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Deformations of Super-Minimal J-Holomorphic Curves of a 6-Dimensional Sphere

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Dedicated to Professor kouei SEKIGAWA on his sixtieth birthday

Abstract. We shall give a representation formula of super-minimal J-holomorphic curves of a nearly Kähler 6-dimensional sphere and construct a deformation of such J-holomorphic curves.

1. Introduction

In 1982, R. L. Bryant ([Br1]) obtained the construction of super-minimal J-holomorphic curves in the nearly Kähler 6-dimensional sphere S^6. We explain this method (so-called twistor theory) as follows. The exceptional Lie group $\mathbf{G}_2$ coincides with the principal $SU(3)$-bundle over $S^6 = \mathbf{G}_2/SU(3)$, we take the $SU(3)$-connection on this bundle. Let $\pi : Q^5 = \mathbf{G}_2/U(2) \to S^6 = \mathbf{G}_2/SU(3)$ be the associated $P^2(\mathbf{C})$ bundle of the principal $SU(3)$-bundle over S^6, where π denotes the projection. Then we can define the holomorphic horizontal distribution $\mathcal{H}$ on Q^5 with respect to the $SU(3)$-connection over S^6. Next, we define the exterior differential system $\mathcal{L}$ of the holomorphic cotangent bundle $T^{*(1,0)}Q^5$ (which is dual to the subbundle $\mathcal{L}^\natural$ of complex dimension 2 of $\mathcal{H}$ over S^6 on which π_* is complex linear with respect to the canonical complex structure of Q^5 and the almost complex structure of S^6). If we take a holomorphic map Ξ from a Riemann surface M^2 to Q^5 which is an integral curve of $\mathcal{L}$, then we can obtain a J-holomorphic curve $\pi \circ \Xi : M \to S^6$. Such a J-holomorphic curve is called super-minimal (or null torsion). In particular, R. L. Bryant proved that the corresponding differential equation of the integral curve can be reduced to a 1st order linear differential equation of one complex variable. The differential equation always has a solution which can be represented by an arbitrary holomorphic function. Also, this equation relates to the differential system of E. Cartan ([Ca]). R. L. Bryant gave a representation formula of the integral curves of $\mathcal{L}$ in Q^5 with respect to super-minimal J-holomorphic curves of S^6 almost explicitly, but, to calculate the Gauß curvature of such J-holomorphic curves, we need more detailed information. In this paper, we write down the solution of the integral curves of $\mathcal{L}$ in Q^5, more explicitly, in order to calculate the 1st fundamental form and the Gauß curvature.

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In ([H3]), we proved that the 1st fundamental form of super-minimal J-holomorphic curves (without branch points) of S^6 is the only invariant under the left action of $\mathbf{G}_2$ on S^6. As an application, we obtain the 1-parameter family of super-minimal J-holomorphic curves of S^6 which are not $\mathbf{G}_2$-congruent. The examples include a Boruvka sphere whose Gauß curvature is identically $1/6$. These examples will be useful to construct other invariant submanifolds, for example, Lagrangian, CR 3-dimensional manifolds, and so on. (see [Ej], [HM], [DVV]).

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2. Preliminaries

2.1. Notations. We denote by $M_{p\times q}(\mathbf{C})$ the set of $p \times q$ complex matrices and $[a] \in M_{3\times 3}(\mathbf{C})$ is given by

$$[a] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$$

where $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in M_{3\times 1}(\mathbf{C})$. Then we have

$$[a]b + [b]a = 0$$

where $a, b \in M_{3\times 1}(\mathbf{C})$. Let $\langle\, ,\rangle$ be the canonical inner product of $\mathbf{O}$. For any $x \in \mathbf{O}$, we denote by $\bar{x}$ the conjugate of x. We remark that the octonions may be regarded as the direct sum $\mathbf{Q} \oplus \mathbf{Q}$ where $\mathbf{Q}$ is the quaternions.

2.2. Structure equation of $\mathbf{G}_2$. We recall the structure equations of $(\mathrm{Im}\mathbf{O}, \mathbf{G}_2)$ which were established by R. Bryant ([Br1]). The Lie group $\mathbf{G}_2$ is defined by

$$\mathbf{G}_2 = \{g \in GL_8(\mathbf{R}) : \ g(uv) = g(u)g(v) \text{ for any } u,\ v\ \in \mathbf{O}\}\,.$$

Now, we set a basis of $\mathbf{C}\otimes_R \mathrm{Im}\mathbf{O}$ by $\varepsilon = (0, 1) \in \mathbf{Q} \oplus \mathbf{Q}$, $E_1 = iN$, $E_2 = jN$, $E_3 = -kN$, $\overline{E_1} = i\bar{N}$, $\overline{E_2} = j\bar{N}$ and $\overline{E_3} = -k\bar{N}$, where $N = (1 - \sqrt{-1}\varepsilon)/2$, $\bar{N} = (1 + \sqrt{-1}\varepsilon)/2 \in \mathbf{C} \otimes_R \mathbf{O}$ and $\{1, i, j, k\}$ is the canonical basis of $\mathbf{Q}$. A basis $(u,\ f,\ \bar{f})$ of $\mathbf{C}\otimes_R \mathrm{Im}\mathbf{O}$ is said to be admissible, if there exists $g \in \mathbf{G}_2 \subset M_{7\times 7}(\mathbf{C})$ such that

$$(u,\ f,\ \bar{f}) = (g(\varepsilon),\ g(E),\ g(\bar{E})) = (\varepsilon,\ E,\ \bar{E})g\,.$$

We identify the element of $\mathbf{G}_2$ with the corresponding admissible basis. Then we have

PROPOSITION 2.1. *There exist left invariant* 1*-forms κ and θ on* $\mathbf{G}_2$*; $\theta = (\theta^i)$ with values in* $M_{3\times 1}(\mathbf{C})$ *and $\kappa = (\kappa_j{}^i)$,* $1 \le i, j \le 3$, *with values in the* 3×3 *skew Hermitian* *matrices which satisfy* $tr\kappa = 0$, *and*

$$d(u,\ f,\ \bar{f}) = (u,\ f,\ \bar{f})\begin{pmatrix} 0 & -\sqrt{-1}\,{}^t\bar{\theta} & \sqrt{-1}\,{}^t\theta \\ -2\sqrt{-1}\,\theta & \kappa & [\bar{\theta}] \\ 2\sqrt{-1}\,\bar{\theta} & [\theta] & \bar{\kappa} \end{pmatrix}$$

$$= (u,\ f,\ \bar{f})\Phi\,. \tag{2.1}$$

Then Φ *satisfies* $d\Phi = -\Phi \wedge \Phi$, *or equivalently*,

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta}\,. \tag{2.2}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - ({}^t\theta \wedge \bar{\theta})I_3\,. \tag{2.3}$$

3. The exterior differential system $\mathcal{L}$ of Q^5

In this section, we shall define the exterior differential system $\mathcal{L}$ of Q^5. To define the $\mathcal{L}$, we need the complexification $\mathbf{G}_2(\mathbf{C})$ of the exceptional Lie group $\mathbf{G}_2$ which is defined as follows

$$\mathbf{G}_2(\mathbf{C}) = \{h \in \mathbf{GL}(8, \mathbf{C}) \mid \langle h(\tilde{u}), h(\tilde{v})\rangle = \langle \tilde{u}, \tilde{v}\rangle\,,$$
$$h(\tilde{u})\, h(\tilde{v}) = h(\tilde{u}\tilde{v}) \text{ for any } \tilde{u}, \tilde{v} \in \mathbf{O}\otimes\mathbf{C}\}\,,$$

where $\langle\ ,\ \rangle$ and the product are the complex linear extension of the canonical Euclidean metric and the product of $\mathbf{O}$, respectively. Since $\mathbf{G}_2$ acts transitively on the 5-dimensional complex quadrics $Q^5 = \mathbf{G}_2/U(2)$ in $P^6(\mathbf{C})$, so does $\mathbf{G}_2(\mathbf{C})$. We use a moving frame method of $\mathbf{G}_2(\mathbf{C})$. We remark that, if $g \in \mathbf{G}_2$ then we have $\overline{g(u + \sqrt{-1}v)} = g(u - \sqrt{-1}v)$ for any $u, v \in \mathbf{O}$. However, if $h \in \mathbf{G}_2(\mathbf{C})$, then $\overline{h(u + \sqrt{-1}v)} \neq h(u - \sqrt{-1}v)$ for any $u, v \in \mathbf{O}$ in general. For any $h \in \mathbf{G}_2(\mathbf{C})$, we set

$$(z, f, g) = (h(\varepsilon), h(E), h(\bar{E})) = (\varepsilon, E, \bar{E})\rho(h)$$

where $\rho(h)$ is an element of $\mathbf{GL}(7, \mathbf{C})$. We call (z, f, g) a $\mathbf{G}_2(\mathbf{C})$ admissible frame. If we restrict the element $h \in \mathbf{G}_2(\mathbf{C})$ to $\mathbf{G}_2$, we obtain the $\mathbf{G}_2$-admissible frame which is preserves the exterior product and the complex conjugation. However, a $\mathbf{G}_2(\mathbf{C})$-admissible frame does not preserve the complex conjugation, in general. We shall write $\rho(h)$ to h for brevity. In this case, we see that

$$\langle f_1, f_1\rangle = \langle h(E_1), h(E_1)\rangle = 0\,.$$

REMARK 3.1. We have

$$\langle \tilde{u}\tilde{u}, \tilde{u}\tilde{u}\rangle = \langle \tilde{u}, \tilde{u}\rangle\langle \tilde{u}, \tilde{u}\rangle\,,$$

but

$$\langle \tilde{u}\tilde{v}, \tilde{u}\tilde{v}\rangle \neq \langle \tilde{u}, \tilde{u}\rangle\langle \tilde{v}, \tilde{v}\rangle\,,$$

for $\tilde{u}, \tilde{v} \in \mathrm{Im}\mathbf{O} \otimes \mathbf{C}$, in general.

Next, we write down the structure equations of $\mathbf{G}_2(\mathbf{C})$ which were obtained by R. L. Bryant ([Br1]). Each element (z, f, g) of a $\mathbf{G}_2(\mathbf{C})$ admissible frame can be considered as a $\mathrm{Im}\,\mathbf{O} \otimes \mathbf{C}$- valued function on $\mathbf{G}_2(\mathbf{C})$. Then we have

$$d(z, f, g) = (\varepsilon, E, \bar{E})dh = (z, f, g)\Phi\,,$$

where $\Phi = h^{-1}dh$ is a left invariant $\mathfrak{g}_2(\mathbf{C})$-valued 1-form. Then we have the integrability condition $d\Phi + \Phi \wedge \Phi = 0$. Calculating the multiplication table of the complexified product of $\mathrm{Im}\mathbf{O} \otimes \mathbf{C}$, we have

PROPOSITION 3.2 (cf. Bryant [Br1]). *The left invariant $\mathfrak{g}_2(\mathbf{C})$-valued 1-form is given by the following form*

$$\Phi = \begin{pmatrix} 0 & -\iota\,{}^t\eta & \iota\,{}^t\theta \\ -2\iota\theta & \kappa & [\eta] \\ 2\iota\eta & [\theta] & -\,{}^t\kappa \end{pmatrix}$$

where θ, η are $M_{3\times 1}(\mathbf{C})$-valued holomorphic 1-forms, κ is a $\mathfrak{sl}(3, \mathbf{C})$-valued holomorphic 1-form on $\mathbf{G}_2(\mathbf{C})$ and $\iota = \sqrt{-1}$. The integrability conditions can be rewritten as follows:

$$\begin{aligned} &d\theta + \kappa \wedge \theta - [\eta] \wedge \eta = 0\,, \\ &d\eta - [\theta] \wedge \theta - {}^t\kappa \wedge \eta = 0\,, \\ &d\kappa + \kappa \wedge \kappa - 2\theta \wedge {}^t\eta + [\eta] \wedge [\theta] = 0\,. \end{aligned}$$

REMARK 3.3. If we restrict an element $h \in \mathbf{G}_2(\mathbf{C})$ to $\mathbf{G}_2$, we have $\eta = \bar{\theta}$, κ is a $\mathfrak{su}(3)$-valued 1-form, $z = \bar{z}$ and $g = \bar{f}$, but in this case, the forms are not holomorphic on $\mathbf{G}_2$.

To derive the differential equation of the integral curves of $\mathcal{L}$ of Q^5, we set

$$\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} \quad \text{and} \quad \kappa = \begin{pmatrix} {\kappa_1}^1 & {\kappa_2}^1 & {\kappa_3}^1 \\ {\kappa_1}^2 & {\kappa_2}^2 & {\kappa_3}^2 \\ {\kappa_1}^3 & {\kappa_2}^3 & {\kappa_3}^3 \end{pmatrix}.$$

From the above representation, we have $[f_1] \in Q^5 \subset P^6(\mathbf{C})$. By the structure equation,

$$df_1 = z(-\iota\eta^1) + \sum_{i=1}^{3} f_i{\kappa_1}^i + g_2(-\theta^3) + g_3(\theta^2)\,.$$

From this, we may identify the holomorphic cotangent bundle $T^{*(1,0)}Q^5$ with

$$\mathrm{span}_{\mathbf{C}}\{\eta^1, {\kappa_1}^2, {\kappa_1}^3, -\theta^3, \theta^2\}\,.$$

Hence the distribution $\mathcal{L}^\natural$ is given by

$$\mathcal{L}^\natural = \{v \in T^{(1,0)}Q^5 \mid \eta^1(v) = {\kappa_1}^2(v) = {\kappa_1}^3(v) = 0\}\,.$$

We shall define the exterior differential system $\mathcal{L}$ as the dual bundle of $\mathcal{L}^\natural$, that is, $\mathcal{L} = \mathrm{span}_{\mathbf{C}}\{\theta^2, \theta^3\}$. Since $\mathbf{G}_2(\mathbf{C})$ acts transitively on Q^5, we can define the map $\mu : \mathbf{G}_2(\mathbf{C}) \to Q^5$ as $\mu(h) = [h(E_1)] = [f_1] \in Q^5 \subset P^6(\mathbf{C})$. Then the pull back bundle $\mu^* T^{*(1,0)}Q^5$ can be considered as a subspace of $\mathfrak{g}_2(\mathbf{C})$-valued 1-forms, as follows:

$$\mu^* T^{*(1,0)}\,Q^5 = \left\{ \left(\begin{array}{c|ccc|ccc} 0 & -\imath\eta^1 & 0 & 0 & 0 & \imath\theta^2 & \imath\theta^3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\imath\theta^2 & {\kappa_1}^2 & 0 & 0 & 0 & 0 & \eta^1 \\ -2\imath\theta^3 & {\kappa_1}^3 & 0 & 0 & 0 & -\eta^1 & 0 \\ \hline 2\imath\eta^1 & 0 & \theta^3 & -\theta^2 & 0 & -{\kappa_1}^2 & -{\kappa_1}^3 \\ 0 & -\theta^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^2 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}.$$

The integrability conditions imply that

$$\begin{aligned} d\theta^2 &= 0\,, \\ d\theta^3 &= 0\,, \\ d\eta^1 + 2\,\theta^2 \wedge \theta^3 &= 0\,, \\ d{\kappa_1}^2 - 3\,\theta^2 \wedge \eta^1 &= 0\,, \\ d{\kappa_1}^3 - 3\,\theta^3 \wedge \eta^1 &= 0\,. \end{aligned}$$

From which, we see that (locally)

$$\begin{aligned} \theta^2 &= dx_2\,, \\ \theta^3 &= dx_3\,, \\ \eta^1 &= dy - x_2 dx_3 + x_3 dx_2\,, \\ {\kappa_1}^2 &= dz_2 + 3x_2 dy - (3/2)(x_2)^2 dx_3\,, \\ {\kappa_1}^3 &= dz_3 + 3x_3 dy + (3/2)(x_3)^2 dx_2\,, \end{aligned}$$

where (x_2, x_3, y, z_2, z_3) can be considered as a local holomorphic coordinate system of Q^5 (see [Ca]). More precisely, by Frobenius theorem and the 3rd theorem of Lie, (x_2, x_3, y, z_2, z_3) is the coordinate centered at the identity of the 5-dimensional Lie subgroup H of $\mathbf{G}_2(\mathbf{C})$. Let h be an element of H near the identity, then h is given by the following form with respect to the coordinate;

$h =$

$$\left(\begin{array}{c|ccc|ccc}
1 & -\imath y & 0 & 0 & 0 & \imath x_2 & \imath x_3 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2\imath x_2 & \alpha_1 & 1 & 0 & 0 & {x_2}^2 & y + x_2 x_3 \\
-2\imath x_3 & \alpha_2 & 0 & 1 & 0 & -(y - x_2 x_3) & {x_3}^2 \\
\hline
2\imath y & \alpha_3 & x_3 & -x_2 & 1 & -(z_2 + 2x_2 y - {x_2}^2 x_3/2) & -(z_3 + 2x_3 y + {x_3}^2 x_2/2) \\
0 & -x_3 & 0 & 0 & 0 & 1 & 0 \\
0 & x_2 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

where $\alpha_1 = z_2 + x_2 y - {x_2}^2 x_3/2, \alpha_2 = z_3 + x_3 y + {x_3}^2 x_2/2, \alpha_3 = y^2 + x_3 z_2 - x_2 z_3 - {x_2}^2 {x_3}^2$. In order to write the above form, we need the complexification $\mathbf{G}_2(\mathbf{C})$ of $\mathbf{G}_2$.

LEMMA 3.4. *The differential system of the integral curves of $\mathcal{L}$ can be reduced the following form*:

$$\eta^1 = dw_1 - w_2 d\zeta = 0\,,$$
$$\kappa_1{}^2 = dw - w_1 d\zeta = 0\,,$$
$$\kappa_1{}^3 = (3/4)(dz - {w_2}^2 d\zeta) = 0\,,$$

where $w_1 = y - x_2 x_3,\ w_2 = -2x_3,\ w = (1/3)(z_2 + 3x_2 y - (3/2)\,{x_2}^2 x_3),\ z = (4/3)(z_3 + (3/2){x_3}^2 x_2)$, *and* $x_2 = \zeta$.

4. Integral curves of $\mathcal{L}$ of Q^5

We shall give the explicit (local) solution of integral curves of $\mathcal{L}$ of Q^5 associated to the super-minimal J-holomorphic curves of S^6.

PROPOSITION 4.1. *Let $\Xi : U \to Q^5$ be the integral curve of $\mathcal{L}$ with $\Xi(0) = E_1$. Then $\Xi : U \to Q^5$ can be represented as follows, where U is a simply connected open set of $\mathbf{C}$ which contains the origin $0 \in \mathbf{C}$.*

$$\Xi(\zeta) = \varepsilon\alpha_1(\zeta) + E_1 \cdot 1 + E_2\alpha_2(\zeta) + E_3\alpha_3(\zeta) + \bar{E}_1\alpha_4(\zeta) + \bar{E}_2\alpha_5(\zeta) + \bar{E}_3\zeta$$

where

$\alpha_1(\zeta) = (\sqrt{-1}/2)[\zeta(f''(\zeta) + f''(0)) - 2(f'(\zeta) - f'(0))]\,,$

$\alpha_2(\zeta) = (1/2)\zeta^2 f''(\zeta) - \zeta(2f'(\zeta) + f'(0)) + 3(f(\zeta) - f(0))\,,$

$$\alpha_3(\zeta) = (1/2)f''(\zeta)(f'(0) - f'(\zeta)) + f''(0)[(1/2)(\zeta f''(\zeta)) - f'(\zeta) + f'(0) + (1/4)(f''(0)\zeta)] + (3/4)\int_0^\zeta (f''(z))^2 dz\,,$$

$\alpha_4(\zeta) = f''(\zeta)[-(3/2)f(\zeta) + (1/2)\zeta f'(\zeta) + f'(0)\zeta + (1/4)(f''(0)\zeta^2) + (3/2)f(0)]$

$$+ f'(\zeta)(f'(\zeta) - f''(0)\zeta - 2f'(0)) + (3/2)f''(0)f(\zeta) - (3/4)\,\zeta \int_0^\zeta (f''(z))^2 dz$$

$$- (1/2)(f'(0)f''(0)\zeta) - (3/2)f(0)f''(0) + (f'(0))^2,$$

$$\alpha_5(\zeta) = (1/2)(f''(\zeta) - f''(0)),$$

for an arbitrary holomorphic function $f(\zeta)$ on U.

REMARK 4.2. We note that the quadric $Q^5 \subset P^6(\mathbf{C})$ is defined as follows

$$Q^5 = \{[w_0 : w_1 : \cdots : w_6] \in P^6(\mathbf{C}) | (w_0)^2 + w_1 w_4 + w_2 w_5 + w_3 w_6 = 0\}$$

where [ ] denotes the homogeneous coordinate system of $P^6(\mathbf{C})$.

From Theorem 4.1, we can easily prove the following

PROPOSITION 4.3. *If the function $f(\zeta)$ is a polynomial of the degree not greater than* 3, *then the corresponding immersions are totally geodesic.*

PROOF. If we put $f(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$, then we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, and $\alpha_5 = 3a_3\zeta$, then the map

$$\frac{\iota\,\Xi \times \bar{\Xi}}{\langle \Xi,\ \bar{\Xi} \rangle} = \frac{1}{1 + (9\,|a_3|^2 + 1)\,|\zeta|^2}\,(\varepsilon\,, E\,, \bar{E}\,) \begin{pmatrix} (1/2)\,\{1 - (9\,|a_3|^2 + 1)\,|\zeta|^2)\} \\ 0 \\ -\iota\zeta \\ \iota(3a_3\zeta) \\ 0 \\ \iota\bar{\zeta} \\ -\iota\overline{(3a_3\zeta)} \end{pmatrix}$$

is the corresponding J-holomorphic curve of S^6. If we put $v = \dfrac{E_2\,(-\iota) + E_3\iota\,(3a_3)}{\sqrt{(9\,|a_3|^2 + 1)}}$, the above J-holomorphic curve is contained in $\mathbf{R}^3 \cap S^6$ where $\mathbf{R}^3 = \mathrm{Re}(span_{\mathbf{C}}(\varepsilon\,, v\,, \bar{v}))$. Hence we get the desired result. □

5. Proof of Proposition 4.1

By Lemma 3.4, we obtain the solution of the integral curve of $\mathcal{L}$ in Q^5 as follows:

$$(w_1, w_2, w, z, \zeta) = \left(f'(\zeta), f''(\zeta), f(\zeta), \int^\zeta f''(w)^2 dw, \zeta \right).$$

From this representation, we can get the integral curves of $\mathcal{L}$. However, if we want to calculate the J-holomorphic curves from M^2 to S^6 more explicitly, we need to solve the following 1st order linear differential equation of one complex variable. (The reason for the ambiguity is the choice of the local coordinate system of Q^5 and the $\mathbf{G}_2(\mathbf{C})$-frame on M^2). We may put

$$(z, f, g) = (\varepsilon, E, \bar{E})h$$

where $h = (h_{ij})_{1\leq i,j\leq 7} = (h_1, h_2, \cdots, h_7)$ is a $\mathbf{G}_2(\mathbf{C})$-valued function and each h_i is a $M_{7\times 1}(\mathbf{C})$-valued function for $1 \leq i \leq 7$, respectively. By Lemma 3.4, we have $x_3 = -(1/2)f''$. From this, in order to obtain the holomorphic horizontal curves, we may solve the following:

$$\frac{dh}{d\zeta} = h\left(\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & \imath & -(\imath/2)f''' \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\imath & 0 & 0 & 0 & 0 & 0 & 0 \\ \imath f''' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -(1/2)f''' & -1 & 0 & 0 & 0 \\ 0 & (1/2)f''' & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We can solve the above differential equation with the initial condition $h(0) = I_{7\times 7}$, and from the relation $\Xi = (\varepsilon, E, \bar{E})h_2$, we get the desired result. □

REMARK 5.1. (1) We note that Ξ coincide with f_1. f_1 is one element of the $\mathbf{G}_2(\mathbf{C})$-admissible frame but not $\mathbf{G}_2$-frame, we need the normalization of the length of f_1, to construct a J-holomorphic curve.

(2) We shall write down the above differential equation more precisely, as follows:

$$\frac{dh_1}{d\zeta} = \imath(-2h_3 + f''' h_4), \tag{5.1}$$

$$\frac{dh_2}{d\zeta} = (1/2)f'''h_6 + h_7, \tag{5.2}$$

$$\frac{dh_3}{d\zeta} = -(1/2)f'''h_5, \tag{5.3}$$

$$\frac{dh_4}{d\zeta} = -h_5, \tag{5.4}$$

$$\frac{dh_5}{d\zeta} = 0, \tag{5.5}$$

$$\frac{dh_6}{d\zeta} = \imath h_1, \tag{5.6}$$

$$\frac{dh_7}{d\zeta} = -(\imath/2)f''' h_1. \tag{5.7}$$

We can solve these equations in the following order, (5.5), (5.3), (5.4), (5.1), (5.6), (5.7) and (5.2).

If the map $\Xi : M \to Q^5$ is given as above, then the map

$$\frac{\imath\,\Xi \times \bar{\Xi}}{\langle \Xi,\ \bar{\Xi} \rangle} : M \to S^6 \tag{5.8}$$

is a super-minimal J-holomorphic curve of S^6. For later use, we represent (5.8) more explicitly. We put $\Xi = (\varepsilon, E, \bar{E})\,{}^t(\alpha_1, 1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \zeta)$. Then

$$\frac{\imath\,\Xi \times \bar{\Xi}}{\langle \Xi, \bar{\Xi} \rangle} = \frac{1}{A}(\varepsilon, E, \bar{E}) \begin{pmatrix} (1/2)\{(1 + |\alpha_2|^2 + |\alpha_3|^2) - (|\alpha_4|^2 + |\alpha_5|^2 + |\zeta|^2)\} \\ (\alpha_1\,\overline{\alpha_4} - \overline{\alpha_1}) - \imath(\alpha_5\,\overline{\alpha_3} - \zeta\,\overline{\alpha_2}) \\ (\alpha_1\,\overline{\alpha_5} - \alpha_2\,\overline{\alpha_1}) - \imath(\zeta - \alpha_4\,\overline{\alpha_3}) \\ (\alpha_1\,\bar{\zeta} - \alpha_3\,\overline{\alpha_1}) - \imath(\alpha_4\,\overline{\alpha_2} - \alpha_5) \\ \overline{(\alpha_1\,\overline{\alpha_4} - \overline{\alpha_1}) - \imath(\alpha_5\,\overline{\alpha_3} - \zeta\,\overline{\alpha_2})} \\ \overline{(\alpha_1\,\overline{\alpha_5} - \alpha_2\,\overline{\alpha_1}) - \imath(\zeta - \alpha_4\,\overline{\alpha_3})} \\ \overline{(\alpha_1\,\bar{\zeta} - \alpha_3\,\overline{\alpha_1}) - \imath(\alpha_4\,\overline{\alpha_2} - \alpha_5)} \end{pmatrix} \tag{5.9}$$

where $A = |\alpha_1|^2 + (1/2)\,\{1 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 + |\alpha_5|^2 + |\zeta|^2\}$.

6. The case of monomial of degree 4

We apply the function $f(\zeta) = (\sqrt{15t}/36)e^{\imath\theta}\zeta^4$ to Proposition 4.1, and, after that we change the parameter ζ to $\sqrt{6}z$. Then the corresponding integral curve $\Xi_t : P^1(\mathbf{C}) \to Q^5$ of $\mathcal{L}$ is given by

$$\Xi_t(z) = (\varepsilon, E, \bar{E}) \begin{pmatrix} \imath\sqrt{10t}e^{\imath\theta}z^3 \\ 1 \\ \sqrt{15t}e^{\imath\theta}z^4 \\ -\sqrt{6}t\ e^{2\imath\theta}z^5 \\ te^{2\imath\theta}z^6 \\ \sqrt{15t}e^{\imath\theta}z^2 \\ \sqrt{6}z \end{pmatrix}$$

where $z \in \mathbf{C}, \theta, t \in \mathbf{R}$ and $t \geq 0$. From this and (5.9), we obtain super-minimal J-holomorphic curves of S^6 as follows (also see [H2]):

$$\mathbf{x}_t(z) = \frac{\imath\,\Xi_t(z) \times \overline{\Xi_t(z)}}{\langle \Xi_t(z), \overline{\Xi_t(z)} \rangle}$$

$$= \frac{1}{\langle \Xi_t(z), \overline{\Xi_t(z)} \rangle} \left(\varepsilon, E, \bar{E}\right) \begin{pmatrix} (1/2)(1 - 6p - 15tp^2 + 15tp^4 + 6t^2p^5 - t^2p^6) \\ \imath e^{-\imath\theta}\sqrt{10t}\bar{z}^3(1 + 3p + 3tp^2 + tp^3) \\ -\imath\sqrt{6}z(1 - 5tp^2 - 5tp^3 + t^2p^5) \\ \imath\sqrt{15t}e^{\imath\theta}z^2(1 + 2p - 2tp^3 - tp^4) \\ -\imath e^{\imath\theta}\sqrt{10t}z^3(1 + 3p + 3tp^2 + tp^3) \\ \imath\sqrt{6}\bar{z}(1 - 5tp^2 - 5tp^3 + t^2p^5) \\ -\imath\sqrt{15t}e^{-\imath\theta}\bar{z}^2(1 + 2p - 2tp^3 - tp^4) \end{pmatrix}$$

where $p = z\bar{z} = |z|^2$ and

$$\begin{aligned} &\langle \Xi_t(z), \overline{\Xi_t(z)} \rangle \\ &\quad = (1/2)(1 + 6p + 15tp^2 + 20tp^3 + 15tp^4 + 6t^2p^5 + t^2p^6) = (1/2)\tau_t(p) \geq 1/2 \,. \end{aligned}$$

We shall show that these maps are immersions. To prove this we shall prepare the following:

LEMMA 6.1. *Let $\Xi : M \to Q^5$ be a integral curve of $\mathcal{L}$. The map*

$$\mathbf{x}(\zeta) = \frac{\imath\Xi(\zeta) \times \overline{\Xi(\zeta)}}{\langle \Xi(\zeta), \overline{\Xi(\zeta)} \rangle}$$

is an immersion $\mathbf{x}$ from M to S^6 if and only if

$$\frac{\langle \imath\Xi'(\zeta) \times \overline{\Xi(\zeta)}, \overline{\imath\Xi'(\zeta) \times \overline{\Xi(\zeta)}} \rangle - |\langle \Xi'(\zeta), \overline{\Xi(\zeta)} \rangle|^2}{\langle \Xi(\zeta), \overline{\Xi(\zeta)} \rangle^2} > 0 \,,$$

where $\Xi'(\zeta) = d\Xi\left(\dfrac{\partial}{\partial\zeta}\right)(\zeta)$.

PROOF. Since $\langle \mathbf{x}, \mathbf{x} \rangle = 1$, we have

$$\langle \imath\Xi'(\zeta) \times \overline{\Xi(\zeta)}, \mathbf{x} \rangle \;=\; \langle \Xi'(\zeta)\,, \overline{\Xi(\zeta)} \rangle \,.$$

From this, and calculate the metric $\langle d\mathbf{x}(\frac{\partial}{\partial\zeta})\,, \overline{d\mathbf{x}(\frac{\partial}{\partial\zeta})} \rangle$, we get the desired result. □

By direct calculation, we get

LEMMA 6.2.

$$\begin{aligned} &\langle \imath\Xi_t{}'(z) \times \overline{\Xi_t(z)}, \overline{\imath\Xi_t{}'(z) \times \overline{\Xi_t(z)}} \rangle - |\langle \Xi_t{}'(z), \overline{\Xi_t(z)} \rangle|^2 \\ &\quad = 3(1 + 10tp + 45tp^2 + 120tp^3 + 15t(5t + 9)p^4 + 252t^2p^5 \\ &\qquad + 15t(5t + 9)p^6 + 120t^3p^7 + 45t^3p^8 + 10t^3p^9 + t^4p^{10}) = 3f_t(p) > 0 \,. \end{aligned}$$

REMARK 6.3. (1) If $t = 0$ and $t = 1$, then we have $f_0(p) = 1$ and $f_1(p) = (1 + p)^{10}$, respectively.

(2) Since

$$\lim_{z\to\infty} |z|^4 \frac{24(1+10tp+\cdots+t^4p^{10})}{(1+6p+\cdots+t^2p^6)^2} = \begin{cases} 2/3 & \text{if } t=0 \\ 24 & \text{if } t>0\,, \end{cases}$$

then the map $\mathbf{x}_t(z)$ is an immersion at ∞ for any $t \geq 0$.

By Lemma 6.1, 6.2 and (2) of Remark 6.3, we obtain

PROPOSITION 6.4. *The* 1*-parameter family* $\mathbf{x}_t(z)$ *of super-minimal J-holomorphic curves does not have any branch point, for any* $t \geq 0$*.*

Next, to prove the 1-parameter family $\mathbf{x}_t$ is a deformation of the super-minimal J-holomorphic curves up to the action of $\mathbf{G}_2$, we calculate the Gauß curvature of the $\mathbf{x}_t$. We may put the 1st fundamental form as follows:

$$\rho^2(dx^2+dy^2)$$

where $z = x + \imath y$ is a conformal coordinate of $\mathbf{C}$. Then we have

$$\rho^2 = 2\left\langle d\mathbf{x}\left(\frac{\partial}{\partial z}\right), \quad \overline{d\mathbf{x}\left(\frac{\partial}{\partial z}\right)}\right\rangle. \tag{6.1}$$

The Gauß curvature K is given by

$$K = -\frac{2}{\rho^2}\left\{\frac{\partial}{\partial z}\left(\frac{\partial \log \rho^2}{\partial \bar{z}}\right)\right\}. \tag{6.2}$$

By Lemmas 6.1 and 6.2, we have

$$\rho_t{}^2 = \frac{24(1+10tp+\cdots+t^4p^{10})}{(1+6p+\cdots+t^2p^6)^2}. \tag{6.3}$$

REMARK 6.5. If $t = 0$ and $t = 1$, then we have $\rho^2 = \dfrac{24}{(1+6p)^2}$, $K = 1$ and $\rho^2 = \dfrac{24}{(1+p)^2}$, $K = 1/6$ respectively.

Next we shall give the representation of the Gauß curvature $K_t(z)$ of $\mathbf{x}_t(z)$. We set

$$\sum_{k=0}^{10} b_k p^k = f_t(p) = 1 + 10tp + \cdots + t^4p^{10}, \tag{6.4}$$

$$\sum_{k=0}^{6} a_k p^k = \tau_t(p) = 1 + 6p + \cdots + t^2p^6. \tag{6.5}$$

Then we have

$$\rho_t{}^2 = \frac{24\sum_{k=0}^{10} b_k p^k}{\left(\sum_{k=0}^{6} a_k p^k\right)^2} . \tag{6.6}$$

By (6.2) and (6.6), we have

$$K_t(z) = -\frac{\left(\sum_{k=0}^{6} a_k p^k\right)^2}{12\left(\sum_{k=0}^{10} b_k p^k\right)} \left(\frac{\left(\sum_{k=1}^{10} k^2 b_k p^{k-1}\right)\left(\sum_{l=0}^{10} b_l p^l\right) - \left(\sum_{k=1}^{10} k b_k p^{k-1}\right)^2 p}{\left(\sum_{k=0}^{10} b_k p^k\right)^2} \right.$$
$$\left. - 2\,\frac{\left(\sum_{k=1}^{6} k^2 a_k p^{k-1}\right)\left(\sum_{l=0}^{6} b_l p^l\right) - \left(\sum_{k=1}^{6} k a_k p^{k-1}\right)^2 p}{\left(\sum_{k=0}^{6} a_k p^k\right)^2} \right) .$$

REMARK 6.6. (1) The metric $\rho_t{}^2$ and the Gauß curvature $K_t(z)$ have a S^1-symmetry centered at the origin $0 \in \mathbf{C}$.

(2) From above, we have the following

$$K_t(0) = 1 - \frac{5t}{6} \quad \text{and} \quad K_t(\infty) = 1 - \frac{5}{6t} \tag{6.7}$$

for $t > 0$. It is known that if the Gauß curvature K is constant, then $K = 1, 1/6$ or 0 (see [Se]). Therefore, if $t \neq 0$ or $\neq 1$, K_t is not a constant function. By (6.7), we have

$$\lim_{t \to 0} K_t(\infty) = -\infty .$$

The 1-parameter family of the J-holomorphic curves $\mathbf{f}_t : M \to S^6$ is called a deformation up to the action of $\mathbf{G}_2$ if they are not $\mathbf{G}_2$-congruent. More precisely, two J-holomorphic curves $\mathbf{f}_1 : M \to S^6$ and $\mathbf{f}_2 : N \to S^6$ are $\mathbf{G}_2$-congruent if there exist a $g \in \mathbf{G}_2$ and a diffeomorphism $\varphi : M \to N$ such that

$$g \circ \mathbf{f}_1 \equiv \mathbf{f}_2 \circ \varphi .$$

Then φ is an isometry, since $\mathbf{G}_2 \subset SO(7)$.

LEMMA 6.7. *There exists a $g \in \mathbf{G}_2$ such that $g \circ \mathbf{x}_t(z) = \mathbf{x}_{1/t}(1/z)$.*

PROOF. The corresponding holomorphic horizontal curve with respect to $\mathbf{x}_{1/t}(1/z)$ is given by

$$\varXi_{1/t}(1/z) = (\varepsilon, E, \bar{E}) \begin{pmatrix} \imath\sqrt{10/t}e^{\imath\theta}z^3 \\ z^6 \\ \sqrt{15/t}e^{\imath\theta}z^2 \\ -\sqrt{6}\,(1/t)e^{2\imath\theta}z \\ (1/t)e^{2\imath\theta} \\ \sqrt{15/t}e^{\imath\theta}z^4 \\ \sqrt{6}z^5 \end{pmatrix} \in Q^5 \subset P^6(\mathbf{C})\,.$$

We see that the 1st fundamental forms of $\pi \circ \varXi_{1/t}(1/z) = \mathbf{x}_{1/t}(1/z)$ and of $\mathbf{x}_t(z)$ coincide. By (2) of Theorem 6.5 in ([H3]), we get the desired result. □

By Lemma 6.7, we may take the parameter $0 \leq t \leq 1$.

THEOREM 6.8. *Let* $\mathbf{x}_t : P^1(\mathbf{C}) \to S^6$ *be a* 1*-parameter family of super-minimal J-holomorphic curves of a* 6*-dimensional sphere defined as above. Then* $\mathbf{x}_t$ *is a deformation up to the action of* $\mathbf{G}_2$, *for* $0 \leq t \leq 1$.

PROOF. We may show that $g \circ \mathbf{x}_t \neq \mathbf{x}_s \circ \varphi$ if $t \neq s$ and $t, s \in [0, 1]$. If $K_t(0) = K_s(\infty)$, we have $ts = 1$. Therefore $t = s = 1$, this case does not occur. By Lemma 6.7, we may assume $0 < ts < 1$. We suppose the contrary, and deduce the contradiction. We identify $\mathbf{C} \cup \{\infty\}$ with $P^1(\mathbf{C})$.

We may assume that the Gauß curvature K_t is not constant on $\mathbf{C} \cup \{\infty\} = M$, and $g \circ \mathbf{x}_t \equiv \mathbf{x}_s \circ \varphi$ for $t \neq s, 0 < ts < 1$, where $\varphi : M \to N$ be an isometry from M to $N = \mathbf{C} \cup \{\infty\}$. By (6.7), we see that $\varphi(0) \neq 0 \in N$ and $\varphi(0) \neq \infty \in N$, since $ts \neq 1$. Then there exist a circle $S^1 \subset N$ which includes $\varphi(0)$. We can take another point $w \neq \varphi(0)$ and $w \in S^1$. Then $\varphi^{-1}(w) \in M$ is different from $0 \in M$, so the Gauß curvature is not a constant function (more precisely, the Gauß curvature is a rational function with respect to $p = z\bar{z}$), and (1) of Remark 6.6, $K_t(\varphi^{-1}(w)) \neq K_t(0)$.

On the other hand, since φ is an isometry, which preserve the Gauß curvature, $K_t(\varphi^{-1}(w)) = K_s(w) = K_s(\varphi(0)) = K_t(0)$, which is a contradiction. □

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