TOKYO J. MATH.
VOL. 27, NO. 2, 2004

Lefschetz Theory, Geometric Thom Forms and the Far Point Set

Mihail FRUMOSU and Steven ROSENBERG

Boston University

(Communicated by Y. Maeda)

Abstract. The far point set of a self-map of a closed Riemannian manifold M is defined to be the set of points mapped into their cut locus. We prove that the far point set of a map f with Lefschetz number $L(f) \neq \chi(M)$ is infinite unless M is a sphere. There are homology classes supported near $\mathrm{Far}(f)$ which determine $L(f) - \chi(M)$. Using geometric representatives of Thom classes, we obtain a geometric integral formula for the the Lefschetz number, which specializes to the Chern-Gauss-Bonnet formula when $f = \mathrm{Id}$. We compute this formula explicitly for constant curvature metrics. Finally, we give upper and lower bounds for $L(f)$ in terms of the geometry and topology of M and the differential of f.

1. Introduction

The Lefschetz fixed point formula for a smooth function f on a closed manifold M shows that the fixed point set of f has a topological meaning. In this paper, we show that the set of points mapped far from themselves has a similar topological content.

More precisely, if M has a Riemannian metric and $\mathcal{C}_x$ denotes the cut locus of $x \in M$, we set the *far point set* of f to be $\mathrm{Far}(f) = \{x : f(x) \in \mathcal{C}_x\}$. If $\mathrm{Far}(f) = \emptyset$, then f can be deformed to the identity map along the minimal geodesic joining x and $f(x)$, so $L(f) = \chi(M)$. This indicates that the invariant $L(f) - \chi(M)$ is controlled by $\mathrm{Far}(f)$. We show in Theorem 3.2 that there are homology classes supported arbitrarily close to $\mathrm{Far}(f)$ which determine this difference. We also show in Theorem 2.1 that $|\mathrm{Far}(f)| = \infty$ for $L(f) \neq \chi(M)$ unless M is diffeomorphic to a sphere.

The proof of Theorem 3.2 depends on a simple topological formula for oriented M:

$$L(f) = (-1)^{\dim M} \langle (\mathrm{Id}, f)^* \eta_\Delta, [M] \rangle \tag{1}$$

(Proposition 2.3), where η_Δ is Poincaré dual to the diagonal Δ in $M \times M$. Using geometric representatives of η_Δ, we obtain a geometric integral formula for $L(f)$. This formula is intriguing, as it reduces to the Chern-Gauss-Bonnet theorem for $f = \mathrm{Id}$, it is non-trivial for flat manifolds, and it involves a careful study of Jacobi fields for general Riemannian metrics. The general geometric formula is in Theorem 4.3, and we work out an explicit formula for constant curvature metrics on surfaces in Corollary 4.4. Finally, we use Hodge theory and

Received April 14, 2000; revised March 22, 2004

Cheeger-Gromov type estimates to give two-sided bounds for $L(f)$ in terms of the geometry and topology of M and the norm of df.

We would like to thank the referee for helpful suggestions.

2. The far point set

Let $f : M \to M$ be a smooth map of a closed manifold M. The Lefschetz number of f is

$$L(f) = \sum_q (-1)^q \operatorname{tr} f^q ,$$

where f^q denotes the induced map on the real cohomology group $H^q(M)$. In this section we show that the far point set of f is usually infinite (Theorem 2.1) and give a global expression (Proposition 2.3) for $L(f)$ which will be used in §§3, 4.

2.1. The far point set for $L(f) \neq \chi(M)$. Recall that on a closed Riemannian manifold, a geodesic $\gamma(t)$ is the minimal length curve joining $x = \gamma(0)$ to $\gamma(t)$ for $t \in [0, T]$ up to some maximal time T. The point $\gamma(T)$ is by definition in $\mathcal{C}_x$, the cut locus of x. In particular, if a smooth function $f : M \to M$ has $f(x) \notin \mathcal{C}_x$, for all x, then there is a unique minimal geodesic joining x to $f(x)$. Shrinking this geodesic gives a homotopy from f to the identity, and so the Lefschetz number satisfies $L(f) = L(\mathrm{Id}) = \chi(M)$. This indicates that the difference $L(f) - \chi(M)$ is controlled by the *far point set* of f

$$\mathrm{Far}(f) = \{x : f(x) \in \mathcal{C}_x\} . \tag{2}$$

This set is closed. The following examples show that the far point set is related to, but more complicated than, the cut locus of a fixed point.

EXAMPLES. (i) Small perturbations of the identity map on any manifold have $\mathrm{Far}(f) = \emptyset$. In contrast, the antipodal map on $M = S^n$ has $\mathrm{Far}(f) = M$.

(ii) Let S^1 have the standard metric, and let $a : S^1 \to S^1$ be the antipodal map. For $x_1, \cdots, x_m$ in cyclic order on S^1, we can construct $a_1 : S^1 \to S^1$ with $\mathrm{Far}(a_1) = \{x_1, \cdots, x_m\}$, by composing a with a map that fixes $\{x_1, \cdots, x_m\}$ and sends each interval between $a(x_i)$ and $a(x_{i+1})$ to itself with no fixed point.

(iii) The previous example can be extended to the standard metric on S^2. We extend a_1 from the equator S^1 by a map which replicates a_1 on each meridian circle. $\mathrm{Far}(a_1)$ is still $\{x_1, \cdots, x_m\}$. Note that $L(f) = 0$. This example can then be extended to S^n. For $m = 2$, there is an example with a non-standard metric on S^n by creating a radially symmetric valley near x_2; the cut locus of x_1 is still x_2.

(iv) Fix $x_0 \in M$ and let f be the constant map $f(x) = x_0$. Then $\mathrm{Far}(f) = \{x : x_0 \in \mathcal{C}_x\}$. Since $x \in \mathcal{C}_{x_0}$ iff $x_0 \in \mathcal{C}_x$, $\mathrm{Far}(f) = \mathcal{C}_{x_0}$. It is standard that $\mathcal{C}_x$ has no interior. As the image of a connected set (see below), $\mathcal{C}_x$ is either a single point or an infinite set. In particular, if M is not diffeomorphic to S^n, then $|\mathcal{C}_x| = \infty$.

The goal of this section is to generalize this last fact by proving that only S^n admits maps with $L(f) \neq \chi(M)$ and $\mathrm{Far}(f)$ finite.

THEOREM 2.1. *If $L(f) \neq \chi(M)$ and M is not diffeomorphic to S^n, then* $|\mathrm{Far}(f)| = \infty$.

Before the proof, we recall some facts about the cut locus. The set of vectors $v \in T_xM$ such that the geodesic through x with tangent vector v is minimal up to time one forms a topological ball B_x, whose boundary is called the cut locus of x in T_xM. The exponential map satisfies $\exp_x(\partial B_x) = \mathcal{C}_x$, and we always consider the exponential map $\exp_x : \bar{B}_x \to M$ to have domain $\bar{B}_x$.

Assume $\mathrm{Far}(f) = \{x_1, \cdots, x_n\}$ is finite. f determines a smooth vector field V_f on $M \setminus \mathrm{Far}(f)$ by $V_f(y) = \exp_y^{-1} f(y)$. We can perturb f near the fixed point set so that the fixed point set is finite without altering the far point set. At each fixed point x of f, the local Lefschetz number $L_x(f)$ equals the Hopf index $\mathrm{ind}_x(V_f)$ of V_f [8, p. 135]. We modify V_f by multiplying the vectors in a neighborhood of each x_i by a smooth function which is one on the boundary of the neighborhood and which vanishes to sufficiently high order at x_i. The modified vector field extends to a smooth vector field V'_f on all of M with zeros at the fixed points of f and at the x_i. By the Hopf index formula,

$$\chi(M) = \sum_{\{x : V'_f(x)=0\}} \mathrm{ind}_x(V'_f) = \sum_{\{x : f(x)=x\}} L_x(f) + \sum_i \mathrm{ind}_{x_i}(V'_f)\,,$$

and so

$$L(f) - \chi(M) = -\sum_i \mathrm{ind}_{x_i}(V'_f)\,, \tag{3}$$

by the Lefschetz fixed point formula.

LEMMA 2.2. *Let $x \in \mathrm{Far}(f)$. If $\partial\bar{B}_x \not\supseteq \exp_x^{-1} f(x)$, then* $\mathrm{ind}_x(V'_f) = 0$.

PROOF. Let S_ε^{n-1} be a geodesic sphere in M around x small enough so that (i) there is a unique minimal geodesic from x to all points inside the sphere, and (ii) no far point of f lies on the sphere. $\mathrm{ind}_x V'_f$ is the degree of the map $\alpha_\varepsilon : S_\varepsilon^{n-1} \to \partial\bar{B}_x$, $\alpha_\varepsilon(y) = c_y \|_{y\to x} \exp_y^{-1} f(y)$, where $\|_{y\to x}$ is parallel translation along the geodesic from y to x, and $c_y > 0$ is chosen so that $\alpha_\varepsilon(y) \in \partial\bar{B}_x$. As $\varepsilon \to 0$, $\alpha_\varepsilon(S_\varepsilon^{n-1})$ approaches a subset of $\exp_x^{-1} f(x)$, as the unique minimal geodesic from y_i to $f(y_i)$ converges to a geodesic γ from x to $f(x)$ as $y_i \to x$. (γ will be a minimal geodesic by a continuity argument, but there may be other minimal geodesics.)

If $\exp_x^{-1} f(x) \not\subseteq \partial\bar{B}_x$, then α_ε is not surjective for some ε. Indeed, $\exp_x^{-1} f(x) \cap \partial\bar{B}_x$ is closed in $\partial\bar{B}_x$, so its complement is open. Say $B_{r_0}(z_0)$ is an open ball in the complement. If α_ε is surjective for all ε, take $\varepsilon_i \to 0$ and find $y_i \in S_{\varepsilon_i}^{n-1}$ with $\alpha_{\varepsilon_i}(y_i) = z_0$. Then the minimum geodesic from y_i to $f(y_i)$ converge to a geodesic from x to $\exp_x(z_0) \neq f(x)$, a contradiction.

Once α_ε is not surjective, it is homotopic to the constant map, and hence has degree zero. □

PROOF OF THEOREM 2.1. Say $L(f) \neq \chi(M)$ and $|\mathrm{Far}(f)| < \infty$. Then by (3) and the Lemma, $\exp_x^{-1} f(x) \cap \partial \bar{B}_x = \partial \bar{B}_x$ for some x. Thus M is homeomorphic to the sphere $S = \bar{B}_x / \partial \bar{B}_x$, i.e. $\bar{B}_x$ with its boundary collapsed to a point.

To show that M is diffeomorphic to S^n, let $\theta = (\theta_1, \cdots, \theta_{n-1})$ denote angular coordinates on S^{n-1}. By a standard continuity property of the cut locus at x, B_x is given by $\{(r,\theta) : 0 \leq r < \rho(\theta)\}$ for some continuous function $\rho : S^{n-1} \to \mathbb{R}^+$. Pick ε with $0 < \varepsilon < \min_{\{\theta\}}\{\rho(\theta)\}$. Set $\exp_x : \{(r,\theta) : 0 \leq r < \varepsilon\} \to M$ to be one chart, and cover $M \setminus \{x\}$ with the chart $\exp_x : \{(r,\theta) : 0 < r \leq \rho(\theta)\}/\partial \bar{B}_x \to M \setminus \{x\}$. The domain of the second chart is a topological ball, and the transition maps on the charts' overlap are smooth. Thus M is diffeomorphic to a sphere. □

2.2. A global formula for the Lefschetz number. In this subsection we obtain a formula for the Lefschetz number which is the basis for our geometric generalizations of the Chern-Gauss-Bonnet formula.

We assume for the moment that all manifolds and submanifolds are oriented. By the tubular neighborhood theorem, a neighborhood of a submanifold X of a manifold Y is diffeomorphic to the total space of the normal bundle ν_X of X in Y, so the Thom class $U_X \in H_c^*(\nu_X; \mathbb{Z})$ can be considered as a cohomology class on Y. It is standard [3, Ch. 1.6] that $U_X = \eta_X$, the Poincaré dual of X in Y.

Let $[M]$ be the fundamental class of M, and let Γ be the graph of f, i.e. the image of $(\mathrm{Id}, f) : M \to M \times M$. Γ has a natural orientation induced by (Id, f).

PROPOSITION 2.3. *Let $f : M \to M$ be a smooth map of a closed, oriented manifold. Let Δ be the diagonal of M in $M \times M$, and let η_Δ be the Poincaré dual of the diagonal Δ, Then*

$$L(f) = (-1)^{\dim M} \langle (\mathrm{Id}, f)^* \eta_\Delta, [M] \rangle \,. \tag{4}$$

PROOF. By the Lefschetz fixed point formula, we have

$$\langle \eta_\Gamma \cup \eta_\Delta, [M \times M] \rangle = \Gamma \cap \Delta = L(f)\,.$$

By Poincaré duality, we obtain

$$\begin{aligned}\langle \eta_\Gamma \cup \eta_\Delta\,, [M \times M] \rangle &= (-1)^{\dim M} \langle \eta_\Delta, [\Gamma] \rangle = (-1)^{\dim M} \langle \eta_\Delta, (\mathrm{Id}, f)_*[M] \rangle \\ &= (-1)^{\dim M} \langle (\mathrm{Id}, f)^* \eta_\Delta, [M] \rangle\,,\end{aligned}$$

□

If M is not oriented, let M' be the oriented, connected double cover. f lifts to a map f' covering f. The Lefschetz fixed point formula holds on M, since each fixed point contribution is independent of a choice of local orientation (and consistent local orientation of Γ).

Therefore

$$L(f) = \frac{1}{2}L(f') = \frac{(-1)^{\dim M}}{2}\langle(\mathrm{Id}, f')^*\eta_\Delta, [M']\rangle\,.$$

For convenience, we will from now on assume that M is oriented, with the understanding that the results of the paper carry over for nonorientable M as in the last equation.

3. Cycles on the far point set

In this section, we will find both homology classes F_ε and differential forms on small closed neighborhoods of the far point set which calculate $L(f) - \chi(M)$. This is a precise version of the statement that $L(f) - \chi(M)$ is controlled by the far point set. We also find corresponding classes which control the degree of a map and the Euler characteristic.

Let $\bar{\mathrm{T}}_{TM}$ be a fixed differential form in the Thom class of TM. If 0 denotes the zero section of TM, then $\chi(M) = \int_M 0^*\bar{\mathrm{T}}_{TM}$, since $0^*\bar{\mathrm{T}}_{TM}$ represents the Euler class. In addition, since any section s of TM is homotopic to the zero section, we have $\chi(M) = \int_M (ts)^*\bar{\mathrm{T}}_{TM}$, for $t \geq 0$. As $t \to \infty$, the integrand concentrates on the zero set of s, and we recover the Hopf index formula, as explained in [11, §8].

As in [9], [11], we may construct $\bar{\mathrm{T}}_{TM}$ so that its restriction to M is $\mathrm{Pf}(\Omega)$, the Pfaffian of the curvature of a Riemannian metric on M (cf. §4.1). Thus this construction of geometric representatives of the Thom class gives a proof of the Chern-Gauss-Bonnet theorem: $\chi(M) = \int_M \mathrm{Pf}(\Omega)$.

Motivated by these topological techniques, we will deform f to a function f_t in the basic formula $L(f) = (-1)^{\dim M}\int_M (\mathrm{Id}, f)^*\bar{\mathrm{T}}_\Delta$, where $\bar{\mathrm{T}}_\Delta$ is any representative of the Poincaré dual of the diagonal Δ in $M \times M$. As $t \to \infty$, the Lefschetz fixed point formula can be recovered [5]. However, $f_t(x)$ will be discontinuous at $t = 0$ iff $x \in \mathrm{Far}(f)$. Thus this case is more complicated.

To define f_t, let T be the "cut locus tubular neighborhood" of Δ, whose vertical fiber at (x, x) is $\{x\} \times \exp_x B_x$. T is homeomorphic to the tangent bundle of M, and ignoring differentiability issues at ∂T, we can consider f as a vector field which blows up on $\{x : (x, f(x)) \notin T\} = \mathrm{Far}(f)$. As in Figure 1, the family f_t is analagous to the family of sections ts, except that we freeze f "at infinity."

We define f_t for $t > 0$ by pushing $f(x)$ out the minimal geodesic joining x and $f(x)$ towards ∂T as $t \to \infty$ and pushing $f(x)$ towards x as $t \to 0$, if $(x, f(x)) \in T$, and fixing $f(x)$ otherwise. More precisely, we want

1. $f_1 = f$,
2. $f_t(x) = f(x)$ if $(x, f(x)) \notin T$ or if $f(x) = x$,
3. $\lim_{t\to\infty} f_t(x) \in \partial T$ if $(x, f(x)) \in T$ but $f(x) \neq x$,
4. $\lim_{t\to 0} f_t(x) = x$ if $(x, f(x)) \in T$.

To construct f_t, fix a diffeomorphism $\mu : [0, 1) \to [0, \infty)$ with $\mu(0) = 0$, $\mu(1) = \infty$ and such that the derivative of μ^{-1} grows at most polynomially. Let T denote any smooth

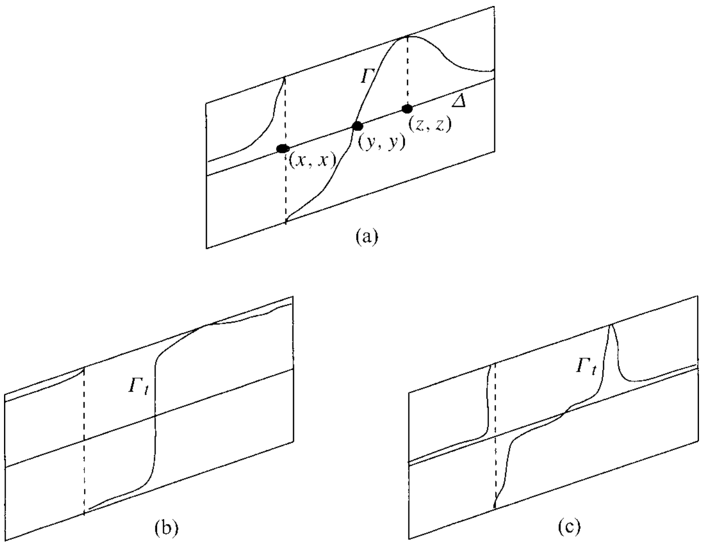

FIGURE 1. (a) The graph Γ of f inside the cut locus tubular neighborhood of the diagonal Δ for $M = S^1$. The top and bottom lines are the boundary of the cut locus tubular neighborhood. $\mathrm{Fix}(f) = \{y\}$ and $\mathrm{Far}(f) = \{x, z\}$. (b) The graph Γ_t of f_t for t large. The support of the Lefschetz integrand $(\mathrm{Id}, f_t)^*\bar{\mathrm{T}}_\Delta$ approaches $\mathrm{Fix}(f)$. (c) The graph Γ_t of f_t for $t \approx 0$. The support of $(\mathrm{Id}, f_t)^*\bar{\mathrm{T}}_\Delta$ approaches a subset of $M \setminus \mathrm{Far}(f)$.

open tubular neighborhood of Δ given by geodesics of the form $(x, y) = (\gamma(t), \gamma(-t))$. For such (x, y), set

$$d_{x,y} = \min\{t : (\gamma(t), \gamma(-t)) \in \partial T\}\,.$$

For $x \in M$ and $t \in [0, \infty)$, define $t_x : M \to M$ by

$$t_x(y) = \begin{cases} \exp_x \left[\mu^{-1}(\mu(d_{x,y}^{-1}|v|)t)d_{x,y}\dfrac{v}{|v|}\right], & (x, y) \in T\,, \quad y = \exp_x v\,, \quad y \neq x\,, \\ y\,, & (x, y) \notin T\,, \\ x\,, & y = x\,. \end{cases}$$

Thus for x, y close but unequal, t_x pushes y towards ∂T as $t \to \infty$ along their minimal geodesic, but fixes y if it is far from x, as measured by T. The complicated expression for $t_x(y)$ in the first case ensures that $1_x(y) = y$.

For $f : M \to M$, define $f_t : M \to M$ by

$$f_t(x) = t_x(f(x))\,.$$

The maps f_t are smooth for $t > 0$. Note that $f_1(x) = f(x)$. Similarly, we have $f_0(x) = x$ if $(x, f(x)) \in T$, and $f_0(x) = f(x)$ otherwise. Thus f_0 is discontinuous on $\{x : (x, f(x)) \in$

$\partial T\}$. This construction extends to the cut locus tubular neighborhood, noting that $d(x, y)$ is now only continuous in (x, y). For this T, we have $M \times M$ equals the closure of T.

Let B_ε be the open ε-neighborhood of $\mathrm{Far}(f)$, and let $\bar{\mathrm{T}}_\Delta$ be a differential form with support in the cut locus tubular neighborhood of Δ and Poincaré dual to Δ.

LEMMA 3.1.

$$\lim_{t\to 0}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta = 0^*\bar{\mathrm{T}}_\Delta$$

uniformly on $M \setminus B_\varepsilon$.

Strictly speaking, we should write $\lim_{t\to 0}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta = i^*0^*\bar{\mathrm{T}}_\Delta$, where $i : M \to M\times M$ is the inclusion $i(x) = (x, x)$.

PROOF. On $M \setminus B_\varepsilon$, $f_t(y) \to y$ uniformly as $t \to 0$. Thus if $\gamma(s)$ is a short curve with $\gamma(0) = y$, $\dot\gamma(0) = w$, then $f_t(\gamma(s)) \to \gamma(s)$ uniformly as $t \to 0$, and

$$\begin{aligned}\lim_{t\to 0}(f_t)_*(w) &= \lim_{t\to 0}\lim_{s\to 0}\frac{f_t(\gamma(s)) - f_t(y)}{s} = \lim_{s\to 0}\lim_{t\to 0}\frac{f_t(\gamma(s)) - f_t(y)}{s}\\ &= \lim_{s\to 0}\frac{\gamma(s) - \gamma(0)}{s} = \dot\gamma(0) = w\,.\end{aligned}$$

This shows that

$$[(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta]_y(v_1, \cdots, v_n) = (\bar{\mathrm{T}}_\Delta)_{(y, f_t(y))}((v_1, (f_t)_*v_1), \cdots, (v_n, (f_t)_*v_n))$$

converges uniformly in y as $t \to 0$ to

$$(\bar{\mathrm{T}}_\Delta)_{(y,y)}((v_1, v_1), \cdots, (v_n, v_n)) = (0^*\bar{\mathrm{T}}_\Delta)_y(v_1, \cdots, v_n)$$

□

Thus

$$\begin{aligned}L(f) &= \int_M (\mathrm{Id},\, f)^*\bar{\mathrm{T}}_\Delta = \int_M (\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta\\ &= \lim_{\varepsilon\to 0}\lim_{t\to 0}\left[\int_{M\setminus B_\varepsilon}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta + \int_{B_\varepsilon}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta\right]\\ &= \lim_{\varepsilon\to 0}\left[\int_{M\setminus B_\varepsilon}0^*\bar{\mathrm{T}}_\Delta + \lim_{t\to 0}\int_{B_\varepsilon}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta\right] \qquad (5)\\ &= \int_M 0^*\bar{\mathrm{T}}_\Delta - \lim_{\varepsilon\to 0}\int_{B_\varepsilon}0^*\bar{\mathrm{T}}_\Delta + \lim_{\varepsilon\to 0}\lim_{t\to 0}\int_{B_\varepsilon}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta\\ &= \chi(M) - \int_{\mathrm{Far}(f)}0^*\bar{\mathrm{T}}_\Delta + \lim_{\varepsilon\to 0}\lim_{t\to 0}\int_{B_\varepsilon}(\mathrm{Id},\, f_t)^*\bar{\mathrm{T}}_\Delta\,.\end{aligned}$$

Notice that $\int_{\mathrm{Far}(f)}0^*\bar{\mathrm{T}}_\Delta = 0$ if the interior of $\mathrm{Far}(f)$ is empty.

We can give a homological interpretation to (5). By the compactness of $\mathrm{Far}(f)$, we can cover the closure $\bar{B}_\varepsilon$ with a finite number of balls of radius ε. Relabeling this union of balls as B_ε, we can easily smooth $\partial \bar{B}_\varepsilon$ so that the resulting space, still called $\bar{B}_\varepsilon$, is a manifold with boundary. Set $M' = M \setminus \bar{B}_\varepsilon$ and $A_\varepsilon = B_\varepsilon \times M \cup M \times B_\varepsilon$. There are restriction maps

$$r_1 : H_*(M \times M; \mathbf{Z}) \to H_*^{\mathrm{lf}}(A_\varepsilon; \mathbf{Z}), \quad r_2 : H_*(M \times M; \mathbf{Z}) \to H_*^{\mathrm{lf}}(M' \times M'; \mathbf{Z}),$$

where H_*^{lf} is locally finite/Borel-Moore homology, the dual to cohomology. These restriction maps are constructed by taking a smooth cycle on $M \times M$, intersecting it with e.g. A_ε (which is a manifold after a little smoothing), and rewriting $c \cap A_\varepsilon$ where c is a simplex in this cycle meeting ∂A_ε, as a locally finite cycle. Under the inclusions i_1, i_2 of A_ε, $M' \times M'$ into $M \times M$, respectively, we have $(i_1)_* r_1 \Delta + (i_2)_* r_2 \Delta = \Delta$, and similarly for Γ. Dropping the inclusion maps, we have

$$\begin{aligned} L(f) - \chi(M) &= \Delta \cdot \Gamma - \Delta \cdot \Delta \\ &= (r_1 \Delta + r_2 \Delta) \cdot (r_1 \Gamma + r_2 \Gamma) - (r_1 \Delta + r_2 \Delta)^2 \\ &= r_1 \Delta \cdot (r_1 \Gamma - r_1 \Delta), \end{aligned}$$

since Γ is homotopic to Δ on $M' \times M'$, and $r_1 \alpha \cdot r_2 \beta = 0$ for all α, β. These intersections numbers are well defined, as they correspond to integrating the compactly supported (in $M \times M$) forms Poincaré dual to Δ, Γ over A_ε or $M' \times M'$.

This gives the following topological formulas for $L(f) - \chi(M)$ in terms of the far point set.

THEOREM 3.2. *Let B_ε be a smoothed epsilon neighborhood of the far point set, let $A_\varepsilon = B_\varepsilon \times M \cup M \times B_\varepsilon$, and let $r_1 : H_*(M \times M; \mathbf{Z}) \to H_*^{\mathrm{lf}}(A_\varepsilon; \mathbf{Z})$ be the corresponding restriction map. For all ε sufficiently small,*

$$\begin{aligned} L(f) - \chi(M) &= -\int_{\mathrm{Far}(f)} 0^* \bar{\mathrm{T}}_\Delta + \lim_{t \to 0} \int_{B_\varepsilon} (\mathrm{Id}, f_t)^* \bar{\mathrm{T}}_\Delta \\ &= r_1 \Delta \cdot (r_1 \Gamma - r_1 \Delta). \end{aligned}$$

In particular, $L(f) - \chi(M)$ can be computed in an arbitrarily small neighborhood of $\mathrm{Far}(f)$.

It is not correct to conclude $\int_{\mathrm{Far}(f)} 0^* \bar{\mathrm{T}}_\Delta = r_1 \Delta \cdot r_1 \Delta$. Identifying B_ε and $(B_\varepsilon, B_\varepsilon) \subset M \times M$, we have $\int_{B_\varepsilon} 0^* \bar{\mathrm{T}}_\Delta = \int_{A_\varepsilon} \bar{T}_\Delta \wedge \mathrm{PD}(B_\varepsilon, B_\varepsilon)$. Although $\bar{T}_\Delta$ is Poincaré dual to the diagonal in $M \times M$, this integral does not equal the integer $r_1 \Delta \cdot r_1 \Delta$, since $0^* \bar{\mathrm{T}}_\Delta$ is not compactly supported in A_ε. Letting $\varepsilon \to 0$ does not improve matters.

The fixed point set analogue to Theorem 3.2 is $L(f) = r_1 \Delta \cdot r_1 \Gamma$, where B_ε is a neighborhood of the fixed point set. This is the Lefschetz fixed point formula if the fixed point set is isolated. Given a vector field with isolated zeros, we can also write $L(f) - \chi(M) = r_1 \Delta \cdot (r_1 \Gamma - r_1 \Delta)$, where B_ε is now a neighborhood of the fixed point set union the zeros of the vector field.

REMARKS. (1) This technique can be used to treat the degree of f, defined by $\deg(f) = \int_M f^*\omega / \int_M \omega$ for a generic top degree form ω. By its homotopy invariance, $\deg(f) = 1$ if $\mathrm{Far}(f) = \emptyset$, so we expect to compute $\deg(f) - 1$ in any neighborhood of $\mathrm{Far}(f)$. Indeed, this follows from

$$\begin{aligned}\mathrm{vol}(M) \cdot \deg(f) &= \lim_{t\to 0} \int_M (f_t)^*\mathrm{dvol} = \lim_{\varepsilon\to 0} \left[\int_{M\setminus \bar{B}_\varepsilon} \mathrm{dvol} + \lim_{t\to 0} \int_{\bar{B}_\varepsilon} (f_t)^*\mathrm{dvol} \right] \\ &= \mathrm{vol}(M) - \int_{\bar{B}_\varepsilon} \mathrm{dvol} + \lim_{t\to 0} \int_{\bar{B}_\varepsilon} (f_t)^*\mathrm{dvol}\,.\end{aligned}$$

We also have $\deg(f) = \langle f^*[M], [M]\rangle$, using $[M]$ to denote fundamental classes in both homology and cohomology. As in the Theorem, this can be rewritten as $\deg(f) - 1 = \langle f^* i_1^*[M], r_1[M]\rangle$.

(2) Instead of comparing f to the identity map, we can compare f to a constant map $c(x) = x_0$. In this case the Lefschetz number (resp. degree) of f is 1 (resp. 0) if the graph of f misses $\mathcal{C}_{x_0}$. Again there are integrals supported near $\mathcal{C}_{x_0}$ which measure $L(f) - 1$ and $\deg(f)$, and a corresponding topological formula. f can also be compared to a fixed map f_0. We obtain that $L(f) - L(f_0)$ can be computed by an integral supported near $\{x : f_0(x) \in \mathcal{C}_{f(x)}\}$. There is a similar result for degrees. As a well known example, if $M = S^n$ and f, f_0 have different Lefschetz numbers (equiv. different degrees), then there exists $x \in S^n$ such that $f(x)$, $f_0(x)$ are antipodal.

(3) Let B_ε be the ε-neighborhood of the zero set of a vector field s on M. Then by the uniform decay of $(ts)^*\bar{\mathrm{T}}_{TM}$ off B_ε,

$$\chi(M) = \lim_{\varepsilon\to 0} \lim_{t\to\infty} \int_{B_\varepsilon} (ts)^*\bar{\mathrm{T}}_{TM}\,.$$

Topologically, we get $\chi(M) = \langle i^*e, r_1[M]\rangle$, where $i : B_\varepsilon \to M$ is the inclusion, and e is the Euler class of M. If the zero set consists of nondegenerate points $x_1, \cdots, x_n$, $\chi(M) = \sum_i \pm\delta_{x_i}$, as in [11, §8]. The sign is determined by the Hopf index at x_i, which recovers the Hopf index formula.

(4) If we let $t \to \infty$ in (5), the integrand becomes concentrated on the fixed point set of f. If the fixed point set is a union of submanifolds, we can with some work recover the Lefschetz fixed submanifold formula [5]; cf. §4.3.

4. Integral geometric formulas and Lefschetz number estimates

In this section we use a geometric representative for the Thom class due to Mathai-Quillen [11] to give a geometric formula for $L(f)$ generalizing the Chern-Gauss-Bonnet theorem. (In fact, Chern's original proof of CGB contains such a geometric representative. For other versions, see [1], [2], [9].) We first recall this construction, following [1], [11]. We give an explicit formula for $L(f)$ for flat manifolds, a formula for general metrics, and specialize to an explicit formula for constant curvature surfaces. Finally, we use Hodge theory techniques to estimate $L(f)$ in terms of geometric data.

4.1. Geometric Thom forms. We adapt the notation of [1], [11] for a geometric expression for the Thom class of an oriented rank n vector bundle $\pi : E \to M$ over a manifold M, where E has an inner product and a compatible connection ∇. Let x be the tautological section of the bundle π^*E over E; in a local orthonormal frame $\{e_i\}$ of E, $x(x^i e_i) = x^i e_i$. Let Ω be the curvature of ∇; we abuse notation by writing ∇, Ω for $\pi^*\nabla$, $\pi^*\Omega$, the induced connection and its curvature on π^*E. Let $\mathcal{B} : \Lambda^* V \to V$ denote the Berezin integral on the exterior algebra of an n-dimensional vector space V, where $\mathcal{B}(\alpha_I v^I) = (-1)^{n(n+1)/2}\alpha_{\{1,2,\cdots,n\}}$ picks out the coefficient of the top component of a "form" of mixed degree. $\mathcal{B}$ extends to $\mathcal{B} : \Lambda^*(E, \Lambda^*\pi^*E)$ by $\mathcal{B}(\alpha \otimes s) = \alpha \cdot \mathcal{B}(s)$.

A representative $\bar{\mathrm{T}} \in H^n_c(E, \mathbb{Z})$ of the Thom class of E is given by

$$\bar{\mathrm{T}} = \pi^{-\frac{n}{2}} \mathcal{B}\left(\exp\left[-\left(|x|^2 + \nabla x + \frac{\Omega}{2}\right)\right]\right) \tag{6}$$

([1, Ch. 1.6], with constants adjusted to agree with [11]). In local coordinates,

$$\bar{\mathrm{T}} = \pi^{-\frac{n}{2}} e^{-|x|^2} \sum_{I, |I| \text{ even}} \varepsilon(I, I')\mathrm{Pf}\left(\frac{1}{2}\Omega_I\right)(\nabla x)^{I'}, \tag{7}$$

where: $x = \sum (x^i)^2$ is an orthornormal fiber coordinate; Ω_I is the submatrix of Ω with respect to the multi-index I with entries in $\{1, 2, \cdots, n\}$; $\mathrm{Pf}(\frac{1}{2}\Omega_I)$ is the Pfaffian of $\frac{1}{2}\Omega_I$; I' denotes the complement of I in $\{1, 2, \cdots, n\}$; $\varepsilon(I, I')$ is the sign of $I \cup I'$ as a permutation, and $(\nabla x)^{I'} = \nabla x^{i_1} \wedge \cdots \wedge \nabla x^{i_q}$, for $I' = \{i_1, i_2, \cdots, i_q\}$. For computations at a point $x \in M$, we may assume that $\{x^i\}$ is a synchronous frame centered at x, in which case $(\nabla x)^I = dx^I$.

This geometric formula contains the Chern-Gauss-Bonnet theorem. By Proposition 2.3, we have

$$L(f) = (-1)^{\dim M} \int_M (\mathrm{Id}, f)^* \bar{T}_\Delta\,, \tag{8}$$

where $\bar{\mathrm{T}}_\Delta$ is the geometric representative of η_Δ, the Thom class of the normal bundle ν_Δ, thought of as a form on $M \times M$, as in §2.2. Thus

$$\chi(M) = L(\mathrm{Id}) = \int_M (\mathrm{Id}, \mathrm{Id})^* \bar{\mathrm{T}}_\Delta = \int_M 0^* \bar{\mathrm{T}}_{TM}\,,$$

since a neighborhood of the zero section in TM is isomorphic to a tubular neighborhood of Δ under an isomorphism taking the zero section 0 to the graph of the identity. For the Levi-Civita connection, we have $0^*\bar{\mathrm{T}}_{TM} = \pi^{-n/2}\mathrm{Pf}(\frac{1}{2}\Omega)$, since $x = 0$ on M implies $0^*(\nabla x)^{I'} = 0$ if

$I' \neq \emptyset$. Thus we obtain the Chern-Gauss-Bonnet theorem

$$\chi(M) = \frac{1}{(2\pi)^{n/2}} \int_M \mathrm{Pf}(\Omega)\,.$$

Since we get CGB for the simplest case $f = \mathrm{Id}$, we expect interesting formulas for more general f. This is the subject of the next two subsections.

4.2. Local expressions for flat manifolds. To calculate explicit expressions for the Lefschetz number from (8), we have to (i) compute $\bar{\mathrm{T}}_\Delta$ in local coordinates, and (ii) compute the action of $(\mathrm{Id}, f)^*$.

Step (i) consists of finding an explicit diffeomorphism $\alpha : \Delta_\varepsilon \to \nu_\Delta$ between an ε-neighborhood of the diagonal and the normal bundle, as $\bar{\mathrm{T}}_\Delta$ in (8) equals $\bar{\mathrm{T}}_{\Delta_\varepsilon} = \alpha^*\bar{\mathrm{T}}_{\nu_\Delta}$, where $\bar{\mathrm{T}}_{\nu_\Delta}$ is the geometric representative of the Thom class of the normal bundle ν_Δ. Even though the exponential map is trivial near the diagonal for flat metrics, we need to compute α carefully for the case of general metrics in the next section.

So fix $(x, y) \in \Delta_\varepsilon$. Since the normal bundle consists of vectors of the form $(-v, v)$, there exists $(\bar{x}, \bar{x}) \in \Delta$ such that

$$(x, y) = \exp_{(\bar{x},\bar{x})}(-v, v) = (\exp_{\bar{x}}(-v), \exp_{\bar{x}} v)\,.$$

Thus $\bar{x}$ is the midpoint of the geodesic $\exp_{\bar{x}}(tv)$, $t \in [-1, 1]$ from x to y. This gives a diffeomorphism $\eta : U \to \Delta_\varepsilon$ for U the ε-neighborhood of the zero section in ν_Δ:

$$\eta(v, -v)_{(\bar{x},\bar{x})} = (\exp_{\bar{x}}(v), \exp_{\bar{x}}(-v)). \tag{9}$$

Let $\rho : [0, \varepsilon) \to [0, \infty)$ be a fixed diffeomorphism with

$$\rho^{(k)}(0) = 0,\ \lim_{z\to\varepsilon} e^{-\rho^2(z)}\rho^{(k)}(z)\rho(z)^n = 0 \tag{10}$$

for all k, n. Set $\rho(z) = \infty$ for $z \geq \varepsilon$.

In the product metric, $d((x, y), (\bar{x}, \bar{x})) = d(x, y)/\sqrt{2}$, so $\Delta_\varepsilon = \{(x, y) \in M \times M : d(x, y) < \sqrt{2}\varepsilon.\}$. Thus $\beta : U \to \nu_\Delta$ given by

$$\beta(v, -v)_{(\bar{x},\bar{x})} = \begin{cases} \left(\rho\left(\dfrac{d(x, y)}{\sqrt{2}}\right)\dfrac{v}{|v|}, -\rho\left(\dfrac{d(x, y)}{\sqrt{2}}\right)\dfrac{v}{|v|}\right), & v \neq 0\,, \\ (\bar{x}, \bar{x})\,, & v = 0\,, \end{cases}$$

is a diffeomorphism, and $\alpha = \beta \circ \eta^{-1} : \Delta_\varepsilon \to \nu_\Delta$ is our desired diffeomorphism. In the flat case, α reduces to $\beta : \mathbf{R}^n \to \mathbf{R}^n$, $v \mapsto \rho(|v|)\frac{v}{|v|}$ for $v \neq 0$, and $\bar{\mathrm{T}}_\Delta = \beta^*\bar{\mathrm{T}}_{\nu_\Delta}$.

On a flat manifold, there exists a local orthornomal frame for which the connection and the curvature forms vanish. The geometric Thom form for the normal bundle to the diagonal is just

$$\bar{\mathrm{T}}_{\nu_\Delta} = \pi^{-n/2}e^{-|x|^2}dx^1 \wedge \cdots \wedge dx^n\,. \tag{11}$$

Thus computing $\beta^*\bar{\mathrm{T}}_{\nu_\Delta}$ reduces to computing β^*dvol, which is easiest in polar coordinates. A short calculation gives $\beta_{*v}(\partial_r) = \rho'(|v|)\partial_r,\ \beta_{*v}(\partial_{\theta^i}) = [\rho(|v|)/|v|]\partial_{\theta^i}$. Therefore $(\beta^* dr)_v = \rho'(|v|)dr,\ \beta^* d\theta^i = d\theta^i$. Thus

$$\begin{aligned}\bar{\mathrm{T}}_{\Delta_\varepsilon} &= \pi^{-n/2} e^{-\rho^2(|v|)} \rho(|v|)^{n-1} \rho'(|v|) \left(\frac{\rho(|v|)}{|v|}\right)^{n-1} dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \\ &= \pi^{-n/2} e^{-\rho^2\left(\frac{d(x,y)}{\sqrt{2}}\right)} \rho'\left(\frac{d(x,y)}{\sqrt{2}}\right) \left(\frac{\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)}{\frac{d(x,y)}{\sqrt{2}}}\right)^{n-1} \mathrm{dvol}\,.\end{aligned}$$

This completes Step (i).

For Step (ii), we calculate

$$\begin{aligned}(\mathrm{Id}, f)^* \bar{\mathrm{T}}_{\Delta\varepsilon} &= \pi^{-n/2} e^{-\rho^2\left(\frac{d(x,f(x))}{\sqrt{2}}\right)} \rho'\left(\frac{d(x,f(x))}{\sqrt{2}}\right) \\ &\quad \times \left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}{\frac{d(x,f(x))}{\sqrt{2}}}\right)^{n-1} (\mathrm{Id}, f)^*(\mathrm{dvol}_{\alpha(x,y)})\,.\end{aligned} \tag{12}$$

Here $\mathrm{dvol}_{\alpha(x,y)}$ is the volume element on the normal bundle, considered as a form near the diagonal.

Since $(\mathrm{Id}, f)^*\bar{\mathrm{T}}_{\Delta_\varepsilon}$ vanishes if $(x, f(x))$ is not in the tubular neighborhood, we may assume there is a unique minimal geodesic from x to $y = f(x)$. Let (x^i) be flat coordinates near x, and let (y^i) be flat coordinates at y given by parallel translating the ∂_{x^i} along the geodesic. Then

$$\mathrm{dvol}_{\alpha(x,y)} = \bigwedge_{i=1}^{n} \left(\frac{-dx^i + dy^i}{\sqrt{2}}\right),$$

since $\nu_{(\bar{x},\bar{x})} = \{(-v, v) : v \in T_{\bar{x}}M\}$. Write $f = (f^1, \cdots, f^n)$ in the $(x^i),\ (y^i)$ coordinates. Then

$$\begin{aligned}(\mathrm{Id}, f)^* \mathrm{dvol}_{\alpha(x,y)} &= (\mathrm{Id}, f)^* \bigwedge_{i=1}^{n} \left(\frac{-dx^i + dy^i}{\sqrt{2}}\right) = 2^{-n/2} \bigwedge_{i=1}^{n} (-dx^i + df^i) \\ &= 2^{-n/2} \bigwedge_{i=1}^{n} \left(-dx^i + \frac{\partial f^i}{\partial x^j} dx^j\right).\end{aligned}$$

The transformation $df \circ \| - \mathrm{Id}$, where $\|$ denotes parallel translation from $f(x)$ to x along their geodesic, has entries

$$a_{ij} = \begin{cases} (\partial f_i/\partial x^i) - 1\,, & i = j\,, \\ \partial f_i/\partial x^j\,, & i \neq j\,, \end{cases}$$

so

$$(\mathrm{Id}, f)^*\mathrm{dvol}_{\alpha(x,y)} = 2^{-n/2}\bigwedge_{i=1}^{n}\sum_{j=1}^{n} a_{ij}dx^j = (-1)^n 2^{-n/2}\det(\mathrm{Id} - df \circ \|)\mathrm{dvol}_M . \qquad (13)$$

Theorem 2.3 and (12), (13) yield:

THEOREM 4.1. *Let $f : M \to M$ be a smooth map of a closed, oriented, flat n-manifold M. Pick $\varepsilon > 0$ sufficiently small, and let $\rho : [0, \varepsilon) \to [0, \infty)$ be a diffeomorphism satisfying* (10). *Then the Lefschetz number of f is given by*

$$L(f) = \frac{1}{(2\pi)^{n/2}}\int_M e^{-\rho^2\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}\rho'\left(\frac{d(x,f(x))}{\sqrt{2}}\right)\left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}{\frac{d(x,f(x))}{\sqrt{2}}}\right)^{n-1}$$
$$\cdot \det\left(\mathrm{Id} - df \circ \|\right)\mathrm{dvol}_M .$$

For f the identity map, $\det(\mathrm{Id} - df \circ \|)$ vanishes and the theorem gives $\chi(M) = L(\mathrm{Id}) = 0$ as expected. Of course, the $\sqrt{2}$ factor in the integrand can be incorporated into the diffeomorphism ρ.

EXAMPLE. Let $f : S^1 \to S^1$ be given by $f(z) = z^n$, so $L(f) = 1 - n$.
For $\varepsilon = \pi/(2\sqrt{2})$, let $\alpha : \Delta_\varepsilon \to \nu_\Delta$ be the diffeomorphism

$$\alpha(\theta_1, \theta_2) = \left(\frac{\theta_1 + \theta_2}{\sqrt{2}}, \rho\left(\frac{\theta_1 - \theta_2}{\sqrt{2}}\right)\right),$$

where (θ_1, θ_2) are the coordinates on $S^1 \times S^1$ and $\rho : \left(-\frac{\pi}{2\sqrt{2}}, \frac{\pi}{2\sqrt{2}}\right) \to (-\infty, \infty)$ is an orientation preserving diffeomorphism given by a fixed odd function ρ. Note that $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}dx = 1$ implies

$$\frac{1}{\sqrt{\pi}}\int_{-\frac{\pi}{2\sqrt{2}}}^{\frac{\pi}{2\sqrt{2}}} \rho'(\theta)e^{-\rho^2(\theta)}d\theta = 1 , \quad \int_0^{\frac{\pi}{2\sqrt{2}}} \rho'(\theta)e^{-\rho^2(\theta)}d\theta = \frac{\sqrt{\pi}}{2} . \qquad (14)$$

The graph of f, drawn on $[0, 2\pi] \times [0, 2\pi]$, consists of n line segments $\theta_2 = n\theta_1 - 2(k-1)\pi,\ k = 1, 2, \cdots, n$. Since the boundary of Δ_ε is given by $\theta_2 = \theta_1 \pm \frac{\pi}{2}$, Γ is in Δ_ε iff

$$\frac{(4k-5)\pi}{2(n-1)} \le \theta_1 \le \frac{(4k-3)\pi}{2(n-1)} ,$$

for $k = 2, \cdots, n-1$, or

$$0 \le \theta_1 \le \frac{\pi}{2(n-1)} , \quad \frac{(4n-5)\pi}{2(n-1)} \le \theta_1 \le 2\pi ,$$

for the first and last segment, respectively. Since df is multiplication by n, Theorem 4.1 gives

$$L(f) = -\frac{1}{\sqrt{\pi}}\int_0^{\frac{\pi}{2(n-1)}} e^{-\rho^2\left(\frac{(n-1)\theta}{\sqrt{2}}\right)}\rho'\left(\frac{(n-1)\theta}{\sqrt{2}}\right)d\theta$$

$$
\begin{aligned}
&-\sum_{k=2}^{n-1}\left[\frac{1}{\sqrt{\pi}}\int_{\frac{(4k-5)\pi}{2(n-1)}}^{\frac{(4k-3)\pi}{2(n-1)}}\left(\frac{n-1}{\sqrt{2}}\right)e^{-\rho^2\left(\frac{(n-1)\theta}{\sqrt{2}}-(k-1)\pi\sqrt{2}\right)}\right.\\
&\left.\times\,\rho'\left(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}}\right)d\theta\right]\\
&-\frac{1}{\sqrt{\pi}}\int_{\frac{(4n-5)\pi}{2(n-1)}}^{2\pi}\left(\frac{n-1}{\sqrt{2}}\right)e^{-\rho^2\left(\frac{(n-1)(\theta-2\pi)}{\sqrt{2}}\right)}\rho'\left(\frac{(n-1)(\theta-2\pi)}{\sqrt{2}}\right)d\theta\,.
\end{aligned}
$$

Under the change of variables $\lambda=[(n-1)\theta/\sqrt{2}]-(k-1)\pi\sqrt{2},\ k=1,\cdots,n$, the first and last integrals become $1/2$, and the integrals inside the sum become 1 by (14). Thus

$$
L(f)=-\frac{1}{2}-\sum_{k=2}^{n-1}1-\frac{1}{2}=1-n\,.
$$

Theorem 4.1 can also be used to estimate $L(f)$ for flat manifolds.

PROPOSITION 4.2. *Let $f: M\to M$ be a smooth map of a closed, oriented, flat n-manifold M. Then*

$$
|L(f)|\le\frac{C}{(2\pi)^{n/2}}(\|df\|+1)^n
$$

for some constant C independent of n.

PROOF. By (10), there exists $C'>0$ such that $0\le e^{-\rho^2(z)}\rho'(z)(\rho(z)/z)^{n-1}\le C'$. For $v\in T_{f(x)}M$,

$$
|(\mathrm{Id}-df\circ\|)(v)|\le(\|df\|+1)|v|\,,
$$

since parallel translation in an isometry. Thus $|\det(\mathrm{Id}-df\circ\|)|\le(\|df\|+1)^n$. By Theorem 4.1, we have

$$
|L(f)|\le\frac{C'}{(2\pi)^{n/2}}\mathrm{vol}(M)(\|df\|+1)^n.
$$

Under a scaling $g\mapsto\lambda g$ of the metric, we send $\rho(z)\mapsto\rho(\lambda^{-1/2}z)$, so that this new diffeomorphism is still defined in the $\lambda^{1/2}\varepsilon$ neighborhood of the diagonal. $e^{-\rho^2(z)}\rho(z)^{n-1}$ stays bounded, ρ' scales by $\lambda^{-1/2}$, $z^{1-n}=(d(x,f(x))/\sqrt{2})^{1-n}$ scales by $\lambda^{(1-n)/2}$, and dvol scales by $\lambda^{n/2}$. Thus $C=C'\cdot\mathrm{vol}(M)$ is bounded above by a constant independent of $\dim(M)$. □

§4.4 contains a similar result for arbitrary manifolds via Hodge theory.

4.3. Local expressions for arbitrary metrics. In this subsection we calculate the local expression for the integrand in (8) for an arbitrary Riemanninan metric. As in §4.2, the main difficulty is in Step (i). Unlike the flat case, the exponential map is nontrivial. We will first calculate the local expression using (7) and then express it more invariantly using (6).

Let $\rho : [0, \varepsilon) \to [0, \infty)$ be a diffeomorphism as in §4.2. Let $\bar{\mathrm{T}}_\nu$ be the geometric Thom form of the normal bundle $\nu = \nu_\Delta$. We want to calculate $(\mathrm{Id}, f)^*(\exp^{-1})^*\rho^*\bar{\mathrm{T}}_\nu$. Here $\exp : \nu_\Delta \to M \times M$. By (7), this splits into terms involving ∇x and terms involving $\mathrm{Pf}(\Omega_I)$.

Note that $\rho^*\mathrm{Pf}(\Omega_I/2) = \mathrm{Pf}(\Omega_I/2)$ in (7), since Ω is really $\pi^*\Omega$, a horizontal form on the total space of $E = TM$; $\rho^* = \mathrm{Id}$ on horizontal forms. Since the ∇x terms are vertical, as in §4.1

$$(\mathrm{Id}, f)^*(\exp^{-1})^*\rho^*(\nabla x)^{I'} = \det\left(\frac{1}{2}(-\mathrm{Id} + df \circ \|)\right)^{I'} (\exp^{-1})^*\rho^*(\nabla x)^{I'},$$

where the I'-minor matrix of $-\mathrm{Id} + \|df$ is computed with respect to orthonormal coordinates at x. Similarly,

$$(\mathrm{Id}, f)^*(\exp^{-1})^*\rho^*\mathrm{Pf}(\Omega_I) = \det\left(\frac{1}{2}(\mathrm{Id} + df \circ \|df)\right)^{I} (\exp^{-1})^*\mathrm{Pf}(\Omega_I),$$

since under the isomorphism $v \mapsto (-v, v)$ of TM and ν_Δ, horizontal vectors X map to (X, X).

Let $(-\mathrm{Id} + df \circ \|)\nabla x$ denote $\sum_i (-\mathrm{Id} + df \circ \|)^i \nabla x^i e_i$, and let $(\mathrm{Id} + df \circ \|)\Omega$ denote $\sum_{i,j} (\mathrm{Id} + df \circ \|)^i \Omega^i_j e^i \wedge e^j$. Then by (6), (7), we have

THEOREM 4.3. *The Lefschetz number of f is given by*

$$\begin{aligned} L(f) = (-1)^{\dim M} \int_M \mathcal{B}\Bigg(\exp\Bigg[-\Bigg(& -\rho^2(d(x, f(x))/\sqrt{2}) \\ & + \frac{1}{2}(-\mathrm{Id} + df \circ \|)(\exp^{-1})^*\rho^*\nabla x \\ & + \frac{1}{2}(\mathrm{Id} + df \circ \|)(\exp^{-1})^*\Omega \Bigg)\Bigg]\Bigg). \end{aligned}$$

It is easy to check that the theorem reduces to the Chern-Gauss-Bonnet theorem and Theorem 4.1 for $f = \mathrm{Id}$ and for flat manifolds, respectively.

EXAMPLE. We illustrate the theorem by calculating the integrand for an oriented surface of constant Gaussian curvature -1.

For x, $f(x)$ close, let $\gamma(t)$ be the minimal unit speed geodesic from x to $f(x)$, with midpoint $\bar{x}$. Recall from §4.1 that $v = \frac{1}{2}d(x, f(x))\dot{\gamma}(\bar{x}) \in T_{\bar{x}}M$ has $(x, f(x)) = (\exp_{\bar{x}}(-v), \exp_{\bar{x}}(v))$.

The only terms in the integrand involve either $(\exp^{-1})^*dA$, a pullback of the area form (since in a synchronous frame $\nabla x^i = dx^i$), or $(\exp^{-1})^*\mathrm{Pf}(\Omega)$. (Although it is tempting to write $\mathrm{Pf}(\Omega) = -dA$, we have to distinguish between the area forms for vertical and horizontal coordinates.) So we need the Jacobians of the exponential map restricted to vertical and horizontal vectors.

The differentials of the exponential map for a vertical or horizontal vector $Y \in T_v T_{\bar{x}} M$ are given respectively by

$$X = d(\exp_{\bar{x}})_v(Y) = \frac{d}{dt}\Big|_{t=d/2} \exp_{\bar{x}}(v + tY)$$

$$\tilde{X} = d(\exp_{\cdot} \|v)(Y) = \frac{d}{dt}\Big|_{t=d/2} \exp_{\exp_{\bar{x}} tY} \|v\,,$$

where $\|$ denotes parallel translation along the curve $\exp_{\bar{x}} tY$ and $d = d(x, f(x))$. X is the endpoint of a Jacobi field J, the variation vector field of the family of geodesics $\gamma_t(s) = \exp_{\bar{x}}(s(v + tY))$, which has $(\nabla J)(d/2) = Y$. Similarly, $\tilde{X}$ is the endpoint of a Jacobi field J with $J(d/2) = Y$ – i.e. J is the variation vector field of the family of geodesics $\gamma_s(t) = \exp_{\eta(s)}(t\|v)$, where $\dot{\eta}(d/2) = Y$ and $t \in [0, 1]$ (cf. [6, Cor. 3.46]).

Thus to find the Jacobian of the exponential map on vertical or horizontal vectors, we have to solve the Jacobi equation with either vanishing initial velocity vector or vanishing initial position, evaluate at $d/2$, and compute the determinant. (As usual, the determinant is computed with respect to e.g. an orthonormal frame of $T_{\bar{x}}M$, considered both as a frame of vertical/horizontal vectors at $T_v T_{\bar{x}} M$ and by parallel translation at $T_x M$.)

The calculation is trivial at a fixed point. Otherwise, let J be a Jacobi field along γ. Plugging $J(t) = a(t)\dot{\gamma} + b(t)\alpha$ into the Jacobi equation $D^2 J/dt^2 + R(\dot{\gamma}, J)\dot{\gamma} = 0$ and using $\langle R(\dot{\gamma}, J)\dot{\gamma}, \dot{\gamma}\rangle = 0$, $\langle R(\dot{\gamma}, \alpha)\dot{\gamma}, \alpha\rangle = -1$ yields $\ddot{a} = 0$, $\ddot{b} - b = 0$. Thus $J(t) = (c_0 + c_1 t)\dot{\gamma} + (d_1 \sinh(t) + d_2 \cosh(t))\alpha$. Imposing the initial conditions easily gives $\det\, d\exp^{-1} = \frac{d}{\sinh(d/2)}$ on vertical vectors, and $\det\, d\exp^{-1} = \frac{2}{\cosh(d/2)}$ on horizontal vectors. This yields:

COROLLARY 4.4. *Let M be an oriented surface of constant curvature -1. Then*

$$L(f) = \frac{1}{2\pi}\int_M e^{-\rho^2\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}\Bigg[\frac{-\det(\frac{1}{2}(df_x \circ \| + \mathrm{Id}))}{\cosh\left(\frac{d(x,f(x))}{2}\right)}$$
$$+ \rho'\left(\frac{d(x, f(x))}{\sqrt{2}}\right)\left(\frac{\rho(d(x, f(x))/\sqrt{2})}{\sqrt{2}\sinh\left(\frac{d(x,f(x))}{2}\right)}\right)\det\left(\frac{1}{2}(df_x \circ \| - \mathrm{Id})\right)\Bigg]dA\,.$$

For constant curvature 1, just replace sinh by sin and change the sign in the first factor in the integrand. It is neither difficult nor particularly insightful to extend this result to higher dimensional constant curvature spaces.

REMARK. We sketch a geometric proof of the Lefschetz fixed submanifold formula based on Theorem 4.3. Assume that the metric on M is a product near a fixed point submanifold N. If the submanifold is given by $\{x^{k+1} = \cdots = x^n = 0\}$ in local coordinates, then as $t \to \infty$, the integrand for $L(f) = L(f_t)$ concentrates on a tubular neighborhood of the fixed submanifold, and the only contribution to the integrand comes from $I = \{1, \cdots, k\}$, since the curvature term vanishes otherwise due to the product metric. Converting back to rectangular coordinates in the normal fiber eliminates the ρ factor and introduces a factor of sgn det$(df_\nu - \mathrm{Id})$. Since $f = \mathrm{Id}$ in submanifold directions, $\det(\frac{1}{2}(d(f_t) + \mathrm{Id})) = 1$ in these directions. Thus the integral splits into the curvature integral over N, yielding $\chi(N)$, and a normal integral, which gives sgn det$(d(f_t)_\nu - \mathrm{Id})$. Plugging these terms into Theorem 4.3 gives the Lefschetz fixed submanifold formula $L(f) = \sum_i \chi(N_i)\mathrm{sgn}\ \det(df_\nu - \mathrm{Id})$, where the fixed point set is the union of submanifolds $\{N_i\}$. This involves less analysis than the heat equation proof [7, Ch. 4].

4.4. Hodge theory and Lefschetz number estimates. The upper bound in Proposition 4.2 for the absolute value of the Lefschetz number of a flat manifold can be extended to arbitrary metrics. Using sectional curvature bounds to control the Jacobi fields and the curvature tensor, one can extract an upper bound from Theorem 4.3 in terms of the sectional curvature. In contrast, there is a Hodge theory argument which constructs better two-sided bounds in terms of Ricci curvature.

Let $\mathbf{N} = \mathbf{N}(n, C, D, V)$ be the class of Riemannian n-manifolds (M, g) with Ricci curvature Ric $\geq C$, $\mathrm{diam}(M) \leq D$ and $\mathrm{vol}(M) \geq V$.

THEOREM 4.5. *There exists a positive constant $D = D(\mathbf{N})$ such that for all $(M, g) \in$ $\mathbf{N}$,*

$$-D \sum_{k \text{ odd}} \beta_k \cdot \sup_{x\in M} |df_x|_\infty^k \leq L(f) \leq 1 + D \sum_{k>0,\text{even}} \beta_k \cdot \sup_{x\in M} |df_x|_\infty^k ,$$

where β_k is the k^{th} Betti number of M.

Before the proof, we compare two norms for differential forms. For $\alpha \in \Lambda^k T_x^* M$, we have the L^2 (Hodge) norm $|\alpha|_2^2 = *(\alpha \wedge *\alpha)$ and the L^∞ norm

$$|\alpha|_\infty = \sup_{v\in (T_x M)^{\otimes k}\setminus\{0\}} \frac{|\alpha(v)|}{|v|} ,$$

where $v_1 \otimes \cdots \otimes v_k$ has norm $\prod |v_i|$. Here we consider α as a linear functional on $(T_x M)^{\otimes k}$.

LEMMA 4.6.

$$\binom{n}{k}^{-1/2} |\alpha|_2 \leq |\alpha|_\infty \leq |\alpha|_2 .$$

PROOF. Let $\{\theta^i\}$ be an orthonormal basis of $T_x^* M$ with dual basis $\{X_i\}$ of $T_x M$. For $\alpha = \alpha_I \theta^I$, we have

$$|\alpha|_\infty \geq \frac{|(\alpha_I\theta^I)(X_{i_1} \otimes \cdots \otimes X_{i_k})|}{|X_{i_1} \otimes \cdots \otimes X_{i_k}|} = |\alpha_{I_0}| ,$$

where $I_0 = (i_1, \cdots, i_k)$. Thus

$$|\alpha|_\infty \geq \sup_I |\alpha_I| \geq \binom{n}{k}^{-1/2} \left(\sum_I |\alpha_I|^2\right)^{1/2} = \binom{n}{k}^{-1/2} |\alpha|_2 .$$

For the other estimate, for $X = \beta^J X_J$,

$$|\alpha|_\infty^2 = \sup_{X\neq 0} \frac{|\alpha_I \theta^I(\beta^J X_J)|^2}{\sum |\beta^J|^2} = \sup_{X\neq 0} \frac{(\sum |\alpha_I \beta^I|)^2}{\sum |\beta^I|^2} \le \sup_{X\neq 0} \frac{\sum |\alpha^I|^2 \sum |\beta^I|^2}{\sum |\beta^I|^2} = |\alpha|_2^2 .$$

□

PROOF OF THE THEOREM. Let $\{\omega_k^i\}$ be an L^2-orthonormal basis of harmonic k-forms. The trace of $f^* : H^k(M;\mathbf{R}) \to H^k(M;\mathbf{R})$ is $\operatorname{tr} f^k = \sum_i \langle f^*\omega_k^i, \omega_k^i\rangle$, so by Cauchy-Schwarz

$$|\operatorname{tr} f^k| \le \sum_i |\langle f^*\omega_k^i, \omega_k^i\rangle| \le \sum_i \|f^*\omega_k^i\| = \sum_i \left[\int_M |(f^*\omega_k^i)_x|_2^2 \mathrm{dvol}(x)\right]^{1/2} . \tag{15}$$

Here $\|\alpha\|^2 = \int_M \alpha \wedge *\alpha$ is the global L^2 norm. When $k = 0$, we have $\|f^*\omega_0^1\| = \|\omega_0^1\| = 1$, so we assume $k \neq 0$ from now on.

By (15) and the lemma, we have

$$|\operatorname{tr} f^k| \le \binom{n}{k} \sum_i \mathrm{vol}^{1/2}(M) \sup_{x\in M} |(f^*\omega_k^i)_x|_\infty . \tag{16}$$

For $X = \beta^J X_J$ and $f_* = df$, we have

$$|(f^*\omega)_x|_\infty = \sup_{X\neq 0} \frac{|(f^*\omega)_x(\beta^J X_J)|}{|\beta_J X^J|} = \sup_{X\neq 0} \frac{|\omega_{f(x)}(\beta^J f_* X_{j_1} \otimes \cdots \otimes f_* X_{j_k})|}{|\beta^J X_J|} .$$

Since the next estimate only improves if $f_* X_j = 0$ for some j, assume $f_* X_j \neq 0$. Then

$$\begin{aligned} |(f^*\omega)_x|_\infty &= \sup_{X\neq 0} \frac{|\omega_{f(x)}(\beta^J f_* X_J)|}{|\beta^J f_* X_J|} \cdot \frac{|\beta^J f_* X_J|}{|\beta^J X_J|} \le |\omega_{f(x)}|_\infty \cdot \sup_{X\neq 0} \frac{|\beta^J| \prod_k |df_x|_\infty \cdot |X_{j_k}|}{(\sum |\beta^J|^2)^{1/2}} \\ &\le |\omega_{f(x)}|_\infty \cdot |df_x|_\infty^k . \end{aligned}$$

By (16) and the lemma, we get

$$\begin{aligned} |\operatorname{tr} f^k| &\le \binom{n}{k} \mathrm{vol}^{1/2}(M) \sum_i \sup_{x\in M} |df_x|_\infty^k \cdot |(\omega_k^i)_{f(x)}|_\infty . \\ &\le \binom{n}{k} \mathrm{vol}^{1/2}(M) \sum_i \sup_{x\in M} |df_x|_\infty^k \cdot |(\omega_k^i)_{f(x)}|_2 . \end{aligned}$$

By [4], [10], there is an explicit constant $D_1(\mathbf{N})$ such that for all $x \in M$,

$$|(\omega_k^i)_x|_2 \le D_1(\mathbf{N}) \|\omega_k^i\| = D_1(\mathbf{N}) .$$

Thus

$$|\operatorname{tr} f^k| \le \binom{n}{k} \beta_k \cdot \mathrm{vol}^{1/2}(M) \cdot D_1(\mathbf{N}) \sup_{x\in M} |df_x|_\infty^k .$$

Finally, vol(M) is bounded above on $\mathbf{N}$ by standard comparison theorems. □

References

[1] N. BERLINE, E. GETZLER and M. VERGNE, *Heat Kernels and Dirac Operators*, Springer (1992).

[2] J. M. BISMUT and W. ZHANG, *An Extension of a Theorem of Cheeger and Müller*, Astérisque **205** (1992) Soc. Math. France.

[3] R. BOTT and L. TU, *Differential Forms in Algebraic Topology*, Springer (1982).

[4] C. CROKE, Some isoperimetric inequalities and eigenvalue estimates, Ann. Sci. Ec. Norm. Super. **13** (1980), 419–435.

[5] M. FRUMOSU, Mathai-Quillen formalism and Lefschetz theory, Ph.D. thesis, Boston University (1997).

[6] S. GALLOT, D. HULIN and J. LAFONTAINE, *Riemannian Geometry*, Springer (1987).

[7] P. B. GILKEY, *Invariance Theory*, *the Heat Equation*, *and the Atiyah-Singer Index Theorem*, Publish or Perish (1984).

[8] V. GUILLEMIN and A. POLLACK, *Differential Topology*, Prentice-Hall (1974).

[9] F. B. HARVEY and H. B. LAWSON, Jr., A theory of characteristic currents associated with a singular connection, Astérisque **213** (1993), 1–268.

[10] P. LI, On the Sobolev constant and the p-spectrum of a compact Riemannian manifold, Ann. Sci. Ec. Norm. Super. **13** (1980), 451–469.

[11] V. MATHAI and D. QUILLEN, Superconnections, Thom classes, and equivariant differential forms, Topology **25** (1986), 85–110.

Present Address:
MIHAIL FRUMOSU
METROPOLITAN COLLEGE, BOSTON UNIVERSITY,
BOSTON, MA 02215, USA.
e-mail: matf@math.bu.edu

STEVEN ROSENBERG
DEPARTMENT OF MATHEMATICS AND STATISTICS,
BOSTON UNIVERSITY,
BOSTON, MA 02215, USA.
e-mail: sr@math.bu.edu