)\,. \end{aligned}$$

This completes the proof of lemma.

COROLLARY 3.2.1 (index theorem for splitting manifolds). *In Theorem* 1.1.1, *if we put* $r = 1$, *then we can get the index theorem for splitting manifolds by J. Roe* [8]. *So, we can see that Theorem* 1.1.1 *gives a generalization of the case of splitting manifolds.*

4. Coarse indices for $\mathbf{R}^n$

Let φ_g be a generator of coarse cohomology group of $\mathbf{R}$. In this section we show that $\chi^c(\varphi_g)$ gives a self-adjoint involution on $L^2(\mathbf{R})$, therefor, gives a one-summable Fredholm module of $Cont(\mathbf{R})$. Farthermore the self-adjoint involution determined by φ_g is nothing but the Hilbert transformation with the canonical Clifford action on $\mathbf{R}$. So we can consider that the generator of coarse cohomology group of $\mathbf{R}$ properly correspond to a operator on $L^2(\mathbf{R})$.

The methods used to prove the theorem for the case of $\mathbf{R}^n$ are almost same as the 1-dimensional case. So, we will give a detail proof for the case of $\mathbf{R}$.

4.1. Generator of $HX^1(\mathbf{R})$ and Hilbert transformation

LEMMA 4.1.1 (generator of $HX^1(\mathbf{R}^n)$). *A generator of $HX^1(\mathbf{R}^n)$ can be written as follows.*

$$\varphi_g(x_0, x_1) = \begin{cases} 0, & for\ \ x_0 geq 0, x_1 geq 0 \\ 1, & for\ \ x_0 < 0, x_1 > 0 \\ 0, & for\ \ x_0 \le 0, x_1 \le 0 \\ -1, & for\ \ x_0 > 0, x_1 < 0\,. \end{cases}$$

PROOF. We first calcurate $BX^1(\mathbf{R})$, the 1-st coarse coboundary group for $\mathbf{R}$. Let $\varphi \in CX^0(\mathbf{R}) = C_c^\infty(\mathbf{R})$ (the forward equation follows from Definition 2.3.1, Remark 2.3.2), then

$$(\delta\varphi)(x_0, x_1) = \varphi(x_1) - \varphi(x_0)\,.$$

Thus $\partial\varphi \in BX^1(\mathbf{R})$ is a anti-symmetric function. We next calcurate $ZX^1(\mathbf{R})$, the 1-st coarse cocycle group. For $\phi \in ZX^1(\mathbf{R})$,

$$\begin{aligned} (\partial\phi)(x_0, x_1, x_2) &= \phi(x_1, x_2) - \phi(x_0, x_2) + \phi(x_0, x_1) \\ &= 0\,, \\ (\delta\phi)(x_1, x_0, x_2) &= \phi(x_0, x_2) - \phi(x_1, x_2) + \phi(x_1, x_0) \\ &= 0\,. \end{aligned}$$

By adding above two equation, we obtain

$$\phi(x_0, x_1) + \phi(x_1, x_0) = 0\,.$$

So, ϕ is also a anti-symmetric function, i.e.

$$\begin{aligned} HX^1(\mathbf{R}) &:= ZX^1(\mathbf{R})/BX^1(\mathbf{R}) \\ &= \mathbf{R}\,. \end{aligned}$$

So the result follows.

REMARK 4.1.2 (virtual potential). Let $h(x)$ be the *Heaviside function*. Then we can describe φ_g, a generator of $HX^1(\mathbf{R})$ as follows, although $h(x)$ itself is not a 0-th coarse cochain.

$$\varphi_g(x_0, x_1) = (\delta h)(x_0, x_1)\,.$$

Let H be the *Hilbert transformation* on $\mathbf{R}$, namely, for $f(x)\in L^2(\mathbf{R})$,

$$(Hf)(x) := \lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{-1}\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy\,,$$

H is a self-adjoint involutive bounded linear operator on $L^2(\mathbf{R})$ and its symbol $\sigma(H)$ is $\mathrm{sgn}(\xi) = \xi/|\xi|$ ([12]). If we put $F := \mathrm{sgn}(D_{\mathbf{R}})$ using the spectral theorem for self-adjoint operators, we can consider a one-summable Fredholm module $(Cont(\mathbf{R}), L^2(\mathbf{R}), \sigma(F))$ by following [4]. And its associated cyclic cocycle $\tau_{\sigma(F)} \in HC^1(Cont(\mathbf{R}))$ is given by

$$\tau_{\sigma(F)}(A_0, A_1) := \frac{1}{4}\mathrm{Tr}(\sigma(F)[\sigma(F), A_0][\sigma(F), A_1]),$$

where $A_0, A_1 \in Cont(\mathbf{R})$.

LEMMA 4.1.3. *Let $\chi^c(\varphi_g)$ be the cyclic character of φ_g, a generator of $HX^1(\mathbf{R})$ and $\tau_{\sigma(F)}$ as above. Then,*

$$\chi^c(\varphi_g) = \tau_{\sigma(F)}.$$

PROOF. We may calcurate the integration of $k_{\sigma(F)[\sigma(F),A_0][\sigma(F),A_1]}$ on diag$(\mathbf{R}\times\mathbf{R})$ instead of $\mathrm{Tr}(\sigma(F)[\sigma(F), A_0][\sigma(F), A_1])$.

$$\begin{aligned}
k_{\sigma(F)[\sigma(F),A_0]}(x_0, x_1) &= \sigma(F)(x_0)\{\sigma(F)(x_0)k_{A_0}(x_0, x_1) - k_{A_0}(x_0, x_1)\sigma(F)(x_1)\} \\
&= k_{A_0}(x_0, x_1) - \sigma(F)(x_0)k_{A_0}(x_0, x_1)\sigma(F)(x_1) \\
k_{[\sigma(F),A_1]}(x_1, x_2) &= \sigma(F)(x_1)k_{A_1}(x_1, x_2) - k_{A_1}(x_1, x_2)\sigma(F)(x_2).
\end{aligned}$$

Now, the composite kernel of the above two integral kernels is given as follows:

$$k_{\sigma(F)[\sigma(F),A_0][\sigma(F),A_1]}(x_0, x_2) = \int k_{\sigma(F)[\sigma(F),A_0]}(x_0, x_1)k_{[\sigma(F),A_1]}(x_1, x_2)dx_1.$$

But what we need is the integration on the diagonal set. Putting $x_2 = x_0$. Namely,

$$\begin{aligned}
k_{\sigma(F)[\sigma(F),A_0][\sigma(F),A_1]}(x_0, x_0) \;=\; & \int \{k_{A_0}(x_0, x_1) \\
& -\sigma(F)(x_0)k_{A_0}(x_0, x_1)\sigma(F)(x_1)\} \\
& \times\{\sigma(F)(x_1)k_{A_1}(x_1, x_0) \\
& -k_{A_1}(x_1, x_0)\sigma(F)(x_0)\}dx_1 \\
=\; & \int k_{A_0}(x_0, x_1)k_{A_1}(x_1, x_0) \cdot 2\{\sigma(F)(x_1) \\
& -\sigma(F)(x_0)\}dx_1 \\
=\; & \int k_{A_0}(x_0, x_1)k_{A_1}(x_1, x_0) \cdot 2 \cdot 2\varphi_g(x_0, x_1)dx_1 \\
=\; & 4\int k_{A_0}(x_0, x_1)k_{A_1}(x_1, x_0)\varphi_g(x_0, x_1)dx_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Tr}(\sigma(F)[\sigma(F), A_0][\sigma(F), A_1]) &= \int k_{\sigma(F)[\sigma(F),A_0][\sigma(F),A_1]}(x_0, x_0)dx_0 \\
&= 4\int k_{A_0}(x_0, x_1)k_{A_1}(x_1, x_0)\varphi_g(x_0, x_1)dx_0dx_1.
\end{aligned}$$

Finally

$$\begin{aligned}\tau_{\sigma(F)}(A_0, A_1) &= \frac{1}{4}\mathrm{Tr}(\sigma(F)[\sigma(F), A_0][\sigma(F), A_1]) \\ &= \int k_{A_0}(x_0, x_1)k_{A_1}(x_1, x_0)\varphi_g(x_0, x_1)dx_0dx_1 \\ &= \chi^c(\varphi_g)(A_0, A_1)\,.\end{aligned}$$

4.2. Proof of Theorem 1.1.2

$$\begin{aligned}\langle c\text{-ind}(D_{\mathbf{R}}), \chi^c(\varphi_g)\rangle &= \langle c\text{-ind}(D_{\mathbf{R}}), \tau_{\sigma(F)}\rangle \\ &\quad \text{([4.1.3])} \\ &= \langle (c\text{-ind}(D_{\mathbf{R}}))^\wedge, (\tau_{\sigma(F)})^\wedge\rangle \\ &\quad \text{(taking Fourier transform)} \\ &= \frac{1}{4}\mathrm{Tr}\left(F\left[F, \frac{\xi - i}{\xi + i} - 1\right]\left[F, \frac{\xi + 1}{\xi - 1} - 1\right]\right) \\ &\quad \text{([4])} \\ &= \frac{1}{4}\mathrm{Tr}\left(F\left[F, \frac{-2i}{\xi + i}\right]\left[F, \frac{2i}{\xi - i}\right]\right)\end{aligned}$$

We think $2i/(\xi - i)$ and $-2i/(\xi + i)$ multiplication operators M and M^{-1}, respectively. Since

$$\begin{aligned}k_{[F,M^{-1}]}(\xi, \eta) &= \frac{2}{\pi}\frac{1}{(\xi + i)(\eta + i)} \\ k_{[F,M]}(\eta, \zeta) &= \frac{-2}{\pi}\frac{1}{(\eta - i)(\zeta - i)}\,,\end{aligned}$$

$$\begin{aligned}k_{[F,M^{-1}][F,M]}(\xi, \zeta) &= \int k_{[F,M^{-1}]}(\xi, \eta)k_{[F,M]}(\eta, \zeta)d\eta \\ &= -\frac{4}{\pi^2}\frac{1}{(\xi + i)(\zeta - i)}\int \frac{1}{\eta^2 + 1}d\eta \\ &= -\frac{4}{\pi^2}\frac{1}{(\xi + 1)(\zeta - 1)}\cdot \pi \\ &= -\frac{4}{\pi}\frac{1}{(\xi + 1)(\zeta - 1)}\,.\end{aligned}$$

So

$$k_{F[F,M^{-1}][F,M]}(\omega, \zeta) = \int k_F(\omega, \xi)k_{[F,M^{-1}][F,M]}(\xi, \zeta)d\xi$$

$$
\begin{aligned}
&= -\frac{4i}{\pi^2}\frac{1}{\zeta - i}\int \frac{1}{(\omega-\xi)}\frac{1}{(\xi+i)}d\xi \\
&= -\frac{4i}{\pi^2}\frac{1}{\zeta - i}\frac{\pi i}{\omega + i} \\
&\quad \text{(Cauchy's p.v.)} \\
&= \frac{4}{\pi}\frac{1}{\omega+1}\frac{1}{\zeta - i}\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{1}{4}\mathrm{Tr}(F[F, M^{-1}][F, M]) &= \frac{1}{4}\int k_{F[F,M^{-1}][F,M]}(\zeta,\zeta)d\zeta \\
&= \frac{1}{\pi}\int \frac{1}{\zeta^2+1}d\zeta \\
&= \frac{1}{\pi}\cdot \pi \\
&= 1\,.
\end{aligned}
$$

This completes the proof.

4.3. Generator of $HX^n(\mathbf{R}^n)$ and Riesz transformation. We can similarly show that $\chi^t : HX^r(\mathbf{R}^r) \to H^r_{dR,c}(\mathbf{R}^r)$ gives an isomorphism as groups. Moreover, there is an isomorphism as rings w.r.t. the cup product in $HX^r(\mathbf{R}^r)$ described as in Definition 2.3.3. So, the generator of $HX^r(\mathbf{R}^r)$ is given as the r-times cup product of the generator of $HX^1(\mathbf{R})$.

Let R_j be the Riesz transformation on $L^2(\mathbf{R}^n)$.

$$
(R_j f)(x) := \lim_{\varepsilon\downarrow 0}\frac{2}{\sqrt{-1}\mathrm{vol}(S^n)}\int_{|x-y|>\varepsilon}\frac{|x_j - y_j|}{|x-y|^{n+1}}f(y)dy\,,
$$

where $\mathrm{vol}(S^n) = 2\pi^{(n+1)/2}/gamma((n+1)/2)$. The Riesz operator satisfies the following well-known facts (see, for example, [12]).

PROPOSITION 4.3.1 (Fourier transform). *Let $\widehat{\ }$ denote the Fourier transformation. Then,*

$$
\widehat{R_j f}(\xi) = \frac{\xi_j}{|\xi|}\widehat{f(\xi)} \quad \left(\sigma(R_j) = \frac{\xi_j}{|\xi|}\right)\,.
$$

So, the R_j are selfadjoint operators of norm 1 that commute with translations. Moreover,

$$
\sum_{j=1}^{n} R_j^2 = 1\,.
$$

Using this property, we can define the symmetry of our Fredholm module.

PROPOSITION 4.3.2 (Symmetry). *Put*

$$F := \mathrm{sgn}(D_{\mathbf{R}^n}) = \sum_{j=1}^{n} gamma_j^{(n)} R_j \,, \quad D_{\mathbf{R}^n} = \sum_{j=1}^{n} gamma_j^{(n)} (-i) \frac{\partial}{\partial x_j}$$

on $L^2(\mathbf{R}^n) \otimes \mathbf{C}^{[n/2]}$. *Here* [ ] *means to take the integer part of real number. Then*

$$F = F^*, \quad F^2 = 1$$

(*i.e., F is a symmetry operator*), *where* $gamma_1^{(n)}, \cdots, gamma_n^{(n)}$ *are* $2^{[n/2]} \times 2^{[n/2]}$*-matrices satisfying the Clifford relations.*

Put $\nu = 2^{[n/2]}$. To be specific we choose the following representation for the $gamma$-matrices:

For $n = 1$, we have $\nu = 1$ and

$$gamma_1^{(1)} := 1_{1\times 1} \,.$$

For all odd n we define inductively

$$gamma_j^{(n+2)} := \begin{pmatrix} 0_{\nu\times\nu} & gamma_j^{(n)} \\ gamma_j^{(n)} & 0_{\nu\times\nu} \end{pmatrix} \quad for\ j = 1, \ldots, n \,,$$

$$gamma_{n+1}^{(n+2)} := \begin{pmatrix} 0_{\nu\times\nu} & -i 1_{\nu\times\nu} \\ i 1_{\nu\times\nu} & 0_{\nu\times\nu} \end{pmatrix}, \quad gamma_{n+2}^{(n+2)} := \begin{pmatrix} 1_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{\nu\times\nu} & -1_{\nu\times\nu} \end{pmatrix}.$$

For all even n we choose

$$gamma_j^{(n)} := gamma_j^{(n+1)} \quad for\ \ j = 1, \ldots, n+1 \,.$$

4.4. Proof of Theorem 1.1.2. Let B be the subalgebra of $C^*(\mathbf{R}^n)$ consisting of all translation invariant smoothing operators. Since $D_{\mathbf{R}^n}$ is translation invariant, c-ind$(D_{\mathbf{R}^n})$ can be thought as an element of $K_*(B)$. In other words, $B = C_c(\mathbf{R}^n; gl(2^{[n/2]}, \mathbf{C}))$, the algebra of all compactly supported smooth functions from $\mathbf{R}^n$ to the Clifford algebra of $\mathbf{R}^n$, which is a dense subalgebra of $C = C_0(\mathbf{R}^n; gl(2^{[n/2]}, \mathbf{C}))$. And let ψ_g and ω_g be the generators of $HX^n(\mathbf{R}^n)$ and $H^n_{dR,c}(\mathbf{R}^n)$, respectively. One can see the following relation between ψ_g and ω_g.

$$\psi_g(x_0, \cdots, x_n) = \int_{\Delta^n(x_0,\cdots,x_n)} \omega_g \,,$$

where $\Delta^n(x_0, \cdots, x_n)$ is the oriented simplex spanned by $(x_0, \cdots, x_n)$. Then, since

$$\int_{\mathbf{R}^n} (T_{-x}(\omega_g)) d\mu(x)$$

is simply the volume form on $\mathbf{R}^n$, where T and μ denote the translation and the Lebesgue measure on $\mathbf{R}^n$, respectively, we can obtain for $f_0, \cdots, f_n \in B$ ([9])

$$\tau_{\psi_g}(f_0, \ \cdots, \ f_n) = c_n' \int_{\mathbf{R}^n} \mathrm{tr}(f_0 df_1 \wedge \cdots \wedge df_n)\,,$$

where

$$c_n' = \frac{(-1)^{[(n+1)/2]}}{n!(2\pi)^n}\,.$$

Next we discuss τ_F, the canonical cyclic cocycle on Fredholm module $(C, F, \mathcal{H})$. Here C and F are as above and $\mathcal{H} = L^2(\mathbf{R}^n) \otimes \mathbf{C}^\nu$. Let

$$gamma^{(n)} = \begin{cases} 1_{\nu\times\nu} & \text{if } n \text{ is odd}\,, \\ gamma_{n+1}^{(n)} & \text{if } n \text{ is even}\,. \end{cases}$$

Then for all $f_0, f_1, \cdots, f_n \in C$,

$$\begin{aligned} \tau_F(f_0, f_1, \cdots, f_n) &= \frac{i^n}{2}\mathrm{Tr}(gamma^{(n)} F[F, f_0] \cdots [F, f_n]) \\ &= c_n'' \int_{\mathbf{R}^n} \mathrm{tr}(f_0 df_1 \wedge \cdots \wedge df_n) \end{aligned}$$

with a normalization constant

$$c_n'' = (2i)^{[n/2]} \frac{1}{n(2\pi)^n} \mathrm{vol}(S^{n-1})\,.$$

$$\begin{aligned} c\text{-ind}_{\varphi_g}(D_{\mathbf{R}^n}) &= \frac{c'}{c''} \langle \tau_{\sigma(F)}, \rho(D_{\mathbf{R}^n}) \rangle \\ &= c_n \end{aligned}$$

where $\tau_{\sigma(F)}$ is a canonical cyclic cocycle for Fredholm module $(C, F, L^2(\mathbf{R}^n))$.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY,
KEIO UNIVERSITY,
HIYOSHI, KOHOKU-KU, YOKOHAMA 223–8522, JAPAN.
e-mail: kamimura@math.keio.ac.jp