TOKYO J. MATH.
VOL. 28, NO. 1, 2005

Hausdorff Dimension of Trees Generated by Piecewise Linear Transformations

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(Communicated by H. Nakada)

Abstract. Trees are constructed by one–dimensional piecewise linear transformations, and the Hausdorff dimension of trees are calculated.

1. Introduction

Let $I = [0, 1]$, and we consider a piecewise linear, expansive and topologically transitive transformation $F : I \to I$, that is, there exists a finite set $\mathcal{A}$, an interval $\langle a \rangle$ corresponds for each $a \in \mathcal{A}$ and

1. $\{\langle a \rangle\}_{a \in \mathcal{A}}$ is a partition of I,
2. $(F|_{\langle a \rangle})'$ is constant. We denote

$$\eta_a = |(F|_{\langle a \rangle})'|^{-1} ,$$

$$\operatorname{sgn} a = \begin{cases} + & F'(x) > 0 \text{ for } x \in \langle a \rangle , \\ - & F'(x) < 0 \text{ for } x \in \langle a \rangle . \end{cases}$$

3. F is expanding:

$$\xi = \liminf_{n \to \infty} \frac{1}{n} \operatorname*{ess\,inf}_{x \in I} \log |F^{n\prime}(x)| > 0 .$$

4. F is topologically transitive.

We call a finite sequence $w = a_1 \cdots a_n$ a word ($a_i \in \mathcal{A}$). For a word $w = a_1 \cdots a_n$, we define

1. $|w| = n$ (the length of a word w),
2. $\langle w \rangle = \bigcap_{i=0}^{n-1} F^{-i}(\langle a_{i+1} \rangle)$,
3. $\operatorname{sgn} w = \prod_{i=1}^{n} \operatorname{sgn} a_i$,
4. $\eta_w = \prod_{i=1}^{n} \eta_{a_i}$.

Received October 24, 2003; revised December 21, 2004

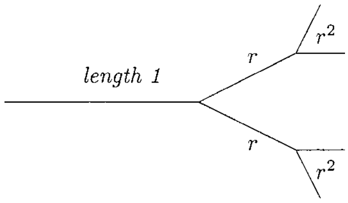

FIGURE 1. Tree corresponding to $F(x) = 2x \pmod 1$

We call a word w admissible if $\langle w\rangle \neq \emptyset$. For convenience, we consider an empty word $\emptyset$ for which we define $|\emptyset| = 0$ and $\langle\emptyset\rangle = I$. We denote by $\mathcal{W}_n$ the set of all the admissible words with length n, and $\mathcal{W} = \bigcup_{n=0}^{\infty} \mathcal{W}_n$.

We fix $0 < r < 1$, and put $R = re^{\xi}$. Now we construct a tree from words in $\mathbb{R}^d$ $(d \geq 2)$. We start from a branch $(\emptyset)$ corresponding to the empty word with its one end point at the origin, and the length of this branch equals $|I|$, where $|J|$ stands for the Lebesgue measure of a set J. From the other end point of $(\emptyset)$, branches (a) corresponding to words $a \in \mathcal{A}$ connect. Then the other endpoints of each (a) branches (ab) $(b \in \mathcal{A})$ connect and so on. Namely for each $w = a_1 \cdots a_n$,

1. (w) is a segment with length $R^{|w|}|\langle w\rangle|$,
2. one endpoint connects to $(a_1 \cdots a_{n-1})$,
3. the other endpoint connects to $(a_1 \cdots a_n b)$ $(b \in \mathcal{A}$ and $a_1 \cdots a_n b \in \mathcal{W})$.

DEFINITION 1. 1. We call the closure of $T^o = \bigcup_{w\in\mathcal{W}}(w)$ a tree and denote it by T.

2. Let $((w)) = \overline{\bigcup_{\langle u\rangle\subset\langle w\rangle}(u)}$, and call it a branch starting from w, where $\overline{J}$ stands for the closure of a set J.

3. We call $T\backslash T^o$ the flowers of T.

We need an assumption that the branches of a tree is sufficiently spread but do not intersect each other.

ASSUMPTION 1. 1. Words $u, v \in \mathcal{W}$ $(|u| < |v|)$ intersects only when $u = a_1 \cdots a_n$ and $v = ua_{n+1}$ with some $a_i \in \mathcal{A}$ $(1 \leq i \leq n+1)$, and they only intersect with their end points.

2. There exists a constant $C_0 > 0$ such that for any word $w \in \mathcal{W}$ the diameter of $((w)) \cap (T\backslash T^o)$ is greater than C_0 times the length of the branch (w).

Remark that each point x in the flowers corresponds to a point in I, because branches do not intersect from Assumption 1.

EXAMPLE 1. Let $F(x) = 2x \pmod 1$. Then the tree corresponding to this transformation looks as in Figure 1.

The main theorem which we will prove in this paper is:

THEOREM 1. *Let $I = [0, 1]$, and F be a piecewise linear, expanding and topologically transitive transformation. Then the Hausdorff dimension of this tree is the maximal solution of* $\det(I - \Phi(R^\alpha, \alpha)) = 0$, *where the definition of $\Phi(z, \alpha)$ is given afterwards.*

2. Notaions and Results on piecewise linear maps

For each $x \in I$, its expansion $s^x = a_1^x a_2^x \cdots$ $(a_i^x \in \mathcal{A})$ is defined by

$$F^{i-1}(x) \in \langle a_i^x \rangle .$$

We identify a point $x \in I$ and its expansion s^x. For a word $w = a_1 \cdots a_n$ and a point $x \in I$, we express by wx an infinite sequence $a_1 \cdots a_n a_1^x a_2^x \cdots$. If there exists a point y which has the same expansion as wx, then we call wx exists, and express $\exists wx$. We can consider two dynamical systems. One is I with F, and the other is a space of infinite sequences

$$\Sigma = \overline{\{s^x : x \in I\}} ,$$

with the shift θ. We can identify these by identifying points in I and their expansions.

LEMMA 1. *Fix any $0 < \xi' < \xi$. Then there exists a constant $C_1 = C_1(\xi')$ such that for any $w \in \mathcal{W}$, we get*

$$\eta_w \le C_1 e^{-\xi'|w|} .$$

The proof directly follows from the definition of ξ. From this lemma, we get also

$$|\langle w \rangle| \le C_1 e^{-\xi'|w|} .$$

We call a piecewise linear transformation F Markov if there exists a finite set $\mathcal{A}$ such that for any words u, v, if $F(\langle u \rangle) \cap \langle v \rangle^o \neq \emptyset$, then $F(\langle u \rangle) \supset \langle v \rangle^0$. Here we denote by J^o the interior of a set J.

For an infinite sequence $s = a_1 a_2 \cdots$ $(a_i \in \mathcal{A})$, we define

$$s[n, m] = a_n \cdots a_m \quad (n \le m) ,$$

$$s[n] = a_n .$$

Especially, for an expansion of x, we denote

$$a^x[1, n] = a_1^x a_2^x \cdots a_n^x .$$

Moreover, by taking limits from the right and the left, we define

$$y^+ = \lim_{x \uparrow y} a_1^x a_2^x \cdots ,$$

$$y^- = \lim_{x \downarrow y} a_1^x a_2^x \cdots .$$

We denote by a^+ and a^- the expansions of the right and the left endpoints of an interval $\langle a \rangle$, respectively.

We will prepare a generating function. Let for $g \in L^{\infty}$, we define for $\alpha > 0$

$$s_g^{y^\sigma}(z,\alpha) = \sum_{n=0}^{\infty} z^n \sum_{w\in\mathcal{W}} \eta_w^\alpha \int_n \sigma(y^\sigma, wx)\delta[\langle w[1]\rangle \supset \langle a_1^y\rangle, \exists\theta wx] g(x)\,dx\,.$$

Here

$$\delta[L] = \begin{cases} 1 & L \text{ is true}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$\sigma(y^\sigma, x) = \begin{cases} +\dfrac{1}{2} & \text{if } y \geq_\sigma x\,, \\ -\dfrac{1}{2} & \text{if } y <_\sigma x\,, \end{cases}$$

$$x <_\sigma y = \begin{cases} x < y & \sigma = +\,, \\ x > y & \sigma = -\,, \end{cases}$$

$$\int_n = \int_{n,y^\sigma} = \begin{cases} \int_{\langle a_1^y\rangle} & n = 0\,, \\ \int_I & n \geq 1\,. \end{cases}$$

Then we put

$$\Phi(z,\alpha)_{y^\sigma,b^\tau} = \sum_{n=1}^{\infty} z^n \sigma(\theta^n y^\sigma, b^\tau)(\operatorname{sgn} y^\sigma[1,n])(\eta_{y^\sigma[1,n]})^\alpha\,,$$

$$\chi_g^{y^\sigma}(z,\alpha) = \sum_{n=0}^{\infty} z^n (\operatorname{sgn} y^\sigma[1,n])(\eta_{y^\sigma[1,n]})^\alpha \int_n g(x)\sigma(\theta^n y^\sigma, x)\,dx\,.$$

We define vectors $s_g(z,\alpha) = (s_g^{\tilde a}(z,\alpha))_{\tilde a\in\tilde{\mathcal{A}}}$ and $\chi_g(z,\alpha) = (\chi_g^{\tilde a}(z,\alpha))_{\tilde a\in\tilde{\mathcal{A}}}$ on $\tilde{\mathcal{A}} = \{a^\sigma : a \in \mathcal{A}, \sigma = \pm\}$. Let $\Phi(z,\alpha)$ be a matrix on $\tilde{\mathcal{A}}$ whose component equals $\Phi(z,\alpha)_{a^\sigma,b^\tau}$ ($a^\sigma, b^\tau \in \tilde{\mathcal{A}}$). We call this matrix the Fredholm matrix associated with F. Note that $\Phi(z,\alpha)$ and $\chi_g(z,\alpha)$ are analytic in $|z| < e^{\xi\alpha}$.

We have a renewal equation of the form (cf. [3] when $\alpha = 1$, and [6] for general α):

$$s_g(z,\alpha) = (I - \Phi(z,\alpha))^{-1}\chi_g(z,\alpha)\,.$$

Set for an interval J

$$s_g^J(z,\alpha) = s_g^{J^+}(z,\alpha) + s_g^{J^-}(z,\alpha)\,.$$

where J^+ and J^- is the expansions of points $\sup\{x \in J\}$ and $\inf\{x \in J\}$, respectively. Then from the definition, we get (cf. also [3] and [6])

$$s_g^{\langle a\rangle}(z,\alpha) = \sum_{n=0}^{\infty} z^n \sum_{w\in\mathcal{W}_n, w[1]=a} \eta_w^\alpha \int 1_{F^n(\langle w\rangle)}(x) g(x)\,dx\,.$$

Then taking $g \equiv 1$, we get for $z > 0$ and $\alpha \geq 1$

$$
\begin{aligned}
s_1^{\langle a \rangle}(z,\alpha) &= \sum_{n=0}^{\infty} z^n \sum_{w \in \mathcal{W}_n, w[1]=a} \eta_w^\alpha |F^n(\langle w \rangle)| \\
&\geq \sum_{n=0}^{\infty} z^n \sum_{w \in \mathcal{W}_n, w[1]=a} \eta_w^\alpha |F^n(\langle w \rangle)|^\alpha \\
&= \sum_{n=0}^{\infty} z^n \sum_{w \in \mathcal{W}_n, w[1]=a} |\langle w \rangle|^\alpha . \qquad (2.1)
\end{aligned}
$$

We define an α-zeta function:

$$
\zeta(z,\alpha) = \exp\left[\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{p=F^n(p)} |F^{n\prime}(p)|^{-\alpha} \right].
$$

Then we get

$$
\zeta(z,\alpha) = [\det(I - \Phi(z,\alpha))]^{-1} .
$$

Moreover, if F is Markov, then a matrix defined by

$$
\tilde{\Phi}(z,\alpha)_{a,b} = \begin{cases} z\eta_a^\alpha & \text{if } F(\langle a \rangle) \supset \langle b \rangle^o , \\ 0 & \text{otherwise} \end{cases}
$$

satisfies

$$
\det(I - \Phi(z,\alpha)) = \det(I - \tilde{\Phi}(z,\alpha)) .
$$

We call this matrix $\tilde{\Phi}(z,\alpha)$ Markov expression of Fredholm matrix. See [5] and [6] for detail.

3. Markov Approximation

For each $a \in \mathcal{A}$ and an integer $M > 0$, we define two infinite sequences of symbols $(a_M^+)^-$ and $(a_M^-)^+$, which coincide with a^+ and a^- until M, and $F^M((a_M^+)^-) = \langle a_M^+ \rangle^-$ and $F^M((a_M^-)^+) = \langle a_M^- \rangle^+$, respectively, that is,

$$
\begin{aligned}
(a_M^+)^- &= a_1^+ \cdots a_{M-1}^+ \langle a_M^+ \rangle^- \\
&= a_1^+ \cdots a_{M-1}^+ (a_M^+)_1^- (a_M^+)_2^- \cdots , \\
(a_M^-)^+ &= a_1^- \cdots a_{M-1}^- \langle a_M^- \rangle^+ \\
&= a_1^- \cdots a_{M-1}^- (a_M^-)_1^+ (a_M^-)_2^+ \cdots .
\end{aligned}
$$

Then for sufficiently large M and $a \in \mathcal{A}$, taking above sequences as points in [0, 1], we get

$$
(a_M^+)^-, (a_M^-)^+ \in \langle a \rangle .
$$

From the construction,

$$(a_M^+)^- \uparrow a^+, \quad (a_M^-)^+ \downarrow a^- .$$

Therefore

$$I_M = \bigcup_{a \in \mathcal{A}} [(a_M^-)^+, (a_M^+)^-] \uparrow [0, 1] .$$

Now let for sufficiently large M

$$C_M = \{x \in I_M : F^n(x) \in I_M \text{ for all } n\} .$$

Then $\overline{C_M}$ is a Cantor set in $[0, 1]$. Take a restriction $F_M = F|_{C_M}$. Then F_M is a Markov transformation on C_M. Because I_M increases to $[0, 1]$, C_M also increases. Thus any periodic orbit belongs to C_M for sufficiently large M. Periodic orbits are dense in $[0, 1]$, so $\overline{C_M} \uparrow [0, 1]$. Thus, since zeta functions are determined by periodic orbits, for sufficiently small z, the α-zeta function associated with F_M converges to the α-zeta function $\zeta(z, \alpha)$ associated with F.

We call a word w a Markov word if $F^{|w|}(\langle w \rangle)$ is a union of $\langle a \rangle$ $(a \in \mathcal{A})$. Let $\mathcal{W}_n^M$ be a set of all the words $w = a_1 \cdots a_n \in \mathcal{W}_n$ which satisfy one of the followings:

1. $n < M$ and there exists a M–Markov word $b_1 \cdots b_M$ such that $\langle w \rangle \supset \langle b_1 \cdots b_M \rangle$,
2. $n \geq M$ and for any $1 \leq i \leq n - M$, $a_i \cdots a_{M+i}$ is a Markov word.

Put $\mathcal{W}^M = \bigcup_{n=0}^{\infty} \mathcal{W}_n^M$.

4. Proof of Theorem 1, Hausdorff dimension of the tree T

Let us estimate the Hausdorff dimension of trees from above. First we prove:

LEMMA 2. *The Hausdorff dimension of T^o equals 1, that is, the main part of the tree T is its flowers.*

PROOF. Let us choose any $0 < p < 1$ and fix it. We will cover a branch $((w))^o$ starting from a word w with length $|w| = n$ by closed discs with radius p^n. The total length of branches $((w))$ with length $|w| = n$ equals R^n. So there exist at most $\frac{R^n}{p^n} + (\#\mathcal{A})^n$ number of disks to cover these branches. Thus by this cover we get an upper estimate:

$$p^{n\alpha}\left(\frac{R^n}{p^n} + (\#\mathcal{A})^n\right) = (Rp^{\alpha-1})^n + (p^\alpha \#\mathcal{A})^n \leq 2(\max\{Rp^{\alpha-1}, p^\alpha \#\mathcal{A}\})^n .$$

Therefore, for any $\alpha > 1$, we can take $\max\{Rp^{\alpha-1}, p^\alpha \#\mathcal{A}\} < 1$ with sufficiently small p. This shows $\dim T^o \leq 1$. On the other hand, T^o contains segments, therefore $\dim T^o \geq 1$. This shows $\dim T^o = 1$. □

There exists a natural one to one and onto correspondence between the symbolic dynamics $s = a_1 a_2 \cdots \in \Sigma$ and $((s)) = \cap_{n=1}^{\infty} ((a_1 \cdots a_n)) \in T \backslash T^o$. So we first cover flowers $T \backslash T^o$ by using closed disks which cover branches $((w))$ $(|w| = n)$ for some n. From the assumption, we get

$$\begin{aligned}
\text{the diameter of } ((w)) &\le \sup_{a_1,a_2,\dots} \sum_{n=0}^{\infty} \text{the length of } (wa_1\cdots a_n) \\
&= \sup_{a_1,a_2,\dots} \sum_{n=0}^{\infty} R^{|w|+n} |\langle wa_1\cdots a_n\rangle| \\
&\le \sum_{n=0}^{\infty} R^{|w|+n} C_1 e^{(-\xi+\varepsilon)n} |\langle w\rangle| \qquad (\xi' = \xi - \varepsilon) \\
&= \sum_{n=0}^{\infty} R^{|w|} r^n C_1 e^{\varepsilon n} |\langle w\rangle| \\
&= R^{|w|} |\langle w\rangle| \frac{C_1}{1 - re^{\varepsilon}}. \qquad (4.1)
\end{aligned}$$

Here we choose any $\varepsilon > 0$ which satisfies $re^{\varepsilon} < 1$.

Using renewal equation

$$s_g(z,\alpha) = (I - \Phi(z,\alpha))^{-1} \chi_g(z,\alpha)\,,$$

and (2.1), we get for $z > 0$

$$\begin{aligned}
&\sum_{n=0}^{\infty} z^n \sum_{|w|=n} [\text{the diameter of } ((w))]^{\alpha} \le \sum_{n=0}^{\infty} z^n \sum_{|w|=n} R^{n\alpha} |\langle w\rangle|^{\alpha} \left(\frac{C_1}{1 - re^{\varepsilon}}\right)^{\alpha} \\
&= \left(\frac{C_1}{1 - re^{\varepsilon}}\right)^{\alpha} \sum_{n=0}^{\infty} (R^{\alpha} z)^n \sum_{|w|=n} |\langle w\rangle|^{\alpha} \\
&\le \left(\frac{C_1}{1 - re^{\varepsilon}}\right)^{\alpha} (1,\dots,1) s_1(R^{\alpha} z, \alpha) \\
&= \left(\frac{C_1}{1 - re^{\varepsilon}}\right)^{\alpha} (1,\dots,1)(I - \Phi(R^{\alpha} z, \alpha))^{-1} \chi_1(R^{\alpha} z, \alpha)\,.
\end{aligned}$$

Let us denote by α_* the maximal solution of the equation

$$\det(I - \Phi(R^{\alpha}, \alpha)) = 0\,.$$

Then the Hausdorff dimension of T is less than or equal to α_* (cf. [6]).

Now we will show the opposite inequality. Let us denote by T_M the tree constructed only by words which belongs to $\mathcal{W}^M$. Because every branches are shorter than that of T and from the assumption, $T_M \subset T$, hence $\dim T_M \le \dim T$.

We first calculate the Hausdorff dimension of T_M. Let $\tilde{\Phi}_M(z,\alpha)$ be the Markov expression of the Fredholm matrix associated with F_M. Let α_M be the maximal solution of $\det(I - \tilde{\Phi}_M(R^{\alpha}, \alpha)) = 0$.

From the Perron-Frobenius' theorem, there exists an eigenvector $(v_a)_{a\in\mathcal{A}}$ such that $v_a > 0$ and $\sum_{a\in\mathcal{A}} v_a = 1$ associated with eigenvalue 1 of $\tilde{\Phi}_M(R^{\alpha_M}, \alpha_M)$. We consider a new partition $\{\langle a\rangle_M\}_{a\in\mathcal{A}}$ of $[0, 1]$ such that $|\langle a\rangle_M| = v_a$. We define a piecewise linear Markov transformation G_M such that $G_M(\langle a\rangle_M) = \bigcup_{b\in F(\langle a\rangle)}\langle b\rangle_M$ and $G'_M|_{\langle a\rangle_M} = (R\eta_a)^{\alpha_M}\operatorname{sgn} a$. Induce the Lebesgue measure on the space where G_M acts to $\cup_x((s^x))$, and denote it by μ_1. We consider another set function μ_2 which is derived by a cover by words $((w))$ of $\{((s_x)) : x \in [0, 1]\}$. We define for a word w $\mu_2[((w))] =$ [the diameter of $((w)) \cap (T\backslash T^o)$]. For these μ_1 and μ_2, we can define a Hausdorff dimension $\dim_{\mu_i}$ $(i = 1, 2)$ by the critical point of

$$\lim_{\delta\downarrow 0} \inf_{\substack{\text{the diameter of } ((w))<\delta \\ \cup_w((w))\supset T\backslash T^o}} \sum_w [\mu_i[((w))]]^\alpha ,$$

and from the assumption

$$\mu_2[((w))] \geq C_0|(w)| = C_0 R^{|w|}|\langle w\rangle| .$$

Therefore

$$T_M \subset \left\{((s_x)) : \liminf_{n\to\infty} \frac{\log \mu_1[((a^x[1, n]))]}{\log \mu_2[((a^x[1, n]))]} \geq \alpha_M\right\} .$$

Now we will appeal to Billingsley's theorem:

THEOREM 2. ([1]) *Assume that*

$$T \subset \left\{((x)) : \liminf_{n\to\infty} \frac{\log \mu_1[((a^x[1, n]))]}{\log \mu_2[((a^x[1, n]))]} \geq \alpha\right\} .$$

Then

$$\dim_{\mu_2} \geq \alpha \dim_{\mu_1} .$$

REMARK 1. To be precise, the set function μ_2 is not a measure, so we can not apply the above theorem to our case directly. But we consider μ_2 only for intervals corresponding to words, so we can easily extend the above theorem to our case just along the same way to prove it (cf. [1]). So, we omit the proof.

Then from this theorem, we get

$$\dim_{\mu_2} \geq \alpha_M \dim_{\mu_1} = \alpha_M .$$

LEMMA 3. $\dim_{\mu_2}$ *equals the Hausdorff dimension of* $T_M\backslash T_M^o$

PROOF. From the definition, we get $\dim_{\mu_2}$ is greater than or equal to the Hausdorff dimension of $T_M\backslash T_M^o$. We will show the opposite inequality. Take α any number which is greater than the Hausdorff dimension of $T_M\backslash T_M^o$. Then there exists a cover $\{D_n\}_n$ of flowers $T_M\backslash T_M^o$ by closed discs such that $\sum_n |D_n|^\alpha < \infty$. To each $D_n \cap (T\backslash T^o)$, an interval $D'_n \subset$

[0, 1] corresponds using symbolic dynamics. Without loss of generality, we can assume

$$|D_n'| < \min_{a\in\mathcal{A}} |\langle a\rangle| \,.$$

Let $w = a_1\cdots a_k \in \mathcal{W}$ be a word with the shortest length for which $\langle w\rangle$ is contained in D_n'. If $\langle a_1\cdots a_{k-1}\rangle$ does not cover D_n', we choose a word $w' = b_1\cdots b_l$ which is the word with the shortest length such that $\langle w'\rangle$ is contained in $D_n'\backslash\langle a_1\cdots a_{k-1}\rangle$. Then $\langle a_1\cdots a_{k-1}\rangle \cup \langle b_1\cdots b_{l-1}\rangle$ covers D_n'. In other words, $((a_1\cdots a_{k-1}))\cup((b_1\cdots b_{l-1}))$ covers $D_n\cap(T_M\backslash T_M^o)$.

Because F_M is Markov, there exist constants $K_1, K_2 > 0$ such that

$$K_1\eta_w \le |\langle w\rangle| \le K_2\eta_w \,.$$

Therefore

$$\begin{aligned}|\langle a_1\cdots a_{k-1}\rangle| &\le K_2\prod_{i=1}^{k-1}\eta_{a_i} \le K_2\max_{a\in\mathcal{A}}(\eta_a)^{-1}\prod_{i=1}^{k}\eta_{a_i}\\ &\le \frac{K_2}{K_1}\max_{a\in\mathcal{A}}(\eta_a)^{-1}|\langle a_1\cdots a_k\rangle| \,.\end{aligned}$$

From the construction and (4.1),

$$\begin{aligned}&\{\text{diameter of } ((a_1\cdots a_{k-1}))\cap(T_M\backslash T_M^o)\}^\alpha\\ &\quad+\{\text{diameter of } ((b_1\cdots b_{l-1}))\cap(T_M\backslash T_M^o)\}^\alpha\}\\ &\quad\le\left(R^k\frac{C_2}{1-re^\varepsilon}\right)^\alpha|\langle a_1\cdots a_{k-1}\rangle|^\alpha+\left(R^l\frac{C_2}{1-re^\varepsilon}\right)^\alpha|\langle b_1\cdots b_{l-1}\rangle|^\alpha\\ &\quad\le\left(R^k\frac{C_2}{1-re^\varepsilon}\frac{K_2}{K_1}\max_{a\in\mathcal{A}}(\eta_a)^{-1}\right)^\alpha|\langle a_1\cdots a_k\rangle|^\alpha\\ &\qquad+\left(R^l\frac{C_2}{1-re^\varepsilon}\frac{K_2}{K_1}\max_{a\in\mathcal{A}}(\eta_a)^{-1}\right)^\alpha|\langle b_1\cdots b_l\rangle|^\alpha) \,.\end{aligned}\tag{4.2}$$

Then from $|((w))| \ge C_0|(w)| = C_0R^{|w|}|\langle w\rangle|$, we get

$$\begin{aligned}&\text{the right hand term of (4.2)}\\ &\le\left(\frac{C_2K_2}{C_0(1-re^\varepsilon)K_1}\max_{a\in\mathcal{A}}(\eta_a)^{-1}\right)^\alpha(|((a_1\cdots a_k))|^\alpha+|((b_1\cdots b_l))|^\alpha)\\ &\le\left(\frac{C_2K_2}{C_0(1-re^\varepsilon)K_1}\max_{a\in\mathcal{A}}(\eta_a)^{-1}\right)^\alpha 2\max\{|((a_1\cdots a_k))|^\alpha, |((b_1\cdots b_l))|^\alpha\}\\ &\le 2\left(\frac{C_2K_2}{C_0(1-re^\varepsilon)K_1}\max_{a\in\mathcal{A}}(\eta_a)^{-1}\right)^\alpha|D_n|^\alpha \,.\end{aligned}$$

Thus we get a cover of $T_M\backslash T_M^o$ by words w_i such that $\sum_i \mu_2[((w_i))]^\alpha < \infty$. This proves the Lemma. □

Therefore α_M equals the Hausdorff dimension of T_M.

The Fredholm determinant $\det(I - \Phi(z,\alpha))$ is the reciprocal of α–zeta function, and the α–zeta function of F_M converges to that of F in $|z| < e^{\xi\alpha}$. Therefore, $\det(I - \Phi_M(z,\alpha)) = \det(I - \tilde{\Phi}_M(z,\alpha))$ converges to $\det(I - \Phi(z,\alpha))$. Hence, we get α_M converges to α_*. This proves Theorem 1.

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