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A Note on the Ampleness of Numerically Positive Log Canonical and Anti-Log Canonical Divisors

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Abstract. In this short note, we consider the conjecture that the log canonical divisor (resp. the anti-log canonical divisor) $K_X+\Delta$ (resp. $-(K_X+\Delta)$) on a pair (X, Δ) consisting of a complex projective manifold X and a reduced simply normal crossing divisor Δ on X is ample if it is numerically positive. More precisely, we prove the conjecture for $K_X+\Delta$ with $\Delta \neq 0$ in dimension 4 and for $-(K_X+\Delta)$ with $\Delta \neq 0$ in dimension 3 or 4.

Every variety is defined over the field of complex numbers throughout the paper. Let X be an n-dimensional nonsingular projective algebraic variety and $\Delta = \sum_{i \in I} \Delta_i$ a reduced simply normal crossing divisor on X (where Δ_i is a prime divisor). We denote the canonical divisor of X by K_X. Thus $K_X + \Delta$ denotes the log canonical divisor on the pair (X, Δ).

By the symbol $\kappa(X, L)$, we mean the Iitaka dimension of a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor L.

DEFINITION 0.1. A $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor L on X is *numerically positive* (*nup* ([5]), for short) if $(L, C) > 0$ for every curve C on X.

REMARK 0.2. In the case where $\dim X = 1$, the nupness means the ampleness.

In this paper we deal with the following four conjectures, which are well known to the specialists of higher dimensional algebraic varieties.

CONJECTURE 0.3. If K_X is nup, then it is ample.

CONJECTURE 0.4. If $K_X + \Delta$ is nup, then it is ample.

CONJECTURE 0.5. If $-K_X$ is nup, then it is ample.

CONJECTURE 0.6. If $-(K_X + \Delta)$ is nup, then it is ample.

Conjectures 0.3 and 0.4 are theorems in dimension $n \leq 3$, by virtue of the abundance and the log abundance theorems (due to Kawamata [2] and [3], Miyaoka [10] and Keel-Matsuki-McKernan [4]). Conjecture 0.5 was proved by Hidetoshi Maeda [6] in dimension $n = 2$ and by Serrano [12] in dimension $n = 3$.

Hironobu Maeda [7] proved Conjecture 0.6 in the case where $\Delta \neq 0$ and $n = 2$, as follows: Assume that $n = 2$, the anti-log canonical divisor $-(K_X + \Delta)$ is nup and $\Delta \neq 0$.

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First we shall show that $(-(K_X + \Delta))^2 > 0$. Let us derive a contradiction, assuming that $(-(K_X + \Delta))^2 = 0$. From the nupness of $-(K_X + \Delta)$, we have $-(K_X + \Delta)\Delta > 0$. Thus $-(K_X + \Delta)K_X < 0$. Then $\kappa(X, -(K_X + \Delta)) = 1$ by virtue of Sakai [11], Theorem 2. Hence the nupness of $-(K_X + \Delta)$ implies that $(-(K_X + \Delta))(-(K_X + \Delta)) > 0$, because a high multiple of $-(K_X + \Delta)$ becomes linearly equivalent to some nonzero effective divisor. This is a contradiction! Consequently we have $(-(K_X + \Delta))^2 > 0$. Next we apply the Nakai criterion to the divisor $-(K_X + \Delta)$ and obtain that it is ample.

By using Wilson's technique [13], Hironobu Maeda [7] proved Conjecture 0.6 also in dimension $n = 3$ under the extra condition $\kappa(X, -(K_X + \Delta)) \geq 1$. (This result was reviewed by Matsuki [8].)

Here we remark that Serrano [12] has implicitly proved Conjecture 0.6 in dimension $n = 3$ under the weaker condition that $\kappa(X, -(K_X + \Delta)) \geq 0$, as follows: Assume that $n = 3$, that the anti-log canonical divisor $-(K_X + \Delta)$ is nup and that $\kappa(X, -(K_X + \Delta)) \geq 0$. Then $\kappa(X, (-1)K_X + 1(-(K_X + \Delta))) = \kappa(X, -2(K_X + \Delta) + \Delta) \geq \kappa(X, -2(K_X + \Delta)) = \kappa(X, -(K_X + \Delta)) \geq 0$. Thus Serrano [12], Proposition 3.1 implies that $-(K_X + \Delta) + \varepsilon K_X$ is ample for a sufficiently small positive rational number ε. Therefore $-(K_X + \Delta) = (1/(1 - \varepsilon))((-(K_X + \Delta) + \varepsilon K_X) + \varepsilon\Delta)$ is big. This satisfies the extra condition stated in the preceding paragraph.

Now we state our main theorem

THEOREM 0.7. (1) *Conjecture* 0.4 *is true in the case where* $\Delta \neq 0$ *and* $n = 4$.
(2) *Conjecture* 0.6 *is true in the case where* $\Delta \neq 0$ *and* $n = 3, 4$.

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1. Proof of Theorem 0.7

We define $\mathbf{Strata}(\Delta) := \{\Gamma \mid \Gamma$ is an irreducible component of $\bigcap_{j \in J} \Delta_j \neq \emptyset$, for some nonempty subset J of $I\}$ and $\mathbf{MS}(\Delta) := \{\Gamma \in \mathbf{Strata}(\Delta) \mid$ If $\Gamma' \in \mathbf{Strata}(\Delta)$ and $\Gamma' \subseteq \Gamma$, then $\Gamma' = \Gamma\}$. We remark that $(K_X + \Delta) |_\Gamma = K_\Gamma$ for every $\Gamma \in \mathbf{MS}(\Delta)$.

Let L be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X.

L is said to be *nef and log big* on (X, Δ), if L is nef, $L^n > 0$ and $(L |_\Gamma)^{\dim \Gamma} > 0$ for any $\Gamma \in \mathbf{Strata}(\Delta)$.

REMARK 1.1. Assume that L is nef.
If $bL - (K_X + \Delta)$ is nef for some $b \geq 0$, then so is $aL - (K_X + \Delta)$ for $a \gg 0$.
If $bL - (K_X + \Delta)$ is nup for some $b \geq 0$, then so is $aL - (K_X + \Delta)$ for $a \gg 0$.
If $bL - (K_X + \Delta)$ is nef and big for some $b \geq 0$, then so is $aL - (K_X + \Delta)$ for $a \gg 0$.
If $bL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $b \geq 0$, then so is $aL - (K_X + \Delta)$ for $a \gg 0$.

We cite two lemmas:

LEMMA 1.2 (An uniruledness theorem of Miyaoka-Mori type, Matsuki [9]). *Let* $D_1, D_2, \cdots, D_n$ *be a sequence of nef Cartier divisors. Suppose* $D_1 \cdot D_2 \cdots D_n = 0$ *and* $-K_X \cdot D_1 \cdot D_2 \cdots D_{n-1} > 0$. *Then* X *is covered by a family of rational curves* C *such that* $D_n \cdot C = 0$.

LEMMA 1.3 (Base point free theorem of Reid type, Fukuda [1]). *If* L *is nef and* $bL - (K_X + \Delta)$ *is nef and log big on* (X, Δ) *for some* $b \geq 0$, *then* L *is semi-ample.*

PROPOSITION 1.4. *Assume that* L *is nef and* $bL - (K_X + \Delta)$ *is nup for some* $b \geq 0$ *and that* $\Delta \neq 0$. *If* $((bL - (K_X + \Delta)) |_\Gamma)^{\dim \Gamma} > 0$ *for any* $\Gamma \in \mathbf{MS}(\Delta)$, *then* $aL - (K_X + \Delta)$ *is nef and log big on* (X, Δ) *for* $a \gg 0$.

PROOF. We prove this proposition by induction on n. If $n = 1$, the statement is trivial. Thus we may assume that $n \geq 2$.

We note that $(aL - (K_X + \Delta)) |_{\Delta_i} = aL |_{\Delta_i} - (K_{\Delta_i} + (\Delta - \Delta_i) |_{\Delta_i})$.

First we shall show that $((aL - (K_X + \Delta)) |_\Gamma)^{\dim \Gamma} > 0$ for any $\Gamma \in \mathbf{Strata}(\Delta)$. If $(\Delta - \Delta_i) |_{\Delta_i} \neq 0$, then the induction hypothesis implies that $((aL - (K_X + \Delta)) |_\Gamma)^{\dim \Gamma} > 0$ for any $\Gamma \subseteq \Delta_i$. Thus we may assume that $(\Delta - \Delta_i) |_{\Delta_i} = 0$. Then $\Delta_i \in \mathbf{MS}(\Delta)$. Therefore $((aL - (K_X + \Delta)) |_{\Delta_i})^{\dim \Delta_i} > 0$.

Next we shall show that $(aL - (K_X + \Delta))^n > 0$. Assuming that $(aL - (K_X + \Delta))^n = 0$ for any $a \gg 0$, we will derive the contradiction. Then we have $L^i (K_X + \Delta)^{n-i} = 0$ for $i = 0, 1, 2, \cdots, n$, by regarding $(aL - (K_X + \Delta))^n$ as a polynomial in the variable a. Thus $-K_X \cdot (aL - (K_X + \Delta))^{n-1} = (-aL + \Delta) \cdot (aL - (K_X + \Delta))^{n-1} = \Delta \cdot (aL - (K_X + \Delta))^{n-1} \geq (aL - (K_X + \Delta))^{n-1} \Delta_i = ((aL - (K_X + \Delta)) |_{\Delta_i})^{\dim \Delta_i} > 0$. Consequently Lemma 1.2 derives the contradiction. □

THEOREM 1.5. *Assume that* L *is nef and* $bL - (K_X + \Delta)$ *is nup for some* $b \geq 0$ *and that* $\Delta \neq 0$. *If* $((bL - (K_X + \Delta)) |_\Gamma)^{\dim \Gamma} > 0$ *for any* $\Gamma \in \mathbf{MS}(\Delta)$, *then* L *is semi-ample.*

PROOF. The assertion follows from Lemma 1.3, because $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for $a \gg 0$ by Proposition 1.4. □

PROPOSITION 1.6. (1) *Conjecture* 0.4 *is true in the case* $\Delta \neq 0$, *if Conjecture* 0.3 *is true in dimension* $\leq n - 1$.

(2) *Conjecture* 0.6 *is true in the case* $\Delta \neq 0$, *if Conjecture* 0.5 *is true in dimension* $\leq n - 1$.

PROOF. (1) Put $L = K_X + \Delta$ in the statement of Theorem 1.5. (2) Put $L = -(K_X + \Delta)$ in the statement of Theorem 1.5. □

PROOF OF THEOREM 0.7. (1) Conjecture 0.3 is true in the case $n \leq 3$ (Miyaoka [10], Kawamata [3]). Thus Proposition 1.6 implies the assertion.

(2) Conjecture 0.5 is true in the case $n \leq 3$ (Hidetoshi Maeda [6], Serrano [12]). Thus Proposition 1.6 implies the assertion. □

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