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On the Interval Maps Associated to the α-mediant Convergents

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1. Introduction

For an irrational number $x \in (0, 1)$, if a non-zero rational number $\frac{p}{q}$, $(p, q) = 1$, satisfies $\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$, then it is the nth regular principal convergent $\frac{p_n}{q_n}$ for some $n \geq 1$. Here, the nth regular principal convergents are defined by the regular continued fraction expansion of x:

$$x = \frac{1\,|}{|\,a_1} + \frac{1\,|}{|\,a_2} + \frac{1\,|}{|\,a_3} + \cdots .$$

We put

$$\begin{cases} p_{-1} = p_{-1}(x) = 1\,, & p_0 = p_0(x) = 0 \\ q_{-1} = q_{-1}(x) = 0\,, & q_0 = q_0(x) = 1 \end{cases}$$

and

$$\begin{cases} p_n = p_n(x) = a_n \cdot p_{n-1} + p_{n-2} \\ q_n = q_n(x) = a_n \cdot q_{n-1} + q_{n-2} \end{cases} \qquad \text{for} \quad n \geq 1\,.$$

Then it is well-known that

$$\frac{p_n}{q_n} = \frac{1\,|}{|\,a_1} + \frac{1\,|}{|\,a_2} + \cdots + \frac{1\,|}{|\,a_n} \qquad \text{for} \quad n \geq 1\,.$$

If $x \in [k, k+1)$ for an integer k, we define its nth regular principal convergent by $\frac{p_n(x-k)}{q_n(x-k)} + k = \frac{p_n(x-k) + k \cdot q_n(x-k)}{q_n(x-k)}$.

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For some $x \in (0,1)$, there exists $\frac{p}{q}$ with $(p,q)=1$ and $\left|x-\frac{p}{q}\right| < \frac{1}{q^2}$, which is not the nth regular principal convergent for any $n \geq 0$. However, we can find such a fraction $\frac{p}{q}$ in the set $\left\{\frac{p_n - p_{n-1}}{q_n - q_{n-1}}, \ \frac{p_n + p_{n-1}}{q_n + q_{n-1}} : n \geq 1\right\}$. This leads us to the notion of the regular mediant convergents of level n, $\frac{u_{n,t}}{v_{n,t}}$, which is defined by

$$\begin{cases} u_{n,t} = t \cdot p_n + p_{n-1} \\ v_{n,t} = t \cdot q_n + q_{n-1} \end{cases} \quad \text{for} \quad 1 \leq t < a_{n+1}, \ n \geq 0 .$$

The regular principal and the regular mediant convergents are obtained by the following maps T and F of $[0,1]$, which are called the Gauss map and the Farey map, respectively, see [2]:

$$T(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if} \quad x \in (0,1] \\ 0 & \text{if} \quad x = 0 \end{cases} \tag{1.1}$$

and

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if} \quad x \in \left[0, \frac{1}{2}\right) \\ \frac{1-x}{x} & \text{if} \quad x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where $[y] = n$ if $y \in [n, n+1)$. We get the coefficients of the regular continued fraction expansion of $x \in [0,1]$ by

$$a_n = a_n(x) = [(T^{n-1}(x))^{-1}], \quad n \geq 1 .$$

We refer to Sh. Ito [3] on the relation between F and the regular mediant convergents. In this paper, we generalize the notion of the mediant convergents to the continued fraction expansion introduced by H. Nakada [5], which are called the α-continued fraction expansion. The α-continued fraction expansion is a generalization of the regular continued fraction expansion and is induced by the following map T_α of $\mathbf{I}_\alpha = [\alpha - 1, \alpha]$ for $\frac{1}{2} \leq \alpha \leq 1$:

$$T_\alpha(x) = \begin{cases} \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right|\right]_\alpha & \text{if } x \in \mathbf{I}_\alpha \setminus \{0\} \\ 0 & \text{if} \quad x = 0 , \end{cases}$$

where $[\, y \,]_\alpha = n$ if $y \in [n-1+\alpha, \ n+\alpha)$. We note that T_1 is the Gauss map. For $n \geq 1$, put

$$\varepsilon_{\alpha,n} = \varepsilon_{\alpha,n}(x) = \operatorname{sgn} T_\alpha^{n-1}(x) ,$$

$$c_{\alpha,n} = c_{\alpha,n}(x) = \left[\left| \frac{1}{T_\alpha^{n-1}(x)} \right| \right]_\alpha \quad (\text{or} = \infty \quad \text{if} \quad T_\alpha^{n-1}(x) = 0)\,.$$

Then we have the α-continued fraction expansion of $x \in \mathbf{I}_\alpha$:

$$x = \frac{\varepsilon_{\alpha,1}|}{|c_{\alpha,1}} + \frac{\varepsilon_{\alpha,2}|}{|c_{\alpha,2}} + \frac{\varepsilon_{\alpha,3}|}{|c_{\alpha,3}} + \cdots\,, \quad c_{\alpha,n} \geq 1\,.$$

Next for $n \geq 1$, we define the nth α-principal convergents $\frac{p_{\alpha,n}}{q_{\alpha,n}}$ by

$$\begin{cases} p_{\alpha,-1} = 1\,, & p_{\alpha,0} = 0 \\ q_{\alpha,-1} = 0\,, & q_{\alpha,0} = 1 \end{cases} \quad \text{and} \quad \begin{cases} p_{\alpha,n} = c_{\alpha,n} \cdot p_{\alpha,n-1} + \varepsilon_{\alpha,n} \cdot p_{\alpha,n-2} \\ q_{\alpha,n} = c_{\alpha,n} \cdot q_{\alpha,n-1} + \varepsilon_{\alpha,n} \cdot q_{\alpha,n-2}\,. \end{cases}$$

We note that the $\{q_{\alpha,n}\}$ is strictly increasing, see [5]. Also we define the α-mediant convergents of level $n \geq 0$, $\left\{ \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} : 1 \leq t < c_{\alpha,n+1} \right\}$, by

$$\begin{cases} u_{\alpha,n,t} = t \cdot p_{\alpha,n} + \varepsilon_{\alpha,n+1} \cdot p_{\alpha,n-1} \\ v_{\alpha,n,t} = t \cdot q_{\alpha,n} + \varepsilon_{\alpha,n+1} \cdot q_{\alpha,n-1} \end{cases} \quad \text{for} \quad 1 \leq t < c_{\alpha,n+1}\,. \tag{1.2}$$

In §2, we define a new map G_α for each α, $\frac{1}{2} \leq \alpha \leq 1$ and show how G_α induces the sequence of the α-principal and the α-mediant convergents. We call G_α the α-Farey map of the first type. We note that G_α is the same as F in the above if $\alpha = 1$. Our idea for getting the mediant convergents is slightly different from the one in [3]. In §3, we give some estimates on the error term of the convergence of $\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}}$ to $x \in \mathbf{I}_\alpha$. The first assertion is its upper estimate:

$$\left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| < \frac{1}{q_{\alpha,n} \cdot q_{\alpha,n-1}}\,.$$

This shows that $\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}}$ converges to x. The second assertion is

$$\limsup_{\substack{1 \leq t < c_{\alpha,n+1} \\ n \to \infty}} v_{\alpha,n,t}^2 \left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| = \infty \quad (\text{a.e.})\,,$$

though

$$\liminf_{\substack{1 \leq t < c_{\alpha,n+1} \\ n \to \infty}} v_{\alpha,n,t}^2 \left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| < 2 \quad \text{if} \quad c_{\alpha,n} \geq 2 \quad \text{occur infinitely often}\,.$$

This means that we can not give any estimates after the normalization by the square of the denominator. For the asymptotic behavior of the values

$$v_{\alpha,n,t}^2 \left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right|\,, \tag{1.3}$$

we give a 2-dimensional map $\widehat{G_\alpha}$ which we call the "natural extension" of G_α in §4. By this map, we can discuss the distribution of (1.3) for almost every x by using the ratio ergodic theorem. In §5, we describe a relation between the α-mediant convergents and the regular mediant convergents. Actually, we show that the set of the α-principal and the α-mediant convergents coincides with the set of the regular's. K. Dajani and C. Kraaikamp [1] showed that Lehner fractions induce the set of the regular principal and the regular mediant convergents. They also showed that this set includes all principal convergents arising from S-expansions, see [4] for the definition of S-expansions. In this sense, they called this set "the mother of all semi-regular continued fractions". Our claim is that we can construct the "mother" from any α-continued fractions, $\frac{1}{2} \le \alpha \le 1$, by producing the α-mediant convergents. Incidentally, it is easy to see that

$$\frac{u_{\alpha,n,1}}{v_{\alpha,n,1}} = \frac{u_{\alpha,n-1,c_{\alpha,n}-1}}{v_{\alpha,n-1,c_{\alpha,n}-1}} \quad \text{if} \quad \varepsilon_{\alpha,n+1} = -1\,.$$

This means that one rational number appears twice when $\varepsilon_{\alpha,n+1} = -1$ in the approximating sequence. In the final part of this paper, we give a new map F_α, $\frac{1}{2} \le \alpha \le 1$, the α-Farey map of the second type, which also induces the α-principal and the α-mediant convergents without $\frac{u_{\alpha,n-1,c_{\alpha,n}-1}}{v_{\alpha,n-1,c_{\alpha,n}-1}}$ if $\varepsilon_{\alpha,n+1} = -1$.

2. The α-Farey maps and the α-mediant convergents

For a real number α, $\frac{1}{2} \le \alpha \le 1$, we put $\mathbf{J}_\alpha = \left[\alpha - 1, \frac{1}{\alpha}\right]$. Define a map G_α of $\mathbf{J}_\alpha$ by

$$G_\alpha(x) = \begin{cases} -\dfrac{x}{1+x} & \text{if} \quad x \in [\alpha - 1, 0) := \mathbf{J}_{\alpha,1} \\ \dfrac{x}{1-x} & \text{if} \quad x \in \left[0, \dfrac{1}{1+\alpha}\right] := \mathbf{J}_{\alpha,2} \\ \dfrac{1-x}{x} & \text{if} \quad x \in \left(\dfrac{1}{1+\alpha}, \dfrac{1}{\alpha}\right] := \mathbf{J}_{\alpha,3}\,. \end{cases}$$

We note that G_1 is the Farey map for the regular continued fractions. In this sense, G_α is a generalization of the Farey map. We call this map the α-Farey map of the first type, because we give a map which will be called the α-Farey map of the second type in the final part of this paper.

In order to get the α-principal and the α-mediant convergents of $x \in \mathbf{J}_\alpha$ by the iterations of G_α, it is convenient to use the following matrices:

$$V_- = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since

$$\frac{ax+b}{cx+d}=\frac{u}{v}\quad\text{with}\quad\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}xz\\ z\end{pmatrix}$$

for any real numbers x and $z \neq 0$, we denote

$$A(x)=\frac{ax+b}{cx+d}\quad\text{and}\quad A(-\infty)=A(\infty)=\frac{a}{c}\quad\text{for}\quad A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Hence, we can write

$$G_\alpha(x)=\begin{cases}V_-^{-1}(x) & \text{if}\quad x\in\mathbf{J}_{\alpha,1}\\ V_+^{-1}(x) & \text{if}\quad x\in\mathbf{J}_{\alpha,2}\\ U^{-1}(x) & \text{if}\quad x\in\mathbf{J}_{\alpha,3}\,.\end{cases}$$

Next, we put

$$M_n(x):=\begin{cases}V_- & \text{if}\quad (G_\alpha)^{n-1}(x)\in\mathbf{J}_{\alpha,1}\\ V_+ & \text{if}\quad (G_\alpha)^{n-1}(x)\in\mathbf{J}_{\alpha,2}\\ U & \text{if}\quad (G_\alpha)^{n-1}(x)\in\mathbf{J}_{\alpha,3}\,.\end{cases}$$

Then, we get a sequence of matrices

$$M_1(x)\,,\quad M_2(x),\;\cdots$$

from the iterations of G_α for each $x \in \mathbf{J}_\alpha$. Here, all matrices M_n's are of determinants ± 1. To investigate relationship between T_α and G_α, we need the following lemmas.

LEMMA 1.

(i) $$\begin{pmatrix}0 & 1\\ 1 & t\end{pmatrix}=\underbrace{\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}}_{t-1}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}=\underbrace{V_+\cdots V_+}_{t-1}U\quad \textit{for}\quad t\geq 1\,.$$

(ii) $$\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\underbrace{\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}}_{t-2}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}=V_-\underbrace{V_+\cdots V_+}_{t-2}U$$

for $t\geq 2\,.$

LEMMA 2. *Suppose that* $x \in \mathbf{J}_\alpha$. *If* $x \in \left[-\frac{1}{j-1+\alpha},-\frac{1}{j+\alpha}\right)\cup\left(\frac{1}{j+\alpha},\frac{1}{j-1+\alpha}\right]$, *then* $G_\alpha(x)\in\left(\frac{1}{j-1+\alpha},\frac{1}{j-2+\alpha}\right]$ *for* $j\geq 2$.

We put

$$k_0(x) := 0 \quad \text{and} \quad k_n(x) := \min\{k > k_{n-1}(x) : (G_\alpha)^{k-1}(x) \in \mathbf{J}_{\alpha,3}\},\ n \geq 1\,.$$

Next proposition shows that T_α is obtained as a jump transformation in the sense of F. Schweiger, see [9].

PROPOSITION 1.

$$(G_\alpha)^{k_1(x)}(x) = T_\alpha(x) \quad \textit{for} \quad x \in \mathbf{I}_\alpha = [\alpha - 1, \alpha]\,.$$

PROOF. If $x \in \left(\frac{1}{j+\alpha}, \frac{1}{j-1+\alpha}\right] \cap \mathbf{I}_\alpha$, then by Lemma 2, we see

$$M_1(x)M_2(x)\cdots M_{k_1(x)}(x) = \underbrace{V_+ \cdots V_+}_{j-1} U \quad \text{for} \quad j \geq 1\,. \tag{2.1}$$

Hence, from Lemma 1, we have

$$(G_\alpha)^{k_1(x)}(x) = U^{-1}\underbrace{V_+^{-1}\cdots V_+^{-1}}_{j-1}(x) = \begin{pmatrix} -j & 1 \\ 1 & 0 \end{pmatrix}(x) = \frac{-jx+1}{x} = T_\alpha(x)\,. \tag{2.2}$$

If $x \in \left[-\frac{1}{j-1+\alpha}, -\frac{1}{j+\alpha}\right) \cap \mathbf{I}_\alpha$, then by Lemma 2 again, we see

$$M_1(x)M_2(x)\cdots M_{k_1(x)}(x) = V_- \underbrace{V_+ \cdots V_+}_{j-2} U \quad \text{for} \quad j \geq 2\,. \tag{2.3}$$

Thus, we also have

$$(G_\alpha)^{k_1(x)}(x) = U^{-1}\underbrace{V_+^{-1}\cdots V_+^{-1}}_{j-2} V_-^{-1}(x) = \begin{pmatrix} j & 1 \\ -1 & 0 \end{pmatrix}(x) = T_\alpha(x)\,. \tag{2.4}$$

□

For any two numbers x and $x' \in \mathbf{I}_\alpha$, their α-continued fraction expansions are different from each other since the expansions converge to x and x', respectively. Thus we have the following.

COROLLARY 1.

$$(M_1(x),\ M_2(x),\ \cdots) \neq (M_1(x'),\ M_2(x'),\ \cdots) \quad \textit{whenever} \quad x \neq x' \in \mathbf{J}_\alpha\,.$$

PROOF. Suppose that $x \neq x' \in \mathbf{J}_\alpha$. If $k_1(x) \neq k_1(x')$ or $M_i(x) \neq M_i(x')$ for some $1 \leq i \leq k_1(x)$, then the assertion is clear. So we assume that $k_1(x) = k_1(x')$ and $M_i(x) = M_i(x')$ for $1 \leq i \leq k_1(x)$. Then Lemma 2 implies that

$$x \text{ and } x' \in \left[-\frac{1}{k_1(x)-1+\alpha}, -\frac{1}{k_1(x)+\alpha}\right) \cup \left(\frac{1}{k_1(x)+\alpha}, \frac{1}{k_1(x)-1+\alpha}\right]$$

and

$$G_\alpha^{k_1(x)}(x) \neq G_\alpha^{k_1(x)}(x') .$$

since $G_\alpha^{k_1(x)}$ is a one-to-one map on $\left[-\dfrac{1}{k_1(x)-1+\alpha}, -\dfrac{1}{k_1(x)+\alpha}\right) \cup \left(\dfrac{1}{k_1(x)+\alpha}, \dfrac{1}{k_1(x)-1+\alpha}\right]$. Then we get sequences

$$M_{k_1(x)+1}(x),\ M_{k_1(x)+2}(x)\,, \cdots$$

and

$$M_{k_1(x)+1}(x'),\ M_{k_1(x)+2}(x')\,, \cdots .$$

Here we note that

$$G_\alpha^{k_1(x)}(x) \in \mathbf{I}_\alpha \quad \text{and} \quad G_\alpha^{k_1(x)}(x') = G_\alpha^{k_1(x')}(x') \in \mathbf{I}_\alpha .$$

By (2.1), (2.2), (2.3) and (2.4), the above sequences correspond to the α-continued fraction expansions of $G_\alpha^{k_1(x)}(x)$ and $G_\alpha^{k_1(x)}(x')$, which are not the same. □

Finally we have the following theorem, which connects the map G_α to the α-mediant convergents explicitly.

THEOREM 1. *For $x \in \mathbf{I}_\alpha$, we have*
(i) *If $l = k_n(x),\ n \geq 1$,*

$$M_1(x)M_2(x)\cdots M_l(x) = \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix} \tag{2.5}$$

(ii) *If $l = k_n(x) + t,\ 1 \leq t < c_{\alpha,n+1},\ n \geq 0$,*

$$M_1(x)M_2(x)\cdots M_l(x) = \begin{pmatrix} u_{\alpha,n,t} & p_{\alpha,n} \\ v_{\alpha,n,t} & q_{\alpha,n} \end{pmatrix} \tag{2.6}$$

PROOF. First, we show (2.5) by induction on n.

[I] $n = 1$

From (2.1), (2.2), (2.3) and (2.4), we have

$$M_1(x)M_2(x)\cdots M_{k_1(x)}(x) = \begin{pmatrix} 0 & \varepsilon_{\alpha,1} \\ 1 & c_{\alpha,1} \end{pmatrix} = \begin{pmatrix} p_{\alpha,0} & p_{\alpha,1} \\ q_{\alpha,0} & q_{\alpha,1} \end{pmatrix} .$$

[II] Suppose we have

$$M_1(x)M_2(x)\cdots M_{k_m(x)}(x) = \begin{pmatrix} p_{\alpha,m-1} & p_{\alpha,m} \\ q_{\alpha,m-1} & q_{\alpha,m} \end{pmatrix}$$

and

$$(G_\alpha)^{k_m(x)}(x) = T_\alpha^m(x) =: y .$$

Then, we see

$$M_{k_m(x)+1}(x)M_{k_m(x)+2}(x)\cdots M_{k_{m+1}(x)}(x) = M_1(y)M_2(y)\cdots M_{k_1(y)}(y)$$

since $k_1(y) = k_{m+1}(x) - k_m(x)$. Thus,

$$M_{k_m(x)+1}(x)\cdots M_{k_{m+1}(x)}(x) = \begin{pmatrix} 0 & \varepsilon_{\alpha,1}(y) \\ 1 & c_{\alpha,1}(y) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_{\alpha,m+1}(x) \\ 1 & c_{\alpha,m+1}(x) \end{pmatrix}.$$

Hence, we have

$$M_1(x)M_2(x)\cdots M_{k_{m+1}(x)}(x) = \begin{pmatrix} p_{\alpha,m-1} & p_{\alpha,m} \\ q_{\alpha,m-1} & q_{\alpha,m} \end{pmatrix}\begin{pmatrix} 0 & \varepsilon_{\alpha,m+1} \\ 1 & c_{\alpha,m+1} \end{pmatrix} = \begin{pmatrix} p_{\alpha,m} & p_{\alpha,m+1} \\ q_{\alpha,m} & q_{\alpha,m+1} \end{pmatrix}.$$

Moreover, we see that

$$(G_\alpha)^{k_{m+1}(x)}(x) = (G_\alpha)^{k_{m+1}(x)-k_m(x)}((G_\alpha)^{k_m(x)}(x)) = T_\alpha(T_\alpha^m(x)) = T_\alpha^{m+1}(x).$$

Consequently, we have

$$M_1(x)M_2(x)\cdots M_{k_n(x)}(x) = \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix} \quad \text{for any} \quad n \geq 1. \tag{2.7}$$

Next, we prove (2.6). If $(G_\alpha)^{k_n(x)}(x) = T_\alpha^n(x) > 0$, then $\varepsilon_{\alpha,n+1}(x) = 1$, otherwise $\varepsilon_{\alpha,n+1}(x) = -1$. So by (2.7), we see that

$$\begin{aligned} M_1(x)M_2(x)\cdots M_{k_n(x)+t}(x) &= \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix}\begin{pmatrix} \varepsilon_{\alpha,n+1} & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{t-1} \\ &= \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix}\begin{pmatrix} \varepsilon_{\alpha,n+1} & 0 \\ t & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_{\alpha,n,t} & p_{\alpha,n} \\ v_{\alpha,n,t} & q_{\alpha,n} \end{pmatrix} \end{aligned}$$

for $1 \leq t < c_{\alpha,n+1}$. □

The following is a direct consequence of Theorem 1.

COROLLARY 2. *We have*

$$(M_1(x)M_2(x)\cdots M_l(x))(\infty) = \begin{cases} \dfrac{p_{\alpha,n-1}}{q_{\alpha,n-1}} & \textit{if} \quad l = k_n(x),\ n \geq 1 \\ \dfrac{u_{\alpha,n,t}}{v_{\alpha,n,t}} & \textit{if} \quad \begin{array}{l} l = k_n(x) + t, \\ 1 \leq t < c_{\alpha,n+1}, \quad n \geq 0. \end{array} \end{cases}$$

REMARK. In [3], the regular mediant convergents were obtained as

$$(M_1(x)M_2(x)\cdots M_{l-1}(x))(1).$$

3. The convergence of the approximation

In this section, we discuss the convergence of the α-mediant convergents to x. We put

$$x_l = (G_\alpha)^l(x) \quad \text{for} \quad l \geq 0 .$$

From the definitions of G_α and M_n in §2, we see that

$$x = (M_1(x)M_2(x)\cdots M_l(x))(x_l) . \tag{3.1}$$

First, we show the fundamental formulas concerning the error of the α-principal and the α-mediant convergents to x.

PROPOSITION 2.
(i) *If* $l = k_n(x)$, $n \geq 1$,

$$q_{\alpha,n-1}^2 \left| x - \frac{p_{\alpha,n-1}}{q_{\alpha,n-1}} \right| = \frac{1}{\left| x_l - \left(-\dfrac{q_{\alpha,n}}{q_{\alpha,n-1}} \right) \right|} .$$

(ii) *If* $l = k_n(x) + t$, $1 \leq t < c_{\alpha,n+1}$, $n \geq 0$,

$$v_{\alpha,n,t}^2 \left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| = \frac{1}{\left| x_l - \left(-\dfrac{q_{\alpha,n}}{v_{\alpha,n,t}} \right) \right|} .$$

REMARK. We note that

$$(M_1(x)M_2(x)\cdots M_l(x))^{-1}(\infty) = \begin{cases} -\dfrac{q_{\alpha,n}}{q_{\alpha,n-1}} & \text{if} \quad l = k_n(x)\,,\ n \geq 1 \\ -\dfrac{q_{\alpha,n}}{v_{\alpha,n,t}} & \text{if} \quad \begin{array}{l} l = k_n(x) + t\,, \\ 1 \leq t < c_{\alpha,n+1}, \quad n \geq 0\,, \end{array} \end{cases}$$

see Theorem 1 .

PROOF. (i) If $l = k_n(x)$, $n \geq 1$, from (3.1) and (2.5), we see that

$$q_{\alpha,n-1}^2 \left| x - \frac{p_{\alpha,n-1}}{q_{\alpha,n-1}} \right| = q_{\alpha,n-1}^2 \left| \frac{p_{\alpha,n-1}x_l + p_{\alpha,n}}{q_{\alpha,n-1}x_l + q_{\alpha,n}} - \frac{p_{\alpha,n-1}}{q_{\alpha,n-1}} \right| = \frac{1}{\left| x_l - \left(-\dfrac{q_{\alpha,n}}{q_{\alpha,n-1}} \right) \right|} .$$

(ii) From (3.1) and (2.6), we conclude the assertion by the same calculation. □

From this proposition, it is possible to show that the sequence of the α-mediant convergents certainly converges to x. However, this convergence also follows from Theorem 2 below. To prove it, we need the following lemma.

LEMMA 3. *For* $n \geq 0$, *we have*
(i) $v_{\alpha,n,1} > q_{\alpha,n-1}$,
(ii) $v_{\alpha,n,t} > (t-1)q_{\alpha,n}$ *for* $2 \leq t < c_{\alpha,n+1}$.

PROOF. If $t \geq 2$, then

$$v_{\alpha,n,t} = t \cdot q_{\alpha,n} \pm q_{\alpha,n-1} > (t-1)q_{\alpha,n}\,,$$

since $q_{\alpha,n}$ is strictly increasing.

Suppose that $t = 1$. If $\varepsilon_{\alpha,n+1} = -1$, then either $\varepsilon_{\alpha,n} = -1$ and $c_{\alpha,n} \geq 3$ or $\varepsilon_{\alpha,n} = 1$ and $c_{\alpha,n} \geq 2$ holds, see H. Nakada [5], p. 403. In the first case, we have

$$v_{\alpha,n,t} = q_{\alpha,n} - q_{\alpha,n-1} \geq 3q_{\alpha,n-1} - q_{\alpha,n-2} - q_{\alpha,n-1} > q_{\alpha,n-1}\,.$$

In the latter case,

$$v_{\alpha,n,t} = q_{\alpha,n} - q_{\alpha,n-1} \geq 2q_{\alpha,n-1} + q_{\alpha,n-2} - q_{\alpha,n-1} > q_{\alpha,n-1}\,.$$

This completes the proof of this lemma. □

From Proposition 2 (ii) and Lemma 3, we have the following theorem which implies the convergence of $\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}}$ to x.

THEOREM 2. *For $n \geq 0$ and $1 \leq t < c_{\alpha,n+1}$, we have*

$$\left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| < \frac{1}{q_{\alpha,n} \cdot q_{\alpha,n-1}}\,.$$

PROOF. If $l \neq k_n(x)$, we see $x_l > 0$. From Proposition 2 (ii), we see

$$\begin{aligned}
\left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| &= \frac{1}{v_{\alpha,n,t}^2} \frac{1}{\left| x_l - \left(-\frac{q_{\alpha,n}}{v_{\alpha,n,t}} \right) \right|} \\
&= \frac{1}{v_{\alpha,n,t}} \left| \frac{1}{v_{\alpha,n,t} x_l + q_{\alpha,n}} \right| \\
&= \frac{1}{v_{\alpha,n,t}} \left| \frac{1}{(t \cdot q_{\alpha,n} + \varepsilon_{\alpha,n+1} \cdot q_{\alpha,n-1}) x_l + q_{\alpha,n}} \right| \\
&< \frac{1}{v_{\alpha,n,t}} \cdot \frac{1}{q_{\alpha,n}}\,.
\end{aligned}$$

Then from Lemma 3, we have the assertion of the theorem. □

The above theorem shows that $\left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right|$ is bounded by $\frac{1}{q_{\alpha,n-1}^2}$. However, next theorem shows that $v_{\alpha,n,t}^2 \left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right|$ is not bounded by any absolute constant.

THEOREM 3. *We have the following*:

(i) $\displaystyle \limsup_{\substack{1 \leq t < c_{\alpha,n+1} \\ n \to \infty}} v_{\alpha,n,t}^2 \left| x - \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} \right| = \infty \quad (a.e.\ x)\,,$

(ii) $\liminf_{\substack{1\le t<c_{\alpha,n+1}\\ n\to\infty}} v_{\alpha,n,t}^2\left|x-\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}}\right|<2$ *if* $c_{\alpha,n+1}\ge 2$ *occur infinitely often*.

PROOF. Suppose $c_{\alpha,n+1}>4M$ for any sufficiently large M. Let $l=k_n(x)+t$, $t=\left[\frac{c_{\alpha,n+1}}{2}\right]$. Since $(G_\alpha)^{k_n(x)}(x)=T_\alpha^n(x)$ and $\left[\left|\frac{1}{T_\alpha^n(x)}\right|\right]_\alpha=c_{\alpha,n+1}$, we see that

$$\frac{1}{c_{\alpha,n+1}+\alpha}<|(G_\alpha)^{k_n(x)}(x)|\le\frac{1}{c_{\alpha,n+1}-1+\alpha}.$$

Hence, from Lemma 2,

$$\frac{1}{c_{\alpha,n+1}-\left[\frac{c_{\alpha,n+1}}{2}\right]+\alpha}<x_l\le\frac{1}{c_{\alpha,n+1}-1-\left[\frac{c_{\alpha,n+1}}{2}\right]+\alpha}.$$

Thus we have

$$x_l<\frac{2}{c_{\alpha,n+1}-4}. \tag{3.2}$$

By Lemma 3 (ii), we have

$$\left(\frac{c_{\alpha,n+1}}{2}-2\right)q_{\alpha,n}<\left(\left[\frac{c_{\alpha,n+1}}{2}\right]-1\right)q_{\alpha,n}<v_{\alpha,n,t}$$

and so

$$\frac{q_{\alpha,n}}{v_{\alpha,n,t}}<\frac{2}{c_{\alpha,n+1}-4}. \tag{3.3}$$

From Proposition 2 (ii), (3.2) and (3.3), we have

$$v_{\alpha,n,t}^2\left|x-\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}}\right|\ge\frac{1}{\left|\frac{2}{c_{\alpha,n+1}-4}+\frac{2}{c_{\alpha,n+1}-4}\right|}=\frac{|c_{\alpha,n+1}-4|}{4}>M-1.$$

By the ergodicity of T_α, there exist infinitely many such n's for almost every x. On the other hand, if we choose $t=1$ with any $n\ge 0$ such that $c_{\alpha,n+1}\ge 2$, we have

$$v_{\alpha,n,1}^2\left|x-\frac{u_{\alpha,n,1}}{v_{\alpha,n,1}}\right|=\frac{1}{\left|x_l-\left(-\frac{q_{\alpha,n}}{v_{\alpha,n,1}}\right)\right|}=\frac{1}{\left|x_l+\frac{q_{\alpha,n}}{q_{\alpha,n}\pm q_{\alpha,n-1}}\right|}<\frac{q_{\alpha,n}\pm q_{\alpha,n-1}}{q_{\alpha,n}}<2,$$

since $x_l>0$ and $q_{\alpha,n}>q_{\alpha,n-1}$. This completes the proof of the theorem. □

Recall $x_l=(M_1(x)M_2(x)\cdots M_l(x))^{-1}(x)$. From the remark after Proposition 2, we see that the explicit value of $q_{\alpha,n}^2\left|x-\frac{p_{\alpha,n}}{q_{\alpha,n}}\right|$ or $v_{\alpha,n,t}^2\left|x-\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}}\right|$ is determined by the images of

x and ∞ by $(M_1(x)M_2(x)\cdots M_l(x))^{-1}$. This leads us to the notion of the natural extension of G_α, which is defined in next section.

4. Natural extension of G_α

In this section, we discuss the distribution of $v^2_{\alpha,n,t}\left|x - \dfrac{u_{\alpha,n,t}}{v_{\alpha,n,t}}\right|$ using a 2-dimensional map $\widehat{G_\alpha}$, which is called the natural extension of G_α.

We define the map $\widehat{G_\alpha}$ of $\widehat{\mathbf{J}_\alpha}$ as follows:

$$\widehat{\mathbf{J}_\alpha} = \begin{cases} \left[\alpha - 1, \dfrac{1-2\alpha}{\alpha}\right) \times \left[-\infty, -\dfrac{\sqrt{5}+3}{2}\right] \cup \left[\dfrac{1-2\alpha}{\alpha}, 0\right) \times [-\infty, -2] \cup [0, \alpha) \\ \quad \times [-\infty, 0] \cup \left[\alpha, \dfrac{1-\alpha}{\alpha}\right) \times \left[-\dfrac{\sqrt{5}+1}{2}, 0\right] \cup \left[\dfrac{1-\alpha}{\alpha}, \dfrac{\alpha}{1-\alpha}\right) \times [-1, 0] \\ \quad \cup \left[\dfrac{\alpha}{1-\alpha}, \dfrac{1}{\alpha}\right] \times \left[-\dfrac{\sqrt{5}-1}{2}, 0\right] & \text{if } \dfrac{1}{2} \le \alpha \le \dfrac{\sqrt{5}-1}{2} \\ [\alpha - 1, 0) \times [-\infty, -2] \cup [0, \alpha) \times [-\infty, 0] \cup \left[\alpha, \dfrac{1}{\alpha}\right] \times [-1, 0] \\ & \text{if } \dfrac{\sqrt{5}-1}{2} \le \alpha \le 1 \end{cases}$$

and

$$\widehat{G_\alpha}(x, y) = \begin{cases} \left(-\dfrac{x}{1+x}, -\dfrac{y}{1+y}\right) & \text{if } x \in \mathbf{J}_{\alpha,1} \\ \left(\dfrac{x}{1-x}, \dfrac{y}{1-y}\right) & \text{if } x \in \mathbf{J}_{\alpha,2} \\ \left(\dfrac{1-x}{x}, \dfrac{1-y}{y}\right) & \text{if } x \in \mathbf{J}_{\alpha,3}\,. \end{cases}$$

From Proposition 2 (ii), we see that the deviation of the α-mediant convergent $\dfrac{u_{\alpha,n,t}}{v_{\alpha,n,t}}$ from x normalized by $v^2_{\alpha,n,t}$ is equal to $\left|\dfrac{1}{x_l - y_l}\right|$ with $(x_l, y_l) = \widehat{G_\alpha}^{\,l}(x, \infty)$, $l = \sum_{i=1}^{n} k_i(x) + t$ for $x \in \mathbf{J}_\alpha$, $1 \le t < c_{\alpha,n+1}$. A number of properties associated to the approximation by α-mediant convergents are obtained by dynamical behaviors of this map. One of the important applications which are obtained from the construction of $\widehat{G_\alpha}$ is the derivation of the density function of the absolutely continuous invariant measure for G_α. In the rest of this section, the most of proofs can be completed by routine calculation, and therefore, we will only sketch the ideas involved.

PROPOSITION 3. *$\widehat{G_\alpha}$ is a one-to-one onto map of $\widehat{\mathbf{J}_\alpha}$ modulo a set of Lebesgue measure* 0.

PROOF. We put

$$\widehat{\mathbf{J}_{\alpha,i}} = \{(x, y) \in \widehat{\mathbf{J}_\alpha} \,:\, x \in \mathbf{J}_{\alpha,i}\}\,, \quad i = 1, 2, 3$$

and

$$\widehat{\mathbf{K}_{\alpha,i}} = \widehat{G_\alpha}\,\widehat{\mathbf{J}_{\alpha,i}}\,, \quad i = 1, 2, 3\,.$$

It is easy to see the following:

case (i) $\quad \dfrac{1}{2} \le \alpha \le \dfrac{\sqrt{5}-1}{2}$

$$\widehat{\mathbf{K}_{\alpha,1}} = \left[0,\, \frac{2\alpha-1}{1-\alpha}\right] \times [-2, -1] \cup \left[\frac{2\alpha-1}{1-\alpha},\, \frac{1-\alpha}{\alpha}\right] \times \left[-\frac{\sqrt{5}+1}{2},\, -1\right],$$

$$\widehat{\mathbf{K}_{\alpha,2}} = \left[0,\, \frac{\alpha}{1-\alpha}\right] \times [-1, 0] \cup \left[\frac{\alpha}{1-\alpha},\, \frac{1}{\alpha}\right] \times \left[-\frac{\sqrt{5}-1}{2},\, 0\right],$$

$$\begin{aligned}\widehat{\mathbf{K}_{\alpha,3}} = &\left[\alpha-1,\, \frac{1-2\alpha}{\alpha}\right] \times \left[-\infty, -\frac{\sqrt{5}+3}{2}\right] \cup \left[\frac{1-2\alpha}{\alpha},\, \frac{2\alpha-1}{1-\alpha}\right] \times [-\infty, -2] \\ &\cup \left[\frac{2\alpha-1}{1-\alpha},\, \alpha\right] \times \left[-\infty, -\frac{\sqrt{5}+1}{2}\right].\end{aligned}$$

case (ii) $\quad \dfrac{\sqrt{5}-1}{2} \le \alpha \le 1$

$$\widehat{\mathbf{K}_{\alpha,1}} = \left[0,\, \frac{1-\alpha}{\alpha}\right] \times [-2, -1]\,,$$

$$\widehat{\mathbf{K}_{\alpha,2}} = \left[0,\, \frac{1}{\alpha}\right] \times [-1, 0]\,,$$

$$\widehat{\mathbf{K}_{\alpha,3}} = \left[\alpha-1,\, \frac{1-\alpha}{\alpha}\right] \times [-\infty, -2] \cup \left[\frac{1-\alpha}{\alpha},\, \alpha\right] \times [-\infty, -1]\,.$$

Then we see

- $\widehat{G_\alpha}$ maps $\widehat{\mathbf{J}_{\alpha,i}}$ to $\widehat{\mathbf{K}_{\alpha,i}}$ one-to-one and onto fashion,
- $\bigcup_{i=1}^{3} \widehat{\mathbf{K}_{\alpha,i}} = \widehat{\mathbf{J}_\alpha}$,
- the interiors of $\widehat{\mathbf{K}_{\alpha,i}}$, $i = 1, 2, 3$, are disjoint from each other.

Hence, we have the assertion of this proposition. □

PROPOSITION 4. *The measure $\widehat{\mu_\alpha}$ given by the density function $\widehat{h_\alpha}(x, y) = \dfrac{1}{(x-y)^2}$ is an invariant measure for $\widehat{G_\alpha}$.*

PROOF. It is easy to check that

$$\widehat{h_\alpha}(\widehat{G_\alpha}(x,y)) \cdot |\det(D\widehat{G_\alpha}(x,y))| \cdot \widehat{h_\alpha}^{-1}(x,y) = 1 \quad \text{(a.e.)},$$

which implies the assertion of this proposition, where $\det(D\widehat{G_\alpha}(x,y))$ denotes the determinant of the Jacobian matrix $D\widehat{G_\alpha}(x,y)$. □

REMARK.

$$\iint_{\widehat{\mathbf{J}_\alpha}} \widehat{h_\alpha}(x,y)dxdy = \infty .$$

THEOREM 4. *The dynamical system* $(\widehat{\mathbf{J}_\alpha}, \widehat{G_\alpha}, \widehat{\mu_\alpha})$ *is ergodic.*

PROOF. We put

$$\widehat{\mathbf{I}_\alpha} = \begin{cases} \left[\alpha-1, \dfrac{1-2\alpha}{\alpha}\right) \times \left[-\infty, -\dfrac{\sqrt{5}+3}{2}\right] \cup \left[\dfrac{1-2\alpha}{\alpha}, \dfrac{2\alpha-1}{1-\alpha}\right) \times \left[-\infty, -2\right] \\ \quad \cup \left[\dfrac{2\alpha-1}{1-\alpha}, \alpha\right] \times \left[-\infty, -\dfrac{\sqrt{5}+1}{2}\right] \qquad \text{if } \dfrac{1}{2} \le \alpha \le \dfrac{\sqrt{5}-1}{2}, \\ \left[\alpha-1, \dfrac{1-\alpha}{\alpha}\right) \times [-\infty, -2] \cup \left[\dfrac{1-\alpha}{\alpha}, \alpha\right] \times [-\infty, -1] \quad \text{if } \dfrac{\sqrt{5}-1}{2} \le \alpha \le 1 . \end{cases}$$

It is not so hard to see that $\widehat{\mathbf{I}_\alpha}$ is invariant under the map $(x,y) \mapsto \widehat{G_\alpha}^{k_1(x)}(x,y)$ and this map is the induced transformation $(\widehat{G_\alpha})_{\widehat{\mathbf{I}_\alpha}}$ of $\widehat{G_\alpha}$. It is also possible to show that $\widehat{G_\alpha}^{k_1(\cdot)}$ is isomorphic to the map $\widehat{T_\alpha}$ of H. Nakada [5], via the isomorphism $(x,w) \mapsto (x, -\frac{1}{w})$. Since $\widehat{T_\alpha}$ is ergodic, so are $(\widehat{G_\alpha})_{\widehat{\mathbf{I}_\alpha}}$ and $\widehat{G_\alpha}^{k_1(x)}(x,y) \in \widehat{\mathbf{I}_\alpha}$ for any $(x,y) \in \widehat{\mathbf{J}_\alpha}$. Then we have the ergodicity of $\widehat{G_\alpha}$ from the ergodicity of $(\widehat{G_\alpha})_{\widehat{\mathbf{I}_\alpha}}$. □

REMARK. For the notion of the induced transformation and its ergodicity, we refer to K. Petersen [8].

We put

$$h_\alpha(x) = \int_{\{y:(x,y)\in\widehat{\mathbf{J}_\alpha}\}} \widehat{h_\alpha}(x,y)dy .$$

Then, we have the following corollaries:

COROLLARY 3. *The measure* μ_α*, which is defined by* $d\mu_\alpha(x) = h_\alpha(x)dx$*, is infinite and* G_α*-invariant.*

COROLLARY 4. *The dynamical system* $(\mathbf{J}_\alpha, G_\alpha, \mu_\alpha)$ *is ergodic.*

Next proposition shows that a number of "$\widehat{\mu_\alpha}$-a.e. (x,y)" properties induce "μ_α-a.e. x".

PROPOSITION 5. *For any* $(x, z) \in \widehat{\mathbf{J}_\alpha}$, *we put* $(x_l, z_l) = \widehat{G_\alpha}^l(x, z)$. *Then we see*

$$\lim_{l \to \infty} |z_l - y_l| = 0\,,$$

where $(x_l, y_l) = \widehat{G_\alpha}^l(x, \infty)$.

PROOF. For $(x, z) \in \widehat{\mathbf{I}_\alpha}$, it is possible to show that

$$\lim_{n \to \infty} |z_{k_n(x)} - y_{k_n(x)}| = 0\,.$$

Then the rest of the proof is easy. □

By this proposition and the ratio ergodic theorem, we can get some metric properties of the α-mediant convergents, which were given by Sh. Ito for the regular case, see [3]. However we do not discuss them in detail. For basic facts on ergodic theory, we refer to K. Petersen [8].

For example, we can get the following:
Put

$$\frac{\eta_l}{\xi_l} := (M_1(x)M_2(x)\cdots M_l(x))(\infty)\,.$$

Then, for almost every $x \in \mathbf{I}_\alpha$, we have

$$\frac{\sharp\left\{n\,:\,1 \le n \le N,\ \xi_l^2\left|x - \dfrac{\eta_l}{\xi_l}\right| < t\right\}}{\sharp\left\{n\,:\,1 \le n \le N,\ \xi_l^2\left|x - \dfrac{\eta_l}{\xi_l}\right| < t'\right\}} = \frac{t}{t'} \quad \text{for any} \quad 0 < t, t' \le c_\alpha\,,$$

where

$$c_\alpha = \begin{cases} \dfrac{\alpha}{\dfrac{\sqrt{5}-1}{2} - \alpha + 1} & \text{if} \quad \dfrac{1}{2} \le \alpha \le \alpha^* \\ 1 - \alpha & \text{if} \quad \alpha^* < \alpha \le \dfrac{\sqrt{5}-1}{2} \\ \dfrac{\alpha}{1+\alpha} & \text{if} \quad \dfrac{\sqrt{5}-1}{2} < \alpha < 1 \\ 1 & \text{if} \quad \alpha = 1 \end{cases}$$

and α^* is the unique positive root of $\alpha^2 + \sqrt{5}\alpha - \dfrac{\sqrt{5}-1}{2} = 0$.

5. Some properties of the α-mediant convergents

The aim of this section is to describe relations between the α-mediant convergents and the regular mediant convergents. At first, we give a coding method which translates the α-continued fraction expansion of x to the regular continued fraction expansion of x (if $0 \le$

$x \le \alpha$) or $1+x$ (if $\alpha - 1 \le x < 0$).

We suppose that

$$x = \frac{\varepsilon_{\alpha,1}|}{|c_{\alpha,1}} + \frac{\varepsilon_{\alpha,2}|}{|c_{\alpha,2}} + \frac{\varepsilon_{\alpha,3}|}{|c_{\alpha,3}} + \cdots \quad \text{for} \quad x \in \mathbf{I}_\alpha$$

and

$$x = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots \quad \text{if} \quad 0 \le x \le \alpha$$

or

$$1 + x = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots \quad \text{if} \quad \alpha - 1 \le x < 0\,.$$

We put $l_n = l_n(x) := \sharp\{1 \le k \le n : \varepsilon_{\alpha,k}(x) = -1\}$. Note that $l_1 = 1$ and $a_1 = 1$ if $x < 0$ (equivalently $\varepsilon_{\alpha,1} = -1$).

LEMMA 4.
(i) $a_{n+l_n} = c_{\alpha,n}$ *and* $l_{n+1} = l_n$ *if* $(\varepsilon_{\alpha,n}, \varepsilon_{\alpha,n+1}) = (1,1)$
(ii) $a_{n+l_n} = c_{\alpha,n} - 1$, $a_{n+l_n+1} = 1$ *and* $l_{n+1} = l_n + 1$ *if* $(\varepsilon_{\alpha,n}, \varepsilon_{\alpha,n+1}) = (1,-1)$
(iii) $a_{n+l_n} = c_{\alpha,n} - 1$ *and* $l_{n+1} = l_n$ *if* $(\varepsilon_{\alpha,n}, \varepsilon_{\alpha,n+1}) = (-1,1)$
(iv) $a_{n+l_n} = c_{\alpha,n} - 2$, $a_{n+l_n+1} = 1$ *and* $l_{n+1} = l_n + 1$ *if* $(\varepsilon_{\alpha,n}, \varepsilon_{\alpha,n+1}) = (-1,-1)$

PROOF. The assertions follow from a discussion in §3, [6]. □

This lemma implies that we can determine $a_1, \cdots, a_{n+l_n}$ when $(\varepsilon_{\alpha,1}, c_{\alpha,1}), \cdots, (\varepsilon_{\alpha,n}, c_{\alpha,n})$ are given and, moreover, $a_{n+l_n+1} = 1$ if $\varepsilon_{\alpha,n+1} = -1$. Now we see that the following repetition of a rational number occurs in the sequence of the α-mediant convergents and such a rational number is a regular principal convergent.

PROPOSITION 6. *Suppose that* $x \in \mathbf{I}_\alpha$ *and* $\varepsilon_{\alpha,n+1}(x) = -1$, *then we have*

$$\frac{u_{\alpha,n,1}(x)}{v_{\alpha,n,1}(x)} = \frac{u_{\alpha,n-1,c_{\alpha,n}-1}(x)}{v_{\alpha,n-1,c_{\alpha,n}-1}(x)} = \frac{p_{n+l_n}(x)}{q_{n+l_n}(x)}\,.$$

PROOF. We note that if $c_{\alpha,n} = 1$, then $\varepsilon_{\alpha,n+1} = 1$. In other words, $\varepsilon_{\alpha,n+1} = -1$ implies $c_{\alpha,n} \ge 2$. From the definition (1.2), we see

$$\begin{aligned}\frac{u_{\alpha,n,1}}{v_{\alpha,n,1}} &= \frac{1 \cdot p_{\alpha,n} - p_{\alpha,n-1}}{1 \cdot q_{\alpha,n} - q_{\alpha,n-1}} \\ &= \frac{c_{\alpha,n} p_{\alpha,n-1} + \varepsilon_{\alpha,n} p_{\alpha,n-2} - p_{\alpha,n-1}}{c_{\alpha,n} q_{\alpha,n-1} + \varepsilon_{\alpha,n} q_{\alpha,n-2} - q_{\alpha,n-1}} \\ &= \frac{(c_{\alpha,n} - 1) p_{\alpha,n-1} + \varepsilon_{\alpha,n} p_{\alpha,n-2}}{(c_{\alpha,n} - 1) q_{\alpha,n-1} + \varepsilon_{\alpha,n} q_{\alpha,n-2}}\end{aligned}$$

$$= \frac{u_{\alpha,n-1,c_{\alpha,n}-1}}{v_{\alpha,n-1,c_{\alpha,n}-1}} .$$

Hence, we have the first equality of the assertion.

Next, we consider the second equality. We only show it in the case of $0 \le x \le \alpha$. For $\alpha - 1 \le x < 0$, the proof is essentially the same since $\frac{p_n(1+x)}{q_n(1+x)} = \frac{p_n(x)}{q_n(x)} + 1$. By using Lemma 4, we have

$$\frac{p_{\alpha,n}}{q_{\alpha,n}} = \begin{cases} \frac{p_{n+l_n}}{q_{n+l_n}} & \text{if} \quad \varepsilon_{\alpha,n+1} = 1 \\ \frac{p_{n+l_n+1}}{q_{n+l_n+1}} & \text{if} \quad \varepsilon_{\alpha,n+1} = -1 . \end{cases} \tag{5.1}$$

In particular, if $\varepsilon_{\alpha,n+1} = -1$, then $a_{n+l_n+1} = 1$ and

$$\begin{cases} p_{n+l_n+1} = p_{n+l_n} + p_{n+l_n-1} \\ q_{n+l_n+1} = q_{n+l_n} + q_{n+l_n-1} . \end{cases} \tag{5.2}$$

On the other hand, by (5.1),

$$\frac{p_{\alpha,n-1}}{q_{\alpha,n-1}} = \frac{p_{n-1+l_n}}{q_{n-1+l_n}} , \tag{5.3}$$

since $l_n = l_{n-1}$ if $\varepsilon_{\alpha,n} = 1$ and $l_n = l_{n-1} + 1$ if $\varepsilon_{\alpha,n} = -1$, respectively. Hence, from (5.1), (5.3) and (5.2), we have

$$\frac{u_{\alpha,n,1}}{v_{\alpha,n,1}} = \frac{p_{\alpha,n} - p_{\alpha,n-1}}{q_{\alpha,n} - q_{\alpha,n-1}} = \frac{p_{n+l_n+1} - p_{n+l_n-1}}{q_{n+l_n+1} - q_{n+l_n-1}} = \frac{p_{n+l_n}}{q_{n+l_n}} .$$

□

Next theorem explains how the α-mediant convergents of level n correspond to the regular mediant and the regular principal convergents.

THEOREM 5. *Suppose that $x \in \mathbf{I}_\alpha$. The set of the α-mediant convergents of level n coincides with the following*:

(i) *the set of the regular mediant convergents of level $n+l_n(x)+1$ and the $(n+l_n(x))$th regular principal convergents if $(\varepsilon_{\alpha,n+1}, \varepsilon_{\alpha,n+2}) = (-1, 1)$*

(ii) *the set of the regular mediant convergents of level $n + l_n(x) + 1$, the $(n + l_n(x) - 1)$th regular principal convergents and the $(n + l_n(x) + 2)$th regular principal convergents if $(\varepsilon_{\alpha,n+1}, \varepsilon_{\alpha,n+2}) = (-1, -1)$*

(iii) *the set of the regular mediant convergents of level $n + l_n(x)$ if $(\varepsilon_{\alpha,n+1}, \varepsilon_{\alpha,n+2}) = (1, 1)$*

(iv) *the set of the regular mediant convergents of level $n+l_n(x)$ and the $(n+l_n(x)+1)$th regular principal convergents if $(\varepsilon_{\alpha,n+1}, \varepsilon_{\alpha,n+2}) = (1, -1)$*

PROOF. (i) We assume $x \geq 0$. If $\varepsilon_{\alpha,n+1} = -1$ and $\varepsilon_{\alpha,n+2} = 1$, then from (5.1), (5.3) and (5.2), we have

$$\begin{aligned} u_{\alpha,n,t} &= t \cdot p_{\alpha,n} - p_{\alpha,n-1} \\ &= t \cdot p_{n+l_n+1} - p_{n+l_n-1} \\ &= t(p_{n+l_n} + p_{n+l_n-1}) - p_{n+l_n-1} \\ &= u_{n+l_n+1,t-1} \end{aligned}$$

and

$$v_{\alpha,n,t} = v_{n+l_n+1,t-1}$$

for $2 \leq t < c_{\alpha,n+1} = a_{n+l_n+2} + 1$. In the case $x < 0$, we use $\dfrac{p_m(1+x)}{q_m(1+x)} = \dfrac{p_m(x)}{q_m(x)} + 1$ and get the same conclusion. Thus

$$\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} = \frac{u_{n+l_n+1,t-1}}{v_{n+l_n+1,t-1}} \quad \text{for} \quad 2 \leq t < a_{n+l_n+2} + 1\,.$$

For $t = 1$, by Proposition 6,

$$\frac{u_{\alpha,n,1}}{v_{\alpha,n,1}} = \frac{p_{n+l_n}}{q_{n+l_n}}\,.$$

Consequently, we have

$$\left\{\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} : 1 \leq t < c_{\alpha,n+1}\right\} = \left\{\frac{p_{n+l_n}}{q_{n+l_n}}\right\} \cup \left\{\frac{u_{n+l_n+1,t-1}}{v_{n+l_n+1,t-1}} : 2 \leq t < a_{n+l_n+2} + 1\right\}.$$

This completes the proof of the assertion (i). (ii), (iii) and (iv) follow in the same way. □

As a corollary of Theorem 5, we claim that the set of the α-principal and the α-mediant convergents coincides with the set of the regular principal and the regular mediant convergents.

COROLLARY 5. *For any* α, $\frac{1}{2} \leq \alpha < 1$ *and* $x \in \mathbf{I}_\alpha$,

$$\begin{aligned} &\left\{\frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} : n \geq 1\right\} \cup \left\{\frac{u_{\alpha,n,t}(x)}{v_{\alpha,n,t}(x)} : 1 \leq t < c_{\alpha,n+1},\ n \geq 0\right\} \\ &\qquad = \left\{\frac{p_n(x)}{q_n(x)} : n \geq 1\right\} \cup \left\{\frac{u_{n,t}(x)}{v_{n,t}(x)} : 1 \leq t < a_{n+1},\ n \geq 0\right\}. \end{aligned}$$

PROOF. The only regular principal convergents which are not α-principal convergents are $\dfrac{p_{n+l_n}}{q_{n+l_n}}$ with $\varepsilon_{\alpha,n+1} = -1$. However, these are α-mediant convergents, see Proposition 6. Since $a_{n+l_n+1} = 1$, there is no regular mediant convergent of level $n + l_n$ when $\varepsilon_{\alpha,n+1} = -1$.

Other mediant convergents are α-mediant convergents because of Theorem 5. Indeed, the α-mediant convergents of level n include the regular mediant convergents of level $n + l_n + 1$ if $\varepsilon_{\alpha,n+1} = -1$. In this case, $l_{n+1} = l_n + 1$. Then the next mediant convergent level is $(n + 1) + l_{n+1}$. The same holds for the case $\varepsilon_{\alpha,n+1} = 1$ (then $l_{n+1} = l_n$). This completes the proof of this corollary. □

Finally we construct a map F_α which we call the α-Farey map of the second type. By this map F_α, we can also get the sequence of the α-mediant convergents without $\dfrac{u_{\alpha,n-1,c_{\alpha,n}-1}}{v_{\alpha,n-1,c_{\alpha,n}-1}}$ associated to $\varepsilon_{\alpha,n+1} = -1$, that is, without the repetition which was stated in Proposition 6.

We put $\mathcal{J}_\alpha = [\alpha - 1, 1]$ for $\dfrac{1}{2} \le \alpha \le 1$ and define a new map F_α of $\mathcal{J}_\alpha$ by

$$F_\alpha(x) = \begin{cases} V_-^{-1}(x) = -\dfrac{x}{1+x} & \text{if} \quad x \in [\alpha - 1, 0) =: \mathcal{J}_{\alpha,1} \\ V_+^{-1}(x) = \dfrac{x}{1-x} & \text{if} \quad x \in \left[0, \dfrac{1}{2}\right) =: \mathcal{J}_{\alpha,2} \\ (V_+U)^{-1}(x) = \dfrac{1-2x}{x} & \text{if} \quad x \in \left[\dfrac{1}{2}, \dfrac{1}{1+\alpha}\right] =: \mathcal{J}_{\alpha,3} \\ U^{-1}(x) = \dfrac{1-x}{x} & \text{if} \quad x \in \left(\dfrac{1}{1+\alpha}, 1\right] =: \mathcal{J}_{\alpha,4}\,. \end{cases}$$

Similarly as in §2, we put

$$\mathcal{M}_n(x) := \begin{cases} V_- & \text{if} \quad (F_\alpha)^{n-1}(x) \in \mathcal{J}_{\alpha,1} \\ V_+ & \text{if} \quad (F_\alpha)^{n-1}(x) \in \mathcal{J}_{\alpha,2} \\ V_+U & \text{if} \quad (F_\alpha)^{n-1}(x) \in \mathcal{J}_{\alpha,3} \\ U & \text{if} \quad (F_\alpha)^{n-1}(x) \in \mathcal{J}_{\alpha,4} \end{cases}$$

and

$$\begin{cases} k_0^*(x) := 0\,, \\ k_n^*(x) := \min\{k > k_{n-1}(x) : (F_\alpha)^{k-1}(x) \in \mathcal{J}_{\alpha,3} \cup \mathcal{J}_{\alpha,4}\},\ n \ge 1\,. \end{cases}$$

This means that we abbreviate V_+UV_- to $(V_+U)V_-$ and get a new sequence $\mathcal{M}_1(x)$, $\mathcal{M}_2(x), \cdots$ from $M_1(x), M_2(x), \cdots$. Then it is easy to see the following.

THEOREM 6. *For* $x \in \mathbf{I}_\alpha$,

$$\mathcal{M}_1(x)\mathcal{M}_2(x)\cdots\mathcal{M}_l(x)$$

$$= \begin{cases} \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix} & \text{if} \quad l = k_n^*(x), \ n \geq 1 \\ \\ \begin{pmatrix} u_{\alpha,n,t} & p_{\alpha,n} \\ v_{\alpha,n,t} & q_{\alpha,n} \end{pmatrix} & \text{if} \quad \begin{array}{l} l = k_n^*(x) + t, \ n \geq 0 \\ \text{with} \begin{cases} 1 \leq t < c_{\alpha,n+1} & \text{if} \quad \varepsilon_{\alpha,n+1} = 1 \\ 1 \leq t < c_{\alpha,n+1} - 1 & \text{if} \quad \varepsilon_{\alpha,n+1} = -1 \,. \end{cases} \end{array} \end{cases}$$

Thus we see that $\dfrac{u_{\alpha,n-1,c_{\alpha,n}-1}}{v_{\alpha,n-1,c_{\alpha,n}-1}}$ is removed whenever $\varepsilon_{\alpha,n+1} = -1$ by this abbreviation.

The author will discuss the α-Farey map of the second type from the ergodic theoretic point of view in [7].

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