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A Note on Anti-Pluricanonical Maps for 5-Folds

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Abstract. We prove that the anti-pluricanonical map $\Phi_{|-mK_X|}$ is birational when $m \geq 16$ for 5-fold X whose anticanonical divisor is nef and big.

1. Introduction

Throughout the ground field k is always supposed to be algebraically closed of characteristic zero. Let X be a non-singular n-fold over k and assume its anticanonical divisor $-K_X$ is a nef and big divisor. It is an interesting problem to find an explicit lower bound $l(n)$ such that the rational map $\Phi_{|-mK_X|}$ associated with the complete linear system $|-mK_X|$ is a birational map onto its image for any $m \geq l(n)$. Ando ([1, Theorem 9]) first gave the bounds $l(2) = 3, l(3) = 5$ and $l(4) = 12$. Fukuda [3] improved Ando's method and got the bounds $l(3) = 4, l(4) = 11$ and

$$l(n) = 2^{n-2} \cdot (n + 4[n/2] - 1) - 2[n/2] - 1$$

for any $n \geq 5$. Chen [2] also used the similar ideas to improve Ando's partial results. Using the Key Lemma in [3], this note proves

MAIN THEOREM. Let X be a smooth 5-fold whose anticanonical divisor $-K_X$ is nef and big. Then $\Phi_{|-mK_X|}$ is a birational map when $m \geq 16$.

2. Preparations

In this note we use the standard terminology as in [4, 5]. For example, $c_i := c_i(TX)$ is the i-th Chern class of the tangential bundle; $H^i(X, \mathcal{F})$ denotes the i-th cohomology with coefficient in a coherent sheaf $\mathcal{F}$, and $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$. We simply denote $H^i(X, \mathcal{O}_X(D))$ by $H^i(X, D)$ if the sheaf is induced by a divisor D.

We will use Lemma 1, a special case of the Key Lemma in [3], which improved the Theorem 5 in [1].

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LEMMA 1 (Ando [1], Fukuda [3], Chen [2]). *Let X be a nonsingular projective variety of dimension n and $-K_X$ is a nef and big divisor. We assume:*

(i) *For each i with $1 \leq i \leq n-2$, there exists a natural number r_i such that* $\dim \Phi_{|-r_i K_X|}(X) \geq i$.

(ii) *There exist an integer $r_0 \geq 3$ such that $H^0(X, -rK_X) \neq 0$ for any $r \geq r_0$.*

Then $\Phi_{|-mK_X|}$ is birational for all $m \geq r_0 + (r_1 + \cdots r_{n-2})$.

PROOF. In the Key Lemma in [3], we let the nef and big divisor $R = -K_X$ and the numerically trivial divisor $T = 0$. By our assumptions we have $H^0(X, -rK_X) = H^0(X, -(r+1)K_X + K_X) \neq 0$ for any $r + 1 \geq r_0 + 1 := \hat{r}_0 \geq 4$. So both (1) and (2) of the Key Lemma are satisfied. Hence $\Phi_{|-mK_X|} = \Phi_{|-(m+1)K_X+K_X|}$ is birational when $m + 1 \geq \hat{r}_0 + (r_1 + \cdots r_{n-2})$, thus $\Phi_{|-mK_X|}$ is birational for all $m \geq r_0 + (r_1 + \cdots r_{n-2})$. □

To use Lemma 1, we need the following Lemma, it is the Proposition 6 in [1], we refer to [1, 2, 3] for reference of it.

LEMMA 2 (Matsusaka & Maehara). *Let D be a nef and big divisor and* $\dim X = n$. *If* $h^0(X, mD) > m^r D^n + r$, *then* $\dim \Phi_{|mD|} > r$.

3. Proof of the Main Theorem

Let

$$P(m) := \chi(\mathcal{O}_X(-mK_X)) = \sum_{i=0}^{5}(-1)^i h^i(X, -mK_X) = h^0(X, -mK_X)\,,$$

Change to:

$$P(m) := \chi(\mathcal{O}_X(-mK_X)) = \sum_{i=0}^{5}(-1)^i h^i(X, \mathcal{O}_X(-mK_X)) = h^0(X, \mathcal{O}_X(-mK_X))\,,$$

since $-K_X$ is nef and big, by the Kawamata-Viehweg vanishing theorem (cf. [5, Corollary 1-2-2]) we have $H^i(X, -mK_X) = 0$ for $m \geq 0$ and $i > 0$. Thus $\chi(\mathcal{O}_X) = 1$. Note by definition, $c_1 = -K_X$, and $c_i = 0$ for $i > 5$. Combine these facts with Hirzebruch-Riemann-Roch formula ([4, P_{432}]), we have

$$\begin{aligned} P(m) &= \int_X \mathrm{ch}(\mathcal{O}_X(-mK_X))\mathrm{Td}(X) \\ &= m(m+1)(2m+1)(3m^2+3m-1)\frac{(-K_X)^5}{720} + m(m+1)(2m+1)\frac{(-K_X)^3 \cdot c_2}{144} \\ &\quad + (2m+1) \\ &= (2m+1)\{(m(m+1)[(3m^2+3m-1)a+b]+1\}\,, \end{aligned}$$

where $720a = (-K_X)^5$ and $144b = (-K_X)^3 \cdot c_2$.

To use Lemma 1 and Lemma 2, we need priori estimates of $P(m)$ for $m \geq 0$.

PROPOSITION 1.

$$\begin{aligned} &\text{(i)} \quad P(1)=0\,,\; P(2)\geq 0 \Rightarrow P(3)\geq 35\,; \\ &\text{(ii)} \quad P(1)=1\,,\; P(2)\geq 1 \Rightarrow P(3)\geq 21\,; \\ &\text{(iii)} \quad P(1)=2\,,\; P(2)\geq 2 \Rightarrow P(3)\geq 7\,; \\ &\text{(iv)} \quad P(1)=3 \Rightarrow P(2)\geq 6\,; \\ &\text{(v)} \quad P(1)=3\,,\; P(2)=6 \Rightarrow P(3)=49\,; \\ &\text{(vi)} \quad P(m+1)>P(m) \text{ when } m>3 \text{ and } P(3)\geq 7\,. \end{aligned}$$

PROOF. Assume $P(1) = 3[2(5a + b) + 1] := l \geq 0$, we have $b = \frac{1}{6}(l - 3) - 5a$. By $P(2) \geq P(1)$ we get $a \geq \frac{1}{360}(10 - 4l)$, so we have $P(3) \geq 35 - 14l$. Hence we get (i)–(iii).

Now we assume $P(1) = 3$ and $P(2) = 5[6(17a + b) + 1] := l$. Then $b = -5a$ and $l = 5(12a + 1) > 5$, and hence $P(2) \geq 6$ and we have (iv). If $P(2) = 2P(1) = 6$. Then we have $a = \frac{1}{60}$ and $b = -\frac{1}{12}$, So $P(3) = 7[12(35a + b) + 1] = 49 > 7$ we get (v).

If $P(2) > 6$, then $P(3) \geq P(2) \geq 7$. Combine with (i)–(v) we always have $P(3) \geq 7$. Since $P(1) \geq 0$, we have $b \geq -5a - \frac{1}{2}$ and $[3m(m + 1) - 1]a + b > 0$ when $m \geq 3$. Thus $P(m+1)-P(m) > (m+1)\{[3m(m+1)-1]a+b+1\}[(m+2)(2m+3)-(m+1)(2m+1)] > 0$ when $m \geq 3$, we get (vi). □

PROPOSITION 2.

$$\begin{aligned} &\text{(i)} \quad \dim \Phi_{|-mK_X|}(X) \geq 1\,, \text{ for any } m \geq 3\,; \\ &\text{(ii)} \quad \dim \Phi_{|-mK_X|}(X) \geq 2\,, \text{ for any } m \geq 4\,; \\ &\text{(iii)} \quad \dim \Phi_{|-mK_X|}(X) \geq 3\,, \text{ for any } m \geq 6\,. \end{aligned}$$

PROOF. By Proposition 1, $P(3) = 7[12(35a + b) + 1] \geq 7$, so we have (i) and $b \geq -35a$. Thus $P(4) = 9[20(59a+b)+1] \geq 180 \times 24a+9 > 6(-K_X)^5+2$. By Lemma 2 we have (ii). $P(6) = 13[42(125a + b) + 1] \geq \frac{13\times 21}{4}(-K_X)^5 + 13 > 36(-K_X)^5 + 3$, we have (iii). □

PROOF OF MAIN THEOREM. By Proposition 1 we have $h^0(X, -3K_X) \geq 7$, so we can put $r_0 = 3$. By Proposition 2, we can set $r_1 = 3, r_2 = 4, r_3 = 6$. By Lemma 1 when $m \geq r_0 + r_2 + r_3 + r_4 = 16$, then $\Phi_{|-mK_X|}$ is a birational map. □

4. An example

EXAMPLE 1. Let $\pi : E = \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \to \boldsymbol{P}^1$ be a rank 5 vector bundle. Let $X = \mathbb{P}(E)$. Then by calculations in the Exercise 8.4 of [4, P_{253}], $K_X = -5L + \pi^*(\det(E) + K_{\boldsymbol{P}^1}) = -5L - H$, where $L \in |\mathcal{O}_X(1)|, H \in |\pi^*\mathcal{O}_{\boldsymbol{P}^1}(1)|$. Clearly X is a Fano manifold and $-K_X$ is a nef and big divisor. Note $L^5 = H \cdot L^4 = 1$, so $(-K_X)^5 = 2 \times 5^5$. By Leray spectral sequence and the fact that $R^i\pi_*(\mathcal{O}(l)) = 0$ for any $i > 0$ and $l > -5$, we have $H^i(X, -mK_X) = 0$ when $i > 0$ and $H^0(X, -mK_X) = H^0(X, \mathcal{O}_X(5m) \otimes \pi^*\mathcal{O}_{\boldsymbol{P}^1}(m)) = H^0(\boldsymbol{P}^1, S^{5m}(E) \otimes \mathcal{O}_{\boldsymbol{P}^1}(m))$ for any $m > 0$. Note that

$$\begin{aligned} S^{5m}(E) &= \bigoplus_{i=0}^{5m} S^{5m-i}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}) \otimes \mathcal{O}_{\boldsymbol{P}^1}(i) \\ &= \bigoplus_{i=0}^{5m} (\mathcal{O}_{\boldsymbol{P}^1}(i) \oplus \mathcal{O}_{\boldsymbol{P}^1}(i) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(i)), \end{aligned}$$

it is a bundle of rank $\frac{1}{6}(5m-i-1)(5m-i)(5m-i+1)$ in the last bracket of above summation. So,

$$\begin{aligned} &h^0(\boldsymbol{P}^1, S^{5m}(E) \otimes \mathcal{O}_{\boldsymbol{P}^1}(m)) \\ &\qquad = \frac{1}{6} \sum_{i=0}^{5m} (5m-i-1)(5m-i)(5m-i+1) h^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(m+i)) \\ &\qquad = \frac{1}{6} \sum_{i=0}^{5m} (m+i+1)(5m-i-1)(5m-i)(5m-i+1) \\ &\qquad = \frac{1}{24} m(5m-1)(5m+1)(5m+2)(10m+3). \end{aligned}$$

It is easy to check that $h^0(\mathbb{P}(E), -K_X) = 91$, we can take $r_0 = r_1 = 3$; and $h^0(\mathbb{P}(E), -4K_X) = 62909 > 10(-K_X)^5 + 2$, we take $r_2 = 4$, and $h^0(\mathbb{P}(E), -5K_X) = 186030 > 5^2(-K_X)^5 + 3$ we can take $r_3 = 5$. So $\Phi_{|-mK_X|}$ is a birational map when $m \geq 15$.

QUESTION. Find out the lowest bound $l(n)$ such that $\Phi_{|-mK_X|}$ is birational when $m \geq l(n)$.

We also don't know how to improve the bounds $l(n)$ given in [3] for $n > 5$ since the Hirzebruch-Riemann-Roch formula is more complicate in these cases.

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