TOKYO J. MATH.
VOL. 27, NO. 1, 2004

Arithmetical Properties of the Leaping Convergents of $e^{1/s}$

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(Communicated by K. Shinoda)

Abstract. Let $p_k/q_k = [a_0; a_1, a_2, \cdots, a_k]$ be the k-th convergent of the continued fraction expansion of a real number α. We shall show several interesting arithmetic properties concerning every third convergent of the continued fraction expansion of $e^{1/s}$ $(s \geq 1)$.

1. Introduction

Let α be real. $p_k/q_k = [a_0; a_1, \cdots, a_k]$ denotes the k-th convergent of the continued fraction expansion of α, $\alpha = [a_0; a_1, a_2, \cdots]$. Namely,

$$\alpha = a_0 + (1/\alpha_1)\,, \qquad a_0 = \lfloor \alpha \rfloor\,,$$
$$\alpha_n = a_n + (1/\alpha_{n+1})\,, \quad a_n = \lfloor \alpha_n \rfloor \quad (n \geq 1)\,.$$

It is well-known that p_k's and q_k's satisfy the recurrence relation:

$$p_k = a_k p_{k-1} + p_{k-2} \quad (k \geq 0)\,, \quad p_{-1} = 1\,, \quad p_{-2} = 0\,,$$
$$q_k = a_k q_{k-1} + q_{k-2} \quad (k \geq 0)\,, \quad q_{-1} = 0\,, \quad q_{-2} = 1\,.$$

They also satisfy

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$$
$$p_k q_{k-2} - p_{k-2} q_k = (-1)^k a_k$$

and so on (See [2], e.g.).

The number $e^{1/s}$ $(s = 1, 2, \cdots)$ has many significant arithmetical properties. For example, every third convergent also has the similar characteristic relations to the original convergent's. Elsner [1] investigated on the case $s = 1$, namely on Euler number $e = [2; 1, 2, 1, 1, 4, 1, \cdots] = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$. Put

$$P_n = p_{3n+1}\,, \quad Q_n = q_{3n+1} \quad (n \geq 0)$$
$$P_{-1} = P_{-2} = Q_{-1} = 1\,, \quad Q_{-2} = -1\,,$$

Received September 26, 2002; revised September 22, 2003
1991 Mathematics Subject Classification. 11A55, 11B50, 11J04, 11J70.

$$P_{-n} = P_{n-3}\,, \quad Q_{-n} = -Q_{n-3} \quad (n \ge 3)\,.$$

Then for any integer n

$$P_n = C_n P_{n-1} + P_{n-2}\,, \quad Q_n = C_n Q_{n-1} + Q_{n-2}$$

with $C_n = 2(2n+1)$,

$$P_{n-1} Q_n - P_n Q_{n-1} = 2(-1)^{n-1}\,,$$

$$P_{n-2} Q_n - P_n Q_{n-2} = 4(2n+1)(-1)^n$$

and some congruent properties.

The similar phenomenon does not necessarily hold concerning the convergents of every real number α. In this paper we shall show some interesting facts on $e^{1/s}$ $(s \ge 2)$.

2. Continued fraction of $e^{1/s}$

The continued fraction expansion of $e^{1/s}$ $(s \ge 2)$ is given by

$$e^{1/s} = [1; \overline{s(2k-1)-1, 1, 1}]_{k=1}^{\infty}$$

(See [3], §31, p. 134, e.g.). Put $P_n = p_{3n}$ and $Q_n = q_{3n}$ $(n \ge 0)$ with

$$P_{-n} = P_{n-1} \quad \text{and} \quad Q_{-n} = -Q_{n-1} \quad (n \ge 1)\,.$$

Put also $A_n = 2s(2n-1)$. Then a series of following properties holds.

THEOREM 1. (i) *For any integer n*

$$P_n = A_n P_{n-1} + P_{n-2}\,, \quad Q_n = A_n Q_{n-1} + Q_{n-2}\,.$$

(ii) $$P_{n-1} Q_n - P_n Q_{n-1} = 2(-1)^n \quad (n = 0, \pm 1, \pm 2, \cdots)\,.$$

(iii) $$P_{n-2} Q_n - P_n Q_{n-2} = 4s(2n-1)(-1)^{n-1} \quad (n = 0, \pm 1, \pm 2, \cdots)\,.$$

(iv) $$[0; A_1, A_2, A_3, \cdots] = \tanh\frac{1}{2s} = \frac{e^{1/s}-1}{e^{1/s}+1}\,.$$

(v) *The n-th convergent of the continued fraction*

$$e^{1/s} = 1 + \cfrac{2}{2s - 1 + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + \cdots}}}}$$

is exactly equal to P_n/Q_n $(n = 0, 1, 2, \cdots)$.

THEOREM 2.

$$\sum_{t=0}^{r} \frac{P_t Q_{r-t}}{t!(r-t)!} = (4s)^r .$$

REMARK. For every prime $r > 2$

$$\sum_{t=0}^{r} \frac{P_t Q_{r-t}}{t!(r-t)!} \equiv 4s \mod r .$$

THEOREM 3. *For every integer $t > 1$ the sequence $\{(P_n, Q_n) \mod t\}_n$ is periodic, whose period is some divisor of t if t is even; $2t$ if t is odd.*

REMARK. In special, for $n = 0, 1, 2, \cdots$

$$(P_{nt}, Q_{nt}) \equiv (1, (-1)^{nt}), \quad (P_{nt-1}, Q_{nt-1}) \equiv (1, (-1)^{nt-1}) \mod t .$$

THEOREM 4. *Let a and t be arbitrary positive integers. Then*

$$\liminf_{\substack{q \geq 1 \\ q \equiv a \mod t}} q \| q e^{1/s} \| = 0 ,$$

where $\| \cdot \|$ denotes the distance from the nearest integer.

3. Proof of Theorems

PROOF OF THEOREM 1. (i) It is sufficient to prove concerning P_n. Since $a_{3n-2} = s(2n-1) - 1$ and $a_{3n-1} = a_{3n} = 1$ $(n \geq 1)$, we have $n \geq 2$

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2} = 2p_{3n-2} + p_{3n-3} \\ &= 2(s(2n-1)-1)p_{3n-3} + 2p_{3n-4} + p_{3n-3} \\ &= 2s(2n-1)p_{3n-3} + p_{3n-4} - p_{3n-5} \\ &= 2s(2n-1)p_{3n-3} + p_{3n-6} \end{aligned}$$

with

$$p_3 = 2(s-1)p_0 + 2p_{-1} + p_0 = 2sp_0 + 1 .$$

Hence, for $n \geq 1$ we have $P_n = 2s(2n-1)P_{n-1} + P_{n-2}$.

$$P_{-n} = 2s(-2n-1)P_{-n-1} + P_{-n-2} \quad (n \geq 0)$$

is equivalent to

$$P_{n-1} = 2s(-2n-1)P_n + P_{n+1} \quad (n \geq 0) .$$

(ii) For $n \geq 1$

$$P_{n-1}Q_n - P_nQ_{n-1} = p_{3n-3}q_{3n} - p_{3n}q_{3n-3}$$

$$= p_{3n-3}(2q_{3n-2} + q_{3n-3}) - (2p_{3n-2} + p_{3n-3})q_{3n-3}$$
$$= 2(p_{3n-3}q_{3n-2} - p_{3n-2}q_{3n-3}) = 2(-1)^{3n-2} = 2(-1)^n .$$

For $n = 0$ we have $P_{-1}Q_0 - P_0Q_{-1} = 1 \cdot 1 - 1 \cdot (-1) = 2$.
For $n \geq 2$ by definition

$$\begin{aligned} P_{-n-1}Q_{-n} - P_{-n}Q_{-n-1} &= -P_{n-2}Q_{n-3} + P_{n-3}Q_{n-2} \\ &= 2(-1)^{n-2} = 2(-1)^n . \end{aligned}$$

(iii) By (ii), for any integer n we have

$$\begin{aligned} P_{n-2}Q_n - P_nQ_{n-2} &= P_{n-2}(A_nQ_{n-1} + Q_{n-2}) - (A_nP_{n-1} + P_{n-2})Q_{n-2} \\ &= A_n(P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}) \\ &= A_n \cdot 2(-1)^{n-1} = 4s(2n-1)(-1)^{n-1} . \end{aligned}$$

(iv) It is known that

$$[0; \overline{(4k-3)u, (4k-1)v}]_{k=1}^{\infty} = \sqrt{\frac{v}{u}} \tanh \frac{1}{\sqrt{uv}}$$

(See [4], (8), e.g.). Thus, setting $u = v = 2s$ yields the result.

(v) By (iv) we have

$$\frac{e^{1/s} - 1}{2} = [0; A_1 - 1, A_2, A_3, \cdots] .$$

The n-th convergent of $(e^{1/s} - 1)/2$,

$$\frac{P'_n}{Q'_n} = [0; A_1 - 1, A_2, \cdots, A_n] ,$$

is given by the recurrence relations

$$\begin{aligned} &P'_0 = 0 , \quad P'_1 = 1 , \quad &&P'_n = A_nP'_{n-1} + P'_{n-2} \quad (n \geq 2) , \\ &Q'_0 = 1 , \quad Q'_1 = A_1 - 1 , \quad &&Q'_n = A_nQ'_{n-1} + Q'_{n-2} \quad (n \geq 2) . \end{aligned}$$

Since $P'_n = (P_n - Q_n)/2$ and $Q'_n = Q_n$ $(n = 0, 1, \cdots)$,

$$\frac{P_n}{Q_n} = 1 + 2\frac{P'_n}{Q'_n} = 1 + 2[0; A_1 - 1, A_2, \cdots, A_n] .$$

PROOF OF THEOREM 2. Put

$$J(x) = \sum_{n=0}^{\infty} \frac{Q_n}{n!} x^n \quad \text{and} \quad K(x) = \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n .$$

Since

$$Q_{n+2} = 4snQ_{n+1} + 6sQ_{n+1} + Q_n$$

and

$$P_{n+2} = 4snP_{n+1} + 6sP_{n+1} + P_n\,,$$

$y = J(x)$ and $y = K(x)$ satisfy the differential equation $-(1-4sx)y'' + 6sy' + y = 0$. Together with the facts

$$J(0) = Q_0 = 1\,, \qquad J'(0) = Q_1 = 2s - 1\,,$$
$$K(0) = P_0 = 1\,, \qquad K'(0) = P_1 = 2s + 1\,,$$

we get

$$J(x) = \frac{1}{\sqrt{1-4sx}} e^{-\frac{1}{2s}(1-\sqrt{1-4sx})} \quad \text{and} \quad K(x) = \frac{1}{\sqrt{1-4sx}} e^{\frac{1}{2s}(1-\sqrt{1-4sx})}\,.$$

Therefore, we have

$$\sum_{r=0}^{\infty}\left(\sum_{t=0}^{r} \frac{P_t Q_{r-t}}{t!(r-t)!}\right)x^r = J(x)K(x) = \frac{1}{1-4sx} = \sum_{r=0}^{\infty}(4sx)^r\,.$$

To prove Theorem 3 we need the following Proposition.

PROPOSITION. (i) *For any integer* $n \ge 1$ *and* $k \ge 0$

$$P_n \equiv P_{k-1}\,, \quad Q_n \equiv (-1)^{n+k-1} Q_{k-1} \mod n+k\,.$$

(ii) *For any integer* $n \ge 1$ *and* $1 \le k < n$

$$P_n \equiv P_k\,, \quad Q_n \equiv (-1)^{n-k} Q_k \mod n-k\,.$$

REMARK. (i) In special,

$$P_n \equiv \begin{cases} P_{-1} = 1\,, & \mod n\,; \\ P_0 = 1\,, & \mod n+1\,; \\ P_1 = 2s+1\,, & \mod n+2 \end{cases}$$

and

$$Q_n \equiv \begin{cases} (-1)^{n-1} Q_{-1} = (-1)^n\,, & \mod n\,; \\ (-1)^n Q_0 = (-1)^n\,, & \mod n+1\,; \\ (-1)^{n+1} Q_1 = (-1)^{n+1}(2s-1)\,, & \mod n+2\,. \end{cases}$$

Proof of Proposition shall be stated in the next section.

PROOF OF THEOREM 3. By Proposition (i) for $n = 1, 2, \cdots, t$ we have

$$P_1 \equiv P_{t-2}\,, \quad P_2 \equiv P_{t-3}, \cdots, \quad P_n \equiv P_{t-n-1}, \cdots,$$

$$P_{t-1} \equiv P_0 = 1\,, \quad P_t \equiv P_{-1} = 1 \mod t\,.$$

By Proposition (ii) for $n = t+1, t+2, \cdots$ we have

$$P_{t+1} \equiv P_1\,, \quad P_{t+2} \equiv P_2, \cdots, \quad P_{2t} \equiv P_t \equiv 1, \cdots \mod t\,.$$

Thus, the sequence $\{P_n\}_n$ is periodic, whose period is some devisor of t.

By Proposition (i) for $n = 1, 2, \cdots, t$ we have

$$Q_1 \equiv (-1)^{t-1}Q_{t-2}\,, \quad Q_2 \equiv (-1)^{t-1}Q_{t-3}, \cdots, \quad Q_n \equiv (-1)^{t-1}Q_{t-n-1}, \cdots,$$
$$Q_{t-1} \equiv (-1)^{t-1}Q_0 = (-1)^{t-1}\,, \quad Q_t \equiv (-1)^{t-1}Q_{-1} = (-1)^t \mod t\,.$$

By Proposition (ii) for $n = t+1, t+2, \cdots$ we have

$$Q_{t+1} \equiv (-1)^t Q_1\,, \quad Q_{t+2} \equiv (-1)^t Q_2, \cdots, \quad Q_{2t} \equiv (-1)^t Q_t \equiv 1, \cdots,$$
$$Q_{2t+1} \equiv (-1)^t Q_{t+1} \equiv Q_1\,, \quad Q_{2t+2} \equiv (-1)^t Q_{t+2} \equiv Q_2, \cdots,$$
$$Q_{3t} \equiv (-1)^t Q_{2t} \equiv Q_t \equiv (-1)^t, \cdots \mod t\,.$$

Thus, the sequence $\{Q_n\}_n$ is periodic, whose period is some devisor of t if t is even; of $2t$ if t is odd.

PROOF OF THEOREM 4. Notice that $aQ_{2t} \equiv a \mod t$ because $Q_{2t} \equiv 1 \mod t$ by Theorem 3. Since

$$\left|\frac{aP_{2t}}{aQ_{2t}} - e^{1/s}\right| < \frac{1}{a_{6t+2}q_{6t+1}^2} = \frac{1}{(2t+1)q_{6t+1}^2}\,,$$

we obtain

$$0 < aQ_{2t}|aQ_{2t}e^{1/s} - aP_{2t}| < \frac{a^2}{2t+1} \to 0 \quad (t \to \infty)\,.$$

Hence,

$$\liminf_{\substack{q \ge 1 \\ q \equiv a \mod s}} q\|qe^{1/s}\| = \lim_{t\to\infty} aQ_{2t}\|aQ_{2t}e^{1/s}\| = 0\,.$$

4. Proof of Proposition

PROOF OF PROPOSITION (i). We shall prove for $k = 0, 1, 2, \cdots, N+1$

(1) $$P_{N-k} \equiv P_{k-1} \mod N\,,$$

(2) $$Q_{N-k} \equiv (-1)^{N-1}Q_{k-1} \mod N\,.$$

Then, setting $N = n+k$ yields the desired results.

This assertion consists of Lemma 1 and Lemma 2.

LEMMA 1. *If N is odd, we have*

(3) $$P_{\frac{N-1}{2}+i} \equiv P_{\frac{N-1}{2}-i} \mod N \quad \left(i = 0, 1, \cdots, \frac{N+1}{2}\right),$$

(4) $$Q_{\frac{N-1}{2}+i} \equiv (-1)^{N-1} Q_{\frac{N-1}{2}-i} \mod N \quad \left(i = 0, 1, \cdots, \frac{N+1}{2}\right).$$

LEMMA 2. *If N is even, we have*

(5) $$P_{\frac{N}{2}+i} \equiv P_{\frac{N}{2}-i-1} \mod N \quad \left(i = 0, 1, \cdots, \frac{N}{2}\right),$$

(6) $$Q_{\frac{N}{2}+i} \equiv (-1)^{N-1} Q_{\frac{N}{2}-i-1} \mod N \quad \left(i = 0, 1, \cdots, \frac{N}{2}\right).$$

PROOF OF LEMMA 1. (3) is clear for $i = 0$. For $i = 1$

$$P_{\frac{N+1}{2}} = 2sNP_{\frac{N-1}{2}} + P_{\frac{N-3}{2}} \equiv P_{\frac{N-3}{2}} \mod N .$$

Suppose that (3) holds for $i = \nu - 2, \nu - 1$. Then

$$\begin{aligned} P_{\frac{N-1}{2}+\nu} &= 2s(N + 2\nu - 2)P_{\frac{N-1}{2}+\nu-1} + P_{\frac{N-1}{2}+\nu-2} \\ &\equiv 2s(2\nu - 2)P_{\frac{N-1}{2}-\nu+1} + P_{\frac{N-1}{2}-\nu+2} \\ &= 2s(2\nu - 2)P_{\frac{N-1}{2}-\nu+1} + 2s(N - 2\nu + 2)P_{\frac{N-1}{2}-\nu+1} + P_{\frac{N-1}{2}-\nu} \\ &\equiv P_{\frac{N-1}{2}-\nu} \mod N . \end{aligned}$$

By induction (3) follows. (4) is similarly proven.

Before proving Lemma 2, we prepare the following Sublemma.

SUBLEMMA. *For $l > 2$ we have*

$$P_{2^{l-1}-1} \equiv P_{2^{l-1}} \equiv 1 \mod 2^l .$$

PROOF OF SUBLEMMA. If $l = 3$,

$$P_3 = 10sP_2 + P_1 \equiv 4s^2 + 4s + 1 = 4s(s + 1) + 1 \equiv 1 \mod 8$$

and

$$P_4 = 14sP_3 + P_2 \equiv 6s \cdot 1 + 2s + 1 \equiv 1 \mod 8 .$$

Suppose that $P_{2^{l-2}-1} \equiv P_{2^{l-2}} \equiv 1 \mod 2^{l-1}$. Then we can show for $\nu = 0, 1, \cdots$

(7) $$P_{2^{l-2}+2\nu-1} \equiv P_{2\nu-1} + P_{2^{l-2}-1} - 1 \mod 2^l ,$$

(8) $$P_{2^{l-2}+2\nu} \equiv P_{2\nu} + P_{2^{l-2}} - 1 \mod 2^l .$$

(7) and (8) are clear when $\nu = 0$. Suppose that they hold for $\nu = \nu'$. Then

$$\begin{aligned} P_{2^{l-2}+2\nu'+1} &= 2s(2^{l-1} + 4\nu' + 1)P_{2^{l-2}+2\nu'} + P_{2^{l-2}+2\nu'-1} \\ &\equiv 2s(4\nu' + 1)(P_{2\nu'} + P_{2^{l-2}} - 1) + P_{2\nu'-1} + P_{2^{l-2}-1} - 1 \\ &\equiv P_{2\nu'+1} + P_{2^{l-2}-1} - 1 \mod 2^l \end{aligned}$$

and

$$\begin{aligned} P_{2^{l-2}+2\nu'+2} &= 2s(2^{l-1} + 4\nu' + 3)P_{2^{l-2}+2\nu'+1} + P_{2^{l-2}+2\nu'} \\ &\equiv 2s(4\nu' + 3)(P_{2\nu'+1} + P_{2^{l-2}-1} - 1) + P_{2\nu'} + P_{2^{l-2}} - 1 \\ &\equiv P_{2\nu'+2} + P_{2^{l-2}} - 1 \mod 2^l . \end{aligned}$$

By induction, (7) and (8) hold for any non-negative integer ν.

Now put $\nu = 2^{l-3}$ in (7) and (8). By the assumption for $l - 1$ we have

$$P_{2^{l-1}-1} \equiv 2P_{2^{l-2}-1} - 1 \equiv 1 , \quad P_{2^{l-1}} \equiv 2P_{2^{l-2}} - 1 \equiv 1 \mod 2^l . \qquad \square$$

PROOF OF LEMMA 2. We shall show that (5) holds for $N = 2N', 2^2N'', \cdots, 2^lN'''$, where $N', N'', \cdots, N'''$ are any odd numbers.

Assume that $N = 2N'$ (N': odd). Let $i = 0$. By (1) for odd N', $P_{N'} - P_{N'-1} \equiv P_{-1} - P_0 = 0 \mod N'$. Since from Theorem 1(i) every P_n is odd, $P_{N'} \equiv P_{N'-1} \mod 2$. Because $\gcd(N', 2) = 1$, we have $P_{N'} \equiv P_{N'-1} \mod 2N'$ or $P_{\frac{N}{2}} \equiv P_{\frac{N}{2}-1} \mod N$. Let $i = 1$. Then

$$\begin{aligned} P_{\frac{N}{2}+1} - P_{\frac{N}{2}-2} &= 2s(N + 1)P_{\frac{N}{2}} + P_{\frac{N}{2}-1} - P_{\frac{N}{2}} + 2s(N - 1)P_{\frac{N}{2}-1} \\ &\equiv (2s - 1)\big(P_{\frac{N}{2}} - P_{\frac{N}{2}-1}\big) \equiv 0 \mod N . \end{aligned}$$

Suppose that (5) holds for $i = \nu - 2, \nu - 1$. Then

$$\begin{aligned} P_{\frac{N}{2}+\nu} &= 2s(N + 2\nu - 1)P_{\frac{N}{2}+\nu-1} + P_{\frac{N}{2}+\nu-2} \\ &\equiv 2s(2\nu - 1)P_{\frac{N}{2}-\nu} + P_{\frac{N}{2}-\nu+1} \\ &= 2s(2\nu - 1)P_{\frac{N}{2}-\nu} + 2s(N - 2\nu + 1)P_{\frac{N}{2}-\nu} + P_{\frac{N}{2}-\nu-1} \\ &\equiv P_{\frac{N}{2}-\nu-1} \mod N . \end{aligned}$$

By induction (5) follows.

Assume that $N = 4N'$ (N': odd). Let $i = 0$. Since $P_{m+2} \equiv 2sP_{m+1} + P_m \mod 4$ and every P_m is odd, we get $P_{m+4} - P_m \equiv 2s(P_{m+3} + P_{m+1}) \equiv 0 \mod 4$. Hence, $P_{2N'} - P_{2N'-1} \equiv P_2 - P_1 = 4s(3s + 1) \equiv 0 \mod 4$. On the other hand, by (1) for $2N'$ (N': odd), $P_{2N'} \equiv P_{2N'-1} \mod 2N'$, so $\mod N'$. From $\gcd(N', 4) = 1$ we have $P_{2N'} \equiv P_{2N'-1} \mod 4N'$ or $P_{\frac{N}{2}} \equiv P_{\frac{N}{2}-1} \mod N$. By the similar step to the case where $N = 2N'$ (N': odd), (5) can be proven for $N = 4N'$ (N': odd) by induction.

Suppose that (5) holds in the case where $N = 2^{l-1}N'$ (N': odd) with $l > 2$.

Then by (1) for $N = 2^{l-1}N'$, $P_{2^{l-1}N'} \equiv P_{2^{l-1}N'-1} \mod 2^{l-1}N'$, so $\mod N'$. Thus, if

$$P_{2^{l-1}N'} \equiv P_{2^{l-1}N'-1} \mod 2^l\,, \tag{9}$$

by $\gcd(2^{l-1}, N')$ we have $P_{2^{l-1}N'} \equiv P_{2^{l-1}N'-1} \mod 2^l N'$ or $P_{\frac{N}{2}} \equiv P_{\frac{N}{2}-1} \mod N$. By the similar step to the case where $N = 2N'$ (N': odd), (5) can be proven for $N = 2^l N'$ (N': odd) by induction. It follows that (1) holds for any even number N.

Finally, we shall prove (9). By repeating the same step in the proof of Sublemma, we have for a positive integer M

$$P_{2^{l-1}M+\mu} \equiv P_\mu \mod 2^l \quad (1 \le \mu \le 2^{l-1})\,.$$

Hence, by Sublemma we obtain

$$P_{2^{l-1}N'} \equiv P_{2^{l-1}} \equiv 1 \equiv P_{2^{l-1}N'-1} \equiv P_{2^{l-1}N'-1} \mod 2^l\,.$$

(6) is similarly proven. Notice that for $l > 2$

$$Q_{2^{l-1}-1} \equiv -1, \quad Q_{2^{l-1}} \equiv 1 \mod 2^l$$

instead of Sublemma, and for $M = 1, 2, \cdots$ we have

$$Q_{2^{l-1}M+\mu} \equiv Q_\mu \mod 2^l \quad (1 \le \mu \le 2^{l-1})\,.$$

PROOF OF PROPOSITION (ii). By the result of Proposition (i) we have for a non-negative integer N

$$P_N \equiv P_{-1} = 1 = P_0 \quad \text{and} \quad P_{N-1} \equiv P_0 = 1 = P_{-1} \mod N\,.$$

Suppose that

$$P_{N+k-2} \equiv P_{k-2} \quad \text{and} \quad P_{N+k-1} \equiv P_{k-1} \mod N\,.$$

Then

$$\begin{aligned} P_{N+k} &= 2s(2N+2k-1)P_{N+k-1} + P_{N+k-2} \\ &\equiv 2s(2k-1)P_{k-1} + P_{k-2} = P_k \mod N\,. \end{aligned}$$

Therefore, by induction we have $P_{N+k} \equiv P_k \mod N$.

In a similar manner, by Proposition (i)

$$Q_N \equiv (-1)^{N-1}Q_{-1} = (-1)^N = (-1)^N Q_0 \mod N$$

and

$$Q_{N-1} \equiv (-1)^{N-1}Q_0 = (-1)^{N-1} = (-1)^N Q_{-1} \mod N\,.$$

Suppose that

$$Q_{N+k-2} \equiv (-1)^N Q_{k-2} \quad \text{and} \quad Q_{N+k-1} \equiv (-1)^N Q_{k-1} \mod N\,.$$

Then

$$Q_{N+k} = 2s(2N+2k-1)Q_{N+k-1} + Q_{N+k-2}$$
$$\equiv 2s(2k-1)(-1)^N Q_{k-1} + (-1)^N Q_{k-2} = (-1)^N Q_k \mod N\,.$$

Therefore, by induction we have $Q_{N+k} \equiv (-1)^N Q_k \mod N$.

Setting $N = n-k$ yields the results.

5. Generalization

Let $\alpha = [a_0; \overline{c_0 + dk, c_1, \cdots, c_{2l}}]_{k=1}^{\infty}$, where c_i $(i = 0, 1, \cdots, 2l)$ and d are constants so that all of $c_0 + dk$ $(k = 1, 2, \cdots)$ and c_i $(i = 1, 2, \cdots, 2l)$ are positive integers. Put $P_n = p_{(2l+1)n}$ and $Q_n = q_{(2l+1)n}$ $(n = 0, 1, \cdots)$. Then P_n's and Q_n's satisfy the similar relations to those in Theorem 1, even though the congruence relations as seen in Theorems 2, 3 and 4 are no longer guaranteed.

Let positive integers p', q', p'' and q'' satisfy

$$\begin{pmatrix} p' & p'' \\ q' & q'' \end{pmatrix} = \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{2l} & 1 \\ 1 & 0 \end{pmatrix}.$$

Put $A_n = (p'c_0 + p'' + q') + p'dn$ $(n \ge 2)$. Therefore, the following properties hold.

THEOREM 5. (i) $P_n = A_n P_{n-1} + P_{n-2}$ *and* $Q_n = A_n Q_{n-1} + Q_{n-2}$ $(n \ge 2)$.

(ii)
$$P_{n-1}Q_n - P_n Q_{n-1} = p'(-1)^n \quad (n \ge 1)\,.$$

(iii)
$$P_{n-2}Q_n - P_n Q_{n-2} = p' A_n (-1)^{n-1} \quad (n \ge 2)\,.$$

(iv) *The n-th convergent of the continued fraction*

$$\alpha = a_0 + \cfrac{p'}{A_1 - p'' + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cfrac{1}{A_4 + \cdots}}}}$$

is exactly equal to P_n/Q_n $(n = 0, 1, 2, \cdots)$.

PROOF. (i) First, by the relation between continued fractions and matrices,

$$\frac{p_{(2l+1)n}}{q_{(2l+1)n}} = [a_0; a_1, a_2, \cdots, a_{(2l+1)n}]$$

yields

$$\begin{pmatrix} p_{(2l+1)n} & p_{(2l+1)n-1} \\ q_{(2l+1)n} & q_{(2l+1)n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{(2l+1)n} & 1 \\ 1 & 0 \end{pmatrix}$$

$$
= \begin{pmatrix} p_{(2l+1)(n-1)} & p_{(2l+1)(n-1)-1} \\ q_{(2l+1)(n-1)} & q_{(2l+1)(n-1)-1} \end{pmatrix} \begin{pmatrix} c_0+dn & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{2l} & 1 \\ 1 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} p_{(2l+1)(n-1)} & p_{(2l+1)(n-1)-1} \\ q_{(2l+1)(n-1)} & q_{(2l+1)(n-1)-1} \end{pmatrix} \begin{pmatrix} p'(c_0+dn)+q' & p''(c_0+dn)+q'' \\ p' & p'' \end{pmatrix} .
$$

Hence,

$$
p_{(2l+1)n} = (p'(c_0+dn)+q')p_{(2l+1)(n-1)} + p' p_{(2l+1)(n-1)-1} . \tag{10}
$$

Similarly, by

$$
\begin{aligned}
&\begin{pmatrix} p_{(2l+1)(n-1)} & p_{(2l+1)(n-1)-1} \\ q_{(2l+1)(n-1)} & q_{(2l+1)(n-1)-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} p_{(2l+1)(n-2)} & p_{(2l+1)(n-2)-1} \\ q_{(2l+1)(n-2)} & q_{(2l+1)(n-2)-1} \end{pmatrix} \\
&\qquad \times \begin{pmatrix} p'(c_0+d(n-1))+q' & p''(c_0+d(n-1))+q'' \\ p' & p'' \end{pmatrix}
\end{aligned}
$$

and $p''q - p'q'' = -1$, we get

$$
\begin{aligned}
&\begin{pmatrix} p_{(2l+1)(n-2)} & p_{(2l+1)(n-2)-1} \\ q_{(2l+1)(n-2)} & q_{(2l+1)(n-2)-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} p_{(2l+1)(n-1)} & p_{(2l+1)(n-1)-1} \\ q_{(2l+1)(n-1)} & q_{(2l+1)(n-1)-1} \end{pmatrix} \\
&\qquad \times \begin{pmatrix} p'(c_0+d(n-1))+q' & p''(c_0+d(n-1))+q'' \\ p' & p'' \end{pmatrix}^{-1} \\
&\quad = \begin{pmatrix} p_{(2l+1)(n-1)} & p_{(2l+1)(n-1)-1} \\ q_{(2l+1)(n-1)} & q_{(2l+1)(n-1)-1} \end{pmatrix} \begin{pmatrix} -p'' & p''(c_0+d(n-1))+q'' \\ p' & -p'(c_0+d(n-1))-q' \end{pmatrix} .
\end{aligned}
$$

Hence,

$$
p_{(2l+1)(n-2)} = -p'' p_{(2l+1)(n-1)} + p' p_{(2l+1)(n-1)-1} . \tag{11}
$$

(10) and (11) entail that

$$
p_{(2l+1)n} - p_{(2l+1)(n-2)} = (p'(c_0+dn)+q'+p'')p_{(2l+1)(n-1)} .
$$

(ii) Since

$$
\begin{pmatrix} p_{2l+1} & p_{2l} \\ q_{2l+1} & q_{2l} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_0+d & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{2l} & 1 \\ 1 & 0 \end{pmatrix} ,
$$

we get $P_1 = p_{2l+1} = a_0 p'(c_0+d) + a_0 q' + p'$ and $Q_1 = q_{2l+1} = p'(c_0+d)+q'$. Therefore, by (i), for $n \ge 1$

$$
P_{n-1}Q_n - P_n Q_{n-1} = P_{n-1}(A_n Q_{n-1} + Q_{n-2}) - (A_n P_{n-1} + P_{n-2})Q_{n-1}
$$

$$
\begin{aligned}
&= -(P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}) = \cdots \\
&= (-1)^{n-1}(P_0Q_1 - P_1Q_0) \\
&= (-1)^{n-1}\big(a_0\big(p'(c_0+d)+q'\big) - \big(a_0p'(c_0+d)+a_0q'+p'\big)\big) \\
&= (-1)^n p' .
\end{aligned}
$$

(iii) By using (ii), for $n \ge 2$

$$
\begin{aligned}
P_{n-2}Q_n - P_nQ_{n-2} &= P_{n-2}(A_nQ_{n-1} + Q_{n-2}) - (A_nP_{n-1} + P_{n-2})Q_{n-2} \\
&= A_n(P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}) \\
&= A_n(-1)^{n-1}p' .
\end{aligned}
$$

(iv) The 0-th convergent is $a_0 = P_0/Q_0$. The first convergent is

$$
a_0 + \frac{p'}{A_1 - p''} = \frac{a_0p'(c_0+d) + a_0q' + p'}{p'(c_0+d)+q'} = \frac{P_1}{Q_1} .
$$

For $n \ge 2$ the n-th convergent is equal to P_n/Q_n because of the relation

$$
\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 - p'' & 1 \\ p' & 0 \end{pmatrix} \begin{pmatrix} A_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} A_n & 1 \\ 1 & 0 \end{pmatrix} .
$$

The property corresponding to Theorem 1(iv) does not exist because there is no way to find a concrete real number β satisfying $\beta = [0; A_1, A_2, A_3, \cdots]$ in this general case.

Notice that the recurrence relations above are only one-sided. They do not hold for negative n. The properties like Theorems 2, 3 and 4 also do not hold in the general case.

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