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Structure of Jackson Integrals of BC_n Type

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Abstract. The finiteness of non-symmetric and symmetric cohomologies associated with Jackson integrals of type BC_n is studied. The explicit bases of the cohomologies are also given. These bases determine parameter-dependent Jackson integral, and it is shown that they satisfy holonomic systems of linear q-difference equations with respect to the parameters.

1. Introduction

q-hypergeometric Jackson integrals are defined as an infinite sum over a lattice of a q-multiplicative function $\Phi(z)$ defined on the n-dimensional algebraic torus $(\mathbf{C}^*)^n$. Jackson integrals with Weyl group symmetry are especially interesting. In particular, such integrals with symmetry associated with A-type root system have already been investigated by several authors, and they have found important applications in product formulae, q-KZ equations, Yang–Baxter equations, orthogonal polynomials, etc (see [10, 25, 27, 34]). However, very little is known about BC_n-type integrals.

One of us has developed the rational q-de Rham cohomology theory for studying Jackson integrals giving general q-hypergeometric functions (see [2, 3, 4, 5]). The purpose of this paper is to show the finite dimensionality of non-symmetric and symmetric cohomologies $H^n(X, \Phi, \nabla_q)$, $H^n_{\text{sym}}(X, \Phi, \nabla_q)$ associated with Jackson integrals of type BC_n, and to find their explicit bases. Futhermore, we show that the parameter-dependent integrals determined by these bases satisfy linear holonomic q-difference equations with respect to the parameters (see Theorems 1.6–1.9). For the generic case, finite dimensionality was proved in full generality in [9, 31].

The symmetric cohomology $H^n_{\text{sym}}(X, \Phi, \nabla_q)$, which has a basis represented by symplectic Schur functions, is particularly important. When the dimension $\kappa = \dim H^n_{\text{sym}}(X, \Phi, \nabla_q)$ is equal to 1, the BC_n-type Jackson integrals (or Gustafson–Macdonald type sums) are expressible as an infinite product of q-gamma functions and theta functions (see [14, 15, 21, 22, 26]).

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At present, the explicit form of q-difference equations arising from BC_n-type Jackson integrals has not been found except for the BC_1 case (see (45) and (46)). Instead, using the definition (1) of Jackson integral below, we can evaluate the determinants of the pairings of the cohomology $H^n_{\mathrm{sym}}(X, \Phi, \nabla_q)$ and the corresponding nth homology (represented by lattice orbits in $(\mathbf{C}^*)^n$). Some of these results are given in Section 4 without proof. Further details are discussed in [7, 8, 24, 23]. Finally, in the appendix we prove the dimension formula for $\tilde{\kappa} = \dim H^n(X, \Phi, \nabla_q)$ in a purely analytic way, using generating functions for counting the number of special tree-like graphs.

To explain the main theorems we first introduce the Jackson integrals and their cohomologies.

1.1. Jackson integral. Throughout this paper, q is a real number such that $0 < q < 1$ and we use the symbols $(a; q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i)$ and $(a; q)_k = (a; q)_\infty/(aq^k; q)_\infty$ for $k \in \mathbf{Z}$.

Let m be an even positive integer defined by $m = 2s + 2$, $s = -1, 0, 1, 2, 3, \ldots$ and $a_1, a_2, \ldots, a_m, t_1, t_2, \ldots, t_l$ be arbitrary constants in $\mathbf{C}^*$. We denote by $\Phi(z) = \Phi(z_1, z_2, \ldots, z_n)$ the q-multiplicative function of B_n type

$$\Phi(z) = \prod_{r=1}^{n} z_r^{\frac{m}{2}-\delta+(n-r)(l-2\tau)} \prod_{k=1}^{m}\prod_{r=1}^{n} \frac{(qa_k^{-1}z_r; q)_\infty}{(a_k z_r; q)_\infty}$$
$$\times \prod_{k=1}^{l} \prod_{1\le i<j\le n} \frac{(qt_k^{-1}z_i/z_j; q)_\infty (qt_k^{-1}z_i z_j; q)_\infty}{(t_k z_i/z_j; q)_\infty (t_k z_i z_j; q)_\infty}$$

defined on $X = (\mathbf{C}^*)^n$, where $q^\delta = a_1 a_2 \cdots a_m$ and $q^\tau = t_1 t_2 \cdots t_l$ (see [18, 19]). For an arbitrary $z = (z_1, z_2, \ldots, z_n) \in X$, we define the q-shift $z \to zq^\nu$ by the lattice point $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbf{Z}^n$ as

$$zq^\nu := (z_1 q^{\nu_1}, z_2 q^{\nu_2}, \ldots, z_n q^{\nu_n}) \in X\,.$$

The set $\Lambda_z := \{zq^\nu \in X\,;\ \nu \in \mathbf{Z}^n\}$ forms an orbit of the lattice $\mathbf{Z}^n$ in X.

DEFINITION 1.1. For a function $\varphi(z)$ on X and an arbitrary point $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in X$, the Jackson integral over Λ_ξ is defined as the pairing of difference n-forms and lattice orbits

$$\int_{\Lambda_\xi} \Phi(z)\varphi(z)\frac{d_q z_1}{z_1} \wedge \cdots \wedge \frac{d_q z_n}{z_n} := (1-q)^n \sum_{\nu\in\mathbf{Z}^n} \Phi(\xi q^\nu)\varphi(\xi q^\nu) \tag{1}$$

provided this sum exists. The left-hand side of (1) is simply denoted by $\langle \varphi, \xi \rangle$.

REMARK 1.1.1. If m is an odd integer, i.e., $m = 2s + 1$, then it is sufficient to take $a_{2s+2} = \sqrt{q}$ in the case $m = 2s + 2$. The main results at the end of this section are not changed by this substitution.

By definition, the Jackson integral $\langle\varphi,\xi\rangle$ is invariant under the q-shift $\xi\to\xi q^\nu$, $\nu\in\mathbf{Z}^n$. Since $\langle 1,\xi\rangle$ is convergent if

$$|a_1a_2\cdots a_m(t_1t_2\cdots t_l)^{n+i-2}|>q^{\frac{m}{2}+(n+i-2)\frac{l}{2}}\quad\text{for}\quad i=1,2,\dots,n\,, \tag{2}$$

and

$$\begin{cases} a_k\xi_r\notin q^{\mathbf{Z}} & \text{for}\ \ 1\le r\le n,\ 1\le k\le m\,,\\ t_k\xi_i/\xi_j,\, t_k\xi_i\xi_j\notin q^{\mathbf{Z}} & \text{for}\ \ 1\le i<j\le n,\ 1\le k\le l\,,\end{cases}$$

we assume the above conditions for $a_1,a_2,\dots,a_m,t_1,t_2,\dots,t_l$ and $\xi\in X$.

Let $\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n\}$ be the standard basis of $\mathbf{R}^n$ satisfying $\langle\varepsilon_i,\varepsilon_j\rangle=\delta_{ij}$ for all $i,j=1,2,\dots,n$, where δ_{ij} denotes the Kronecker delta. The Weyl group W of type C_n is generated by the reflections $\sigma_{\varepsilon_1-\varepsilon_2},\sigma_{\varepsilon_2-\varepsilon_3},\dots,\sigma_{\varepsilon_{n-1}-\varepsilon_n},\sigma_{2\varepsilon_n}$, where

$$\sigma_\alpha(x):=x-2\frac{\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\alpha\quad\text{for}\quad x\in\mathbf{R}^n\,.$$

The scalar product $\langle\cdot,\cdot\rangle$ is uniquely extended linearly to $\mathbf{C}^n$. For the variable $z\in X$, if we put $z=(q^{\langle\varepsilon_1,x\rangle},\dots,q^{\langle\varepsilon_n,x\rangle})$, then the transformation of z by $\sigma\in W$ is defined by $\sigma z:=(q^{\langle\varepsilon_1,\sigma x\rangle},\dots,q^{\langle\varepsilon_n,\sigma x\rangle})$. For instance, the transformations of $z\in X$ by the generators of W are given by

$$\begin{aligned}\sigma_{\varepsilon_i-\varepsilon_{i+1}}\colon z_i &\longleftrightarrow z_{i+1}\ \ (1\le i\le n-1),\\ \sigma_{2\varepsilon_n}\colon z_n &\longleftrightarrow z_n^{-1}.\end{aligned}$$

The group W acts on a space of functions on X:

$$\sigma(\in W)\colon\quad f(z)\longrightarrow\sigma f(z):=f(\sigma^{-1}z).$$

Let $\Theta(z)$ be the function on X defined by

$$\Theta(z):=\frac{(1-q)^n\prod_{r=1}^n z_r^{s+1-\delta+(n-r)(l-2\tau)}}{\prod_{h=1}^m\prod_{r=1}^n\vartheta(a_hz_r;q)\prod_{k=1}^l\prod_{1\le i<j\le n}\vartheta(t_kz_i/z_j;q)\vartheta(t_kz_iz_j;q)} \tag{3}$$

where $\vartheta(z;q)$ is the Jacobi theta function $(z;q)_\infty(q/z;q)_\infty(q;q)_\infty$. Since the theta function has the property $\vartheta(qz;q)=-\vartheta(z;q)/z$, if we put

$$U_\sigma(z):=\frac{\sigma\Theta(z)}{\Theta(z)}\quad\text{for}\ \sigma\in W\,, \tag{4}$$

then $U_\sigma(z)$ is the *cocycle of pseudo-constants*, i.e., a constant with respect to the q-shifts $z\to zq^\nu$, $\nu\in\mathbf{Z}^n$. More precisely, by definition of $\Phi(z)$, it follows that the function $\sigma\Phi(z)$ is equal to $\Phi(z)$ up to the pseudo-constant $U_\sigma(z)$ as follows:

$$\sigma\Phi(z)=\Phi(z)U_\sigma(z)\,. \tag{5}$$

In this sense, we regard the function $\Phi(z)$ as *symmetric* with respect to W, and both $\Phi(z)$ and $\sigma\Phi(z)$ satisfy the same q-difference equations with respect to the q-shift $z\to zq^\nu$, $\nu\in\mathbf{Z}^n$.

From (1) and (5), we immediately have the following lemma:

LEMMA 1.2. *If $\sigma \in W$, then*

$$\sigma\langle\varphi, \xi\rangle = U_\sigma(\xi)\langle\sigma\varphi, \xi\rangle\,. \tag{6}$$

In particular, if $\varphi(z)$ is skew-symmetric under the action of W, i.e.,

$$\sigma\varphi(z) = \mathrm{sgn}(\sigma)\,\varphi(z)\,,$$

then

$$\sigma\langle\varphi, \xi\rangle = \mathrm{sgn}(\sigma)U_\sigma(\xi)\langle\varphi, \xi\rangle\,. \tag{7}$$

1.2. Rational de Rham cohomology of BC_n type. We denote by L the ring of Laurent polynomials $\mathbf{C}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$ over $\mathbf{C}$. Let R be the L-module generated by the following set of rational functions of z:

$$\bigcup_{h\ge 0}\left\{\prod_{k=1}^{m}\prod_{j=1}^{n}\frac{(a_k z_j; q)_{-h}}{(qa_k^{-1}z_j; q)_h}\prod_{k=1}^{l}\prod_{1\le i<j\le n}\frac{(t_k z_i/z_j; q)_{-h}\,(t_k z_i z_j; q)_{-h}}{(qt_k^{-1}z_i/z_j; q)_h\,(qt_k^{-1}z_i z_j; q)_h}\right\}$$

and R_{alt} be the part of R consisting of the elements which are skew-symmetric under the action of W, i.e.,

$$R_{\mathrm{alt}} := \{\varphi(z) \in R\,;\ \sigma\varphi(z) = \mathrm{sgn}(\sigma)\,\varphi(z) \text{ for } \sigma \in W\}\,.$$

LEMMA 1.3. *For $\varphi(z) \in R$ and $\xi \in X$, the Jackson integral $\langle\varphi, \xi\rangle$ is written as*

$$\langle\varphi, \xi\rangle = f_\varphi(\xi)\Theta(\xi)$$

where $f_\varphi(z)$ is a holomorphic function on X. Moreover, if $\varphi(z) \in R_{\mathrm{alt}}$, then there exists a holomorphic function $g_\varphi(z)$ on X such that

$$\langle\varphi, \xi\rangle = g_\varphi(\xi)\Theta(\xi)\theta_{\mathrm{alt}}(\xi)$$

where

$$\theta_{\mathrm{alt}}(z) := \prod_{i=1}^{n}\frac{\vartheta(z_i^2; q)}{z_i}\prod_{1\le j<k\le n}\frac{\vartheta(z_j/z_k; q)\vartheta(z_j z_k; q)}{z_j}\,.$$

Note that the function $\theta_{\mathrm{alt}}(z)$ is obviously skew-symmetric, i.e.,

$$\sigma\theta_{\mathrm{alt}}(z) = \mathrm{sgn}(\sigma)\,\theta_{\mathrm{alt}}(z)\,. \tag{8}$$

We denote by T_u and T_u^- the shift operators on a parameter $u \to uq$ and $u \to uq^{-1}$ respectively. The cocycle function associated with $\Phi(z)$ is defined by

$$b_\nu(z) := \Phi(zq^\nu)/\Phi(z) = T_{z_1}^{\nu_1}\cdots T_{z_n}^{\nu_n}\Phi(z)/\Phi(z) \quad \text{for} \quad \nu \in \mathbf{Z}^n\,,$$

which is the so-called *b-function*. The b-function $b_\nu(z)$ satisfies the relation

$$\sigma b_\nu(z) = b_{\sigma(\nu)}(z) \quad \text{for} \quad \sigma \in W. \tag{9}$$

In particular, if $\nu = \varepsilon_r$, $r = 1, 2, \ldots, n$, we have

$$b_{\varepsilon_r}(z) = q^{\frac{m}{2}-\delta+(n-r)(l-2\tau)} \prod_{k=1}^{m} \frac{1 - a_k z_r}{1 - q a_k^{-1} z_r}$$

$$\times \prod_{k=1}^{l} \left(\prod_{j=1}^{r-1} \frac{(1 - t_k^{-1} z_j/z_r)(1 - t_k z_j z_r)}{(1 - q^{-1} t_k z_j/z_r)(1 - q t_k^{-1} z_j z_r)} \times \prod_{j=r+1}^{n} \frac{(1 - t_k z_r/z_j)(1 - t_k z_j z_r)}{(1 - q t_k^{-1} z_r/z_j)(1 - q t_k^{-1} z_j z_r)} \right),$$

which is simply denoted by $b_r(z)$.

Let ∇_q be the n-dimensional covariant q-difference operator defined by

$$\nabla_q : (\psi_1(z), \psi_2(z), \ldots, \psi_n(z)) \in R^n \longrightarrow \sum_{j=1}^{n} \nabla_{q,j} \psi_j(z) \in R$$

where $\nabla_{q,j} \psi(z) := \psi(z) - b_j(z) T_{z_j} \psi(z)$. We denote by $\mathcal{A}$ the alternation

$$\mathcal{A} : f(z) \longrightarrow \sum_{\sigma \in W} \mathrm{sgn}(\sigma)\, \sigma f(z)$$

for a function $f(z)$ on X. Then we have

$$R_{\mathrm{alt}} = \mathcal{A} R\,, \tag{10}$$

$$\mathcal{A}\nabla_q(R^n) = \nabla_q(R^n) \cap R_{\mathrm{alt}}\,. \tag{11}$$

DEFINITION 1.4. The quotients $H = R/\nabla_q(R^n)$ and $H_{\mathrm{sym}} = R_{\mathrm{alt}}/\mathcal{A}\nabla_q(R^n)$ define the n-dimensional non-symmetric and symmetric rational de Rham cohomologies $H^n(X, \Phi, \nabla_q)$, $H^n_{\mathrm{sym}}(X, \Phi, \nabla_q)$ associated with the Jackson integrals (1) respectively. They are isomorphic (see also [5, 9] for the definitions of these cohomologies).

It should be remarked that here we call the *symmetric cohomology* for "cohomology consisting of skew-symmetric elements." The reader should not be confused by this in the sequel.

REMARK 1.4.1. By symmetry, it follows that

$$\mathcal{A}\nabla_q(R^n) \subset \nabla_q(R^n)\,,$$

and $\mathcal{A}\nabla_{q,i} = \mathcal{A}\nabla_{q,j}$ for all $i, j \in \{1, 2, \ldots, n\}$, so that we have

$$\mathcal{A}\nabla_q(R^n) = \mathcal{A}\nabla_{q,r} R\,.$$

This implies that H_{sym} is identified with the linear subspace of H consisting of the elements which are skew-symmetric under the Weyl group W.

LEMMA 1.5. *Suppose $\varphi(z) \in \nabla_q(R^n)$. Then $\langle \varphi, \xi \rangle = 0$ and $\langle \mathcal{A}\varphi, \xi \rangle = 0$ if they are summable.*

This lemma shows that the integral $\langle\varphi, \xi\rangle$ for $\varphi(z) \in R$ and that for $\varphi(z) \in R_{\mathrm{alt}}$ depend only on the quotients H and H_{sym} respectively.

1.3. Main results. For a sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$ we set $z^\lambda := z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_n^{\lambda_n}$ and denote the skew-symmetric Laurent polynomials in z

$$\mathcal{A}z^\lambda := \sum_{\sigma \in W} \mathrm{sgn}(\sigma)\, \sigma(z^\lambda)\,,$$

which are equal to the $n \times n$ determinant with (k, j)-entry $z_k^{\lambda_j} - z_k^{-\lambda_j}$. It is obvious that the set of skew-symmetric Laurent polynomials is spanned by $\{\mathcal{A}z^\lambda\,;\ \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0\}$.

In the sequel we assume that

$(\mathcal{C})$ all the parameters $a_1, a_2, \ldots, a_m$ and $t_1, t_2, \ldots, t_l$ are generic.

The purpose of this paper is to prove the following four theorems:

THEOREM 1.6. *Let Q be the set defined by*

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n; -s-1-(n-1)l \le \lambda_i \le s+(n-1)l \ \ for\ 1 \le i \le n\}\,. \tag{12}$$

Under the condition $(\mathcal{C})$, $H^n(X, \Phi, \nabla_q)$ has dimension $\tilde{\kappa} := \{m + 2(n-1)l\}^n$ and is spanned by the basis $\{z^\lambda\,;\ \lambda \in Q\}$.

THEOREM 1.7. *Let Q_{sym} be the set defined by*

$$\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n\,;\ s+(n-1)l \ge \lambda_1 > \lambda_2 > \cdots > \lambda_n \ge 1\}\,. \tag{13}$$

Under the condition $(\mathcal{C})$, $H^n_{\mathrm{sym}}(X, \Phi, \nabla_q)$ has dimension $\kappa := \binom{s+(n-1)l}{n}$ and is spanned by the basis $\{\mathcal{A}z^\lambda\,;\ \lambda \in Q_{\mathrm{sym}}\}$.

We obtain the following holonomic q-difference equations for $\langle z^\lambda, \xi\rangle$ and $\langle\mathcal{A}z^\lambda, \xi\rangle$, with respect to the q-shift of the parameters $a_1, \ldots, a_m$ and $t_1, \ldots, t_l$:

THEOREM 1.8. *There exist invertible matrices $\mathcal{Y}_{a_k}, \mathcal{Y}_{t_j}$ whose entries $\eta^{(a_k)}_{\lambda,\nu}, \eta^{(t_j)}_{\lambda,\nu}$ are rational functions of $a_1, \ldots, a_m$ and $t_1, \ldots, t_l$ respectively, such that*

$$T_{a_k}\langle z^\lambda, \xi\rangle = \sum_{\nu \in Q} \eta^{(a_k)}_{\lambda,\nu} \langle z^\nu, \xi\rangle\,,$$

$$T_{t_j}\langle z^\lambda, \xi\rangle = \sum_{\nu \in Q} \eta^{(t_j)}_{\lambda,\nu} \langle z^\nu, \xi\rangle$$

where λ runs over the set Q satisfying (12).

THEOREM 1.9. *There exist invertible matrices Y_{a_k}, Y_{t_j} whose entries $y^{(a_k)}_{\lambda,\nu}$, $y^{(t_j)}_{\lambda,\nu}$ are rational functions of $a_1, \dots, a_m$ and $t_1, \dots, t_l$ respectively, such that*

$$T_{a_k}\langle \mathcal{A}z^\lambda, \xi\rangle = \sum_{\nu\in Q_{\mathrm{sym}}} y^{(a_k)}_{\lambda,\nu}\langle \mathcal{A}z^\nu, \xi\rangle\,, \tag{14}$$

$$T_{t_j}\langle \mathcal{A}z^\lambda, \xi\rangle = \sum_{\nu\in Q_{\mathrm{sym}}} y^{(t_j)}_{\lambda,\nu}\langle \mathcal{A}z^\nu, \xi\rangle \tag{15}$$

where λ runs over the set Q_{sym} satisfying (13).

REMARK 1.9.1. If $(m, l) = (2n+2, 0)$ or $(4, 1)$ in Theorem 1.7, then $\kappa = 1$ and hence the matrices Y_{a_k} and Y_{t_1} in Theorem 1.9 reduce to scalars which are explicitly expressible as a ratio of product of q-gamma functions. These coincide with some results in [19, 13, 5, 20, 21, 22, 25, etc]. See also Theorems 4.1 and 4.2 in Section 4.

The proofs of Theorems 1.6–1.9 will be given in the next two sections, based on the results in [4, 9].

2. The dimension of $H^n(X, \Phi, \nabla_q)$

In this section, we prove Theorems 1.6 and 1.8 for non-symmetric case.

2.1. Proof of $\dim H^n(X,\Phi,\nabla_q) \geq \tilde{\kappa}$**.** Let Δ^+ be the set of positive roots of type B_n relative to the simple root $\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \dots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_n\}$. The set Δ^+ is written as

$$\Delta^+ = \Delta^+_{\mathrm{short}} \cup \Delta^+_{\mathrm{long}}$$

where $\Delta^+_{\mathrm{short}} := \{\varepsilon_i\,;\ 1 \leq i \leq n\}$ and $\Delta^+_{\mathrm{long}} := \{\varepsilon_i \pm \varepsilon_j\,;\ 1 \leq i < j \leq n\}$. Let $\mathcal{R}$ be the set defined by

$$\mathcal{R} := \{\langle\beta, x\rangle - \alpha_k\,;\ \beta\in\Delta^+_{\mathrm{short}}, 1\leq k\leq m\} \cup \{\langle\beta, x\rangle - \tau_k\,;\ \beta\in\Delta^+_{\mathrm{long}}, 1\leq k\leq l\}\,, \tag{16}$$

which consists of all affine forms of $x = (x_1, x_2, \dots, x_n) \in \mathbf{C}^n$ corresponding to each factor in the numerator of $\Phi(z)$ where $a_k = q^{\alpha_k}, t_k = q^{\tau_k}$ and $z_i = q^{\langle\varepsilon_i, x\rangle}$. We express by $\bar{\mu}$ and $-\mu_0$ the homogeneous part and the constant term of $\mu \in \mathcal{R}$ respectively, i.e., we have $\mu(x) = \langle\bar{\mu}, x\rangle - \mu_0$. Let $\widetilde{\mathcal{R}}$ be the set defined by

$$\widetilde{\mathcal{R}} := \left\{\{\mu^{(1)}, \dots, \mu^{(n)}\} \subset \mathcal{R}\,;\ [\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}] \neq 0\right\}$$

where $[\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]$ denotes the determinant $\det\big(\langle\bar{\mu}^{(i)}, \varepsilon_j\rangle\big)_{1\leq i,j\leq n}$, i.e., $\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}$ are linearly independent if $\{\mu^{(1)}, \dots, \mu^{(n)}\} \in \widetilde{\mathcal{R}}$.

LEMMA 2.1.

$$\tilde{\kappa} = \sum_{\{\mu^{(1)},\dots,\mu^{(n)}\}\in\widetilde{\mathcal{R}}} [\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2\,. \tag{17}$$

REMARK 2.1.1. The system $\{\mu^{(1)}, \ldots, \mu^{(n)}\}$ is associated with a special graph with n edges, and the identity (17) can be explained in a graphical sense. See [4, 9] and the Appendix for more details.

PROOF. Let M be the positive definite symmetric matrix of degree n such that

$$M := \Big(\sum_{\mu \in \mathcal{R}} \langle \bar{\mu}, \varepsilon_i \rangle \langle \bar{\mu}, \varepsilon_j \rangle \Big)_{1 \le i, j \le n}.$$

The matrix M is written as $M = mA + lB$ where

$$A = \Big(\sum_{\beta \in \Delta^+_{\text{short}}} \langle \beta, \varepsilon_i \rangle \langle \beta, \varepsilon_j \rangle \Big)_{1 \le i, j \le n} = \Big(\sum_{k=1}^{n} \langle \varepsilon_k, \varepsilon_i \rangle \langle \varepsilon_k, \varepsilon_j \rangle \Big)_{1 \le i, j \le n},$$

$$\begin{aligned} B &= \Big(\sum_{\beta \in \Delta^+_{\text{long}}} \langle \beta, \varepsilon_i \rangle \langle \beta, \varepsilon_j \rangle \Big)_{1 \le i, j \le n} \\ &= \Big(\sum_{1 \le k < k' \le n} \langle \varepsilon_k - \varepsilon_{k'}, \varepsilon_i \rangle \langle \varepsilon_k - \varepsilon_{k'}, \varepsilon_j \rangle \Big)_{1 \le i, j \le n} \\ &\quad + \Big(\sum_{1 \le k < k' \le n} \langle \varepsilon_k + \varepsilon_{k'}, \varepsilon_i \rangle \langle \varepsilon_k + \varepsilon_{k'}, \varepsilon_j \rangle \Big)_{1 \le i, j \le n}. \end{aligned}$$

Since $B = 2(n-1)A$ (see [17, Lemma 4.6]) and A is the identity matrix, we have

$$\det M = \det((m + 2(n-1)l)A) = \{m + 2(n-1)l\}^n = \tilde{\kappa}. \tag{18}$$

On the other hand, we have the following identity of Gram determinant:

$$\det M = \sum_{\{\mu^{(1)}, \ldots, \mu^{(n)}\} \in \widetilde{\mathcal{R}}} [\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(n)}]^2 \tag{19}$$

From (18) and (19), we obtain (17). This completes the proof. □

Using Lemma 1.3, for $\varphi(x) \in R$ the Jackson integral $\langle \varphi, z \rangle$ is written as

$$\langle \varphi, z \rangle = f_\varphi(z) \Theta(z) \tag{20}$$

where $f_\varphi(z)$ denotes a holomorphic function on X. Since the integral $\langle \varphi, z \rangle$ is invariant under the q-shift $z \to zq^\nu$, $\nu \in \mathbf{Z}^n$, calculating the quasi-periodicity of $\Theta(z)$ explicitly from (3), we see the function $f_\varphi(z)$ satisfies the following functional equation:

$$T_{z_i} f(z) = (-1)^m q^{-\frac{m}{2} - (n-1)l} z_i^{-m-2(n-1)l} f(z) \quad \text{for} \quad 1 \le i \le n. \tag{21}$$

We denote by $\mathcal{H}$ the linear space of holomorphic functions $f(z)$ on X satisfying (21). $\mathcal{H}$ has dimension $\tilde{\kappa}$ and its basis consists of the theta functions

$$\prod_{i=1}^{n} z_i^{\lambda_i} \vartheta\big(-(-1)^m z_i^{m+2(n-1)l} q^{\lambda_i + \frac{m+2(n-1)l}{2}} ; q^{m+2(n-1)l} \big) \tag{22}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ runs over the set Q.

REMARK 2.1.2. This fact follows from a general result on theta functions of several variables (see for example, [32, Theorem 12A] or [12, §35, p91–94]). In fact the space $\mathcal{H}_i$ of holomorphic functions depending on the single variable z_i satisfying (21) has dimension $m + 2(n-1)l$ and the basis consisting of the functions

$$z_i^{\lambda_i} \vartheta\big(-(-1)^m z_i^{m+2(n-1)l} q^{\lambda_i + \frac{m+2(n-1)l}{2}}; q^{m+2(n-1)l}\big).$$

Then $\mathcal{H}$ is isomorphic to the tensor product of $\mathcal{H}_i$ $(1 \le i \le n)$.

From (20), the following map $\mathcal{M}$ is well-defined:

$$\begin{aligned} \mathcal{M}\colon\ R &\longrightarrow \quad \mathcal{H} \\ \varphi(z) &\longmapsto \mathcal{M}\varphi(z) := \frac{\langle \varphi, z\rangle}{\Theta(z)}\,. \end{aligned} \tag{23}$$

The following is a key lemma for the proof:

LEMMA 2.2. *There exist $\phi_i(z) \in L$ $(1 \le i \le \tilde{\kappa})$ such that $\mathcal{M}\phi_i(z)$'s are linearly independent in $\mathcal{H}$.*

PROOF. The outline of proof is quite similar to that in [9, §3]. (See also [9, Proposition 4].)

We fix a vector $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbf{Z}^n$ such that $\eta_1 \gg \cdots \gg \eta_n \gg 0$. The signs $\epsilon_1, \epsilon_2, \dots, \epsilon_n \in \{+1, -1\}$ for $\{\mu^{(1)}, \dots, \mu^{(n)}\} \in \widetilde{\mathcal{R}}$ relative to η are uniquely determined as follows:

$$\epsilon_i := \begin{cases} +1 & \text{if } \langle \eta, x\rangle > 0\,, \\ -1 & \text{if } \langle \eta, x\rangle < 0 \end{cases}$$

where $x \in \mathbf{R}^n$ is a vector satisfying $\langle \bar{\mu}^{(i)}, x\rangle > 0$ and

$$\langle \bar{\mu}^{(1)}, x\rangle = 0, \dots, \langle \bar{\mu}^{(i-1)}, x\rangle = 0, \langle \bar{\mu}^{(i+1)}, x\rangle = 0, \dots, \langle \bar{\mu}^{(n)}, x\rangle = 0\,.$$

Note that, for all $\{\mu^{(1)}, \dots, \mu^{(n)}\} \in \widetilde{\mathcal{R}}$ we have

$$\langle \eta, \nu\rangle \ge 0 \quad \text{if} \quad \langle \epsilon_1\bar{\mu}^{(1)}, \nu\rangle \ge 0, \dots, \langle \epsilon_n\bar{\mu}^{(n)}, \nu\rangle \ge 0\,. \tag{24}$$

Moreover, we set

$$\epsilon\mu^{(i)}(x) := \langle \epsilon_i\bar{\mu}^{(i)}, x\rangle - \mu_0^{(i)} \quad \text{for} \quad \mu^{(i)}(x) = \langle \bar{\mu}^{(i)}, x\rangle - \mu_0^{(i)}\,.$$

For $\{\mu^{(1)}, \dots, \mu^{(n)}\} \in \widetilde{\mathcal{R}}$, we also consider the solutions $z = q^x$ to the following n equations in $(\mathbf{C}^*)^n$

$$q^{\epsilon\mu^{(1)}(x)} = q^{\nu_1}, q^{\epsilon\mu^{(2)}(x)} = q^{\nu_2}, \dots, q^{\epsilon\mu^{(n)}(x)} = q^{\nu_n} \tag{25}$$

for given integers $\nu_i \in \mathbf{Z}$ $(i = 1, 2, \dots, n)$. The set of all solutions

$$\{z = q^x \in (\mathbf{C}^*)^n\,;\ q^{\epsilon\mu^{(1)}(x)} = q^{\nu_1}, \dots, q^{\epsilon\mu^{(n)}(x)} = q^{\nu_n} \text{ and } \nu = (\nu_1, \dots, \nu_n) \in \mathbf{Z}^n\}$$

is divided into $[\epsilon_1\bar{\mu}^{(1)}, \dots, \epsilon_n\bar{\mu}^{(n)}]^2$ disjoint parts modulo the translation by the group $q^{\mathbf{Z}^n}$. Thus the set of all solutions is written as the union of lattice orbits. (See [9, Lemmas 3.2 and 3.3].) Since $[\epsilon_1\bar{\mu}^{(1)}, \dots, \epsilon_n\bar{\mu}^{(n)}]^2 = [\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2$, the set

$$\{z = q^x \in (\mathbf{C}^*)^n;\ q^{\epsilon\mu^{(1)}(x)} = q^{\nu_1}, \dots, q^{\epsilon\mu^{(n)}(x)} = q^{\nu_n} \text{ and } \nu = (\nu_1, \dots, \nu_n) \in \mathbf{Z}^n_{\geq 0}\}$$

is also divided into $[\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2$ disjoint parts. We may take ξ as the point in each disjoint parts such that $|z^\eta|$ is maximum there, and the each disjoint part is written as the fan

$$\Lambda^+_\xi = \{\xi q^\nu;\ \langle\epsilon_1\bar{\mu}^{(1)}, \nu\rangle \geq 0, \dots, \langle\epsilon_n\bar{\mu}^{(n)}, \nu\rangle \geq 0\}.$$

We call such a point ξ the *critical point* relative to the function $|z^\eta|$. From Lemma 2.1 we have in total the $\tilde{\kappa}$ critical points, all of which are different from each other. We denoted by $C_{\mathcal{R}}$ the set of all critical points. By (24), in spite of the choice of $\{\mu^{(1)}, \dots, \mu^{(n)}\} \in \widetilde{\mathcal{R}}$, each critical point $\xi \in C_{\mathcal{R}}$ is the maximum point in Λ^+_ξ of the function $|z^\eta| = |\xi^\eta| q^{\langle\eta,\nu\rangle}$ for the common fixed η.

For example, in case where $n = 2$, the choices of (ϵ_1, ϵ_2) for $(\bar{\mu}^{(1)}, \bar{\mu}^{(2)})$ are given as follows:

$(\bar{\mu}^{(1)}, \bar{\mu}^{(2)})$	(ϵ_1, ϵ_2)	$[\bar{\mu}^{(1)}, \bar{\mu}^{(2)}]^2$	$(\bar{\mu}^{(1)}, \bar{\mu}^{(2)})$	(ϵ_1, ϵ_2)	$[\bar{\mu}^{(1)}, \bar{\mu}^{(2)}]^2$
$(\varepsilon_1, \varepsilon_2)$	$(1, 1)$	1	$(\varepsilon_1, \varepsilon_1 + \varepsilon_2)$	$(1, 1)$	1
$(\varepsilon_1, \varepsilon_1 - \varepsilon_2)$	$(1, -1)$	1	$(\varepsilon_2, \varepsilon_1 + \varepsilon_2)$	$(-1, 1)$	1
$(\varepsilon_2, \varepsilon_1 - \varepsilon_2)$	$(1, 1)$	1	$(\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2)$	$(1, 1)$	4

Corresponding to these choices, we can define

$$\begin{aligned}
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_1 \geq 0,\ \nu_2 \geq 0\} \quad \xi = (a_k, a_j)\,,\\
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_1 \geq 0,\ \nu_1 - \nu_2 \leq 0\} \quad \xi = (a_k, a_k t_j)\,,\\
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_2 \geq 0,\ \nu_1 - \nu_2 \geq 0\} \quad \xi = (a_k t_j, a_k)\,,\\
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_1 \geq 0,\ \nu_1 + \nu_2 \geq 0\} \quad \xi = (a_k, t_j a_k^{-1})\,,\\
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_2 \leq 0,\ \nu_1 + \nu_2 \geq 0\} \quad \xi = (a_k t_j, a_k^{-1})\,,\\
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_1 - \nu_2 \geq 0,\ \nu_1 + \nu_2 \geq 0\} \quad \xi = \pm(\sqrt{t_k t_j}, \sqrt{t_j t_k^{-1}})\,,\\
\Lambda^+_\xi &= \{\xi q^\nu;\ \nu_1 - \nu_2 \geq 0,\ \nu_1 + \nu_2 \geq 0\} \quad \xi = \pm(\sqrt{q t_k t_j}, \sqrt{q t_j t_k^{-1}})\,.
\end{aligned}$$

Though the Jackson integral $\langle\varphi, \xi\rangle$ over Λ_ξ diverges for $\xi \in C_{\mathcal{R}}$ with

$$(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \neq (1, 1, \dots, 1)$$

because $\Phi(z)$ has poles in Λ_ξ^+, the function $\mathcal{M}\varphi(z)$ is still well-defined at $z = \xi$ by analytic continuation. Since the function

$$\frac{\vartheta(q^{\mu_0^{(i)}} z^{\bar{\mu}^{(i)}}; q)}{\vartheta(q^{1-\mu_0^{(i)}} z^{\bar{\mu}^{(i)}}; q)} (z^{\bar{\mu}^{(i)}})^{2\mu_0^{(i)}-1}$$

is invariant under the q-shift $z \to q^\nu z$, $\nu \in \mathbf{Z}^n$, we have the relation

$$\frac{\langle\varphi, z\rangle}{\Theta(z)} = \frac{\mathrm{reg}\langle\varphi, z\rangle}{\mathrm{reg}\Theta(z)} \tag{26}$$

where

$$\begin{aligned} \mathrm{reg}\langle\varphi, \xi\rangle &:= \int_{\Lambda_\xi} \varphi(z)\Phi_{\mathrm{reg}}(z)\frac{d_q z_1}{z_1} \wedge \cdots \wedge \frac{d_q z_n}{z_n} \\ &= (1-q)^n \sum_{\zeta\in\Lambda_\xi} \Phi_{\mathrm{reg}}(\zeta)\varphi(\zeta)\,, \end{aligned} \tag{27}$$

$$\Phi_{\mathrm{reg}}(z) := \Phi(z) \prod_{\substack{i \text{ such that} \\ \epsilon_i=-1}} \frac{\vartheta(q^{\mu_0^{(i)}} z^{\bar{\mu}^{(i)}}; q)}{\vartheta(q^{1-\mu_0^{(i)}} z^{\bar{\mu}^{(i)}}; q)} (z^{\bar{\mu}^{(i)}})^{2\mu_0^{(i)}-1} \tag{28}$$

and

$$\mathrm{reg}\Theta(\xi) := \Theta(\xi) \prod_{\substack{i \text{ such that} \\ \epsilon_i=-1}} \frac{\vartheta(q^{\mu_0^{(i)}} \xi^{\bar{\mu}^{(i)}}; q)}{\vartheta(q^{1-\mu_0^{(i)}} \xi^{\bar{\mu}^{(i)}}; q)} (\xi^{\bar{\mu}^{(i)}})^{2\mu_0^{(i)}-1}\,.$$

Note that for the critical point $\xi \in C_{\mathcal{R}}$ the function $\mathrm{reg}\langle\varphi, \xi\rangle$ is the sum over the fan Λ_ξ^+ instead of Λ_ξ because $\Phi_{\mathrm{reg}}(\zeta) = 0$ if $\zeta \in \Lambda_\xi \setminus \Lambda_\xi^+$. Thus the poles of $\langle\varphi, z\rangle$ and $\Theta(z)$ lying in Λ_ξ^+, $\xi \in C_{\mathcal{R}}$, are cancelled out at $z = \xi$ in (26), and the value of the function $\mathcal{M}\varphi(z)$ at $z = \xi$ is written as

$$\mathcal{M}\varphi(\xi) = \frac{\mathrm{reg}\langle\varphi, \xi\rangle}{\mathrm{reg}\Theta(\xi)}\,.$$

For $\xi \in C_{\mathcal{R}}$ with $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = (1, 1, \ldots, 1)$, the regularizations $\mathrm{reg}\langle\varphi, \xi\rangle$ and $\mathrm{reg}\Theta(\xi)$ coincide with the ordinary $\langle\varphi, \xi\rangle$ and $\Theta(\xi)$ respectively. When we labelled the points in $C_{\mathcal{R}}$ as $C_{\mathcal{R}} = \{\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(\tilde{\kappa})}\}$, we have the following claim which is elementary to prove:

Claim. *There exist $\tilde{\kappa}$ polynomials $\varphi_i(z)$ in $z_1, z_2, \ldots, z_n$ such that*

$$\varphi_i(\xi^{(j)}) = \delta_{ij} \quad \textit{for} \quad 1 \le i, j \le \tilde{\kappa}\,.$$

We define the functions $\phi_i(z) \in L$ by

$$\phi_i(z) := \varphi_i(z) z^{N\eta}$$

where N is an integer and $z^{N\eta} = z_1^{N\eta_1} \cdots z_n^{N\eta_n}$. From (27) and (28), the sum $\mathrm{reg}\langle\phi_i, \xi^{(j)}\rangle$ over the fan $\Lambda^+_{\xi^{(j)}}$ for a large integer N has the asymptotic form

$$\mathrm{reg}\langle\phi_i, \xi^{(j)}\rangle = (1-q)^n \Phi_{\mathrm{reg}}(\xi^{(j)})(\xi^{(j)})^{N\eta}\varphi_i(\xi^{(j)})(1 + O(q^N)) .$$

This means that $\det\left(\mathrm{reg}\langle\phi_i, \xi^{(j)}\rangle\right)_{1\le i,j\le n}$ does not vanish identically, so that

$$\det\left(\mathcal{M}\phi_i(\xi^{(j)})\right)_{1\le i,j\le n} = \det\left(\mathrm{reg}\langle\phi_i, \xi^{(j)}\rangle\right)_{1\le i,j\le n} \Big/ \prod_{j=1}^{\tilde\kappa} \mathrm{reg}\Theta(\xi^{(j)})$$

does not vanish identically. Hence $\mathcal{M}\phi_i(z) \in \mathcal{H}$, $1 \le i \le \tilde\kappa$ are linearly independent. This completes the proof. □

PROPOSITION 2.3. *The map $\mathcal{M}$ is surjective. In particular*, $\dim H \ge \tilde\kappa$

PROOF. From $\dim \mathcal{H} = \tilde\kappa$ and Lemma 2.2, the map $\mathcal{M}$ is surjective. From Lemma 1.5, the kernel of the map $\mathcal{M}$ includes $\nabla_q(R^n)$, so that

$$\dim H = \dim R/\nabla_q(R^n) \ge \dim R/\ker\mathcal{M} = \dim \mathcal{H} = \tilde\kappa$$

This completes the proof. □

REMARK 2.3.1. Since $\ker\mathcal{M} \supset \nabla_q(R^n)$, $\mathcal{M}$ naturally induces the map from H to $\mathcal{H}$, which is surjective. We denote it by $\overline{\mathcal{M}} : H \to \mathcal{H}$. At the end of this section, we will see that $\overline{\mathcal{M}}$ gives the isomorphism $H \xrightarrow{\sim} \mathcal{H}$ and that $\ker\mathcal{M} = \nabla_q(R^n)$.

2.2. Proof of $\dim H^n(X,\Phi,\nabla_q) \le \tilde\kappa$**.** The b-function $b_\nu(z)$ can be written as $b_\nu(z) = b^+_\nu(z)/b^-_\nu(z)$ for $b^\pm_\nu(z) \in L$. In particular, for $\nu = \varepsilon_1$ we have $b_1(z) = b^+_1(z)/b^-_1(z)$ where

$$b^+_1(z) = z_1^{-s-1-(n-1)l} \prod_{k=1}^{m}(a_k^{-1} - z_1) \times \prod_{k=1}^{l}\prod_{j=2}^{n}\left(t_k^{-1} - \frac{z_1}{z_j}\right)(t_k^{-1} - z_1 z_j) , \tag{29}$$

$$b^-_1(z) = (qz_1)^{-s-1-(n-1)l} \prod_{k=1}^{m}(1 - \frac{qz_1}{a_k}) \times \prod_{k=1}^{l}\prod_{j=2}^{n}\left(1 - t_k^{-1}\frac{qz_1}{z_j}\right)(1 - t_k^{-1}qz_1z_j) . \tag{30}$$

Similarly the symmetry (9) gives the identity

$$b_i(z) = b_1(z_i, z_2, \ldots, z_{i-1}, z_1, z_{i+1}, \ldots, z_n) \quad \text{for} \quad i = 2, 3, \ldots, n . \tag{31}$$

The Newton polyhedron $\Delta(b^\pm_i)$ of the function $b^\pm_i(z)$ is the convex polyhedron defined by

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n \,;\, |x_1| + \cdots + |x_n| \le s + 1 + (n-1)l,\ |x_k| \le l\ (k \ne i)\}$$

in the sense of theory of torus embeddings [28]. We define the map $D_i : L \to L$ by

$$\varphi(z) = \sum_\lambda c_\lambda z^\lambda \longmapsto D_i\varphi(z) = \sum_\lambda c_\lambda D_i z^\lambda$$

where

$$D_i z^\lambda := \nabla_{q,i}\{\big(T_{z_i}^- b_i^-(z)\big)z^\lambda\} = \big(T_{z_i}^- b_i^-(z)\big)z^\lambda - b_i^+(z)\, T_{z_i} z^\lambda$$
$$= z^\lambda\Big(T_{z_i}^- b_i^-(z) - b_i^+(z) q^{\lambda_i}\Big) \in L. \tag{32}$$

The Newton polyhedron of $D_i z^\lambda$ is written as $\lambda + \Delta(b_i^\pm)$, which is translation of $\Delta(b_i^\pm)$ by λ.

Let $\mathcal{K}_r$, $r = 0, 1, 2, \ldots$, be the convex polyhedra

$$\mathcal{K}_r := \{(x_1, \ldots, x_n) \in \mathbf{R}^n\,;\ |x_1| + \cdots + |x_n| \le rn\}$$

and $\langle \mathcal{K}_r \rangle$ denote the linear subspace of L such that

$$\langle \mathcal{K}_r \rangle := \bigoplus_{\lambda \in \mathcal{K}_r \cap \mathbf{Z}^n} \mathbf{C} z^\lambda .$$

Suppose $r \ge s + 1 + (n-1)l$. Consider the convex polyhedron

$$\mathcal{K}'_{r,i} := \left\{(x_1, \ldots, x_n) \in \mathbf{R}^n\,;\ \begin{array}{c} |x_1| + \cdots + |x_n| \le rn - s - 1 - (n-1)l\,, \\ |x_k| \le r \quad \text{for} \quad k = 1, \ldots, i-1, i+1, \ldots, n \end{array}\right\}$$

and the linear space

$$\langle \mathcal{K}'_{r,i} \rangle := \bigoplus_{\lambda \in \mathcal{K}'_{r,i} \cap \mathbf{Z}^n} \mathbf{C} z^\lambda .$$

If $\lambda \in \mathcal{K}'_{r,i} \cap \mathbf{Z}^n$, then $\lambda + \Delta(b_i^\pm) \subset \mathcal{K}_r$, so that we have the finite map

$$D_i : \langle \mathcal{K}'_{r,i} \rangle \longrightarrow \langle \mathcal{K}_r \rangle$$

where the domain L of D_i is restricted to $\langle \mathcal{K}'_{r,i} \rangle$. When $a_k^{-1}, t_j^{-1} \to 0$ for all k, j, from (29), (30), (31) and (32), $D_i z^\lambda$ is expressed as

$$\lim_{a_k^{-1}, t_j^{-1} \to 0} D_i z^\lambda = z^\lambda\big(z_i^{-s-1-(n-1)l} - (-1)^m q^{\lambda_i} z_i^{s+1+(n-1)l}\big). \tag{33}$$

This implies that for an arbitrary element $\varphi(z) \in \langle \mathcal{K}_r \rangle$, there exist $\psi_i(z) \in \langle \mathcal{K}'_{r,i} \rangle$ such that the Newton polyhedron of

$$\varphi(z) - \sum_{i=1}^{n} D_i \psi_i(z)$$

contained in a subdomain of $\mathcal{K}_r$ such that

$$-s - 1 - (n-1)l \le x_i \le s + (n-1)l \quad \text{for} \quad i = 1, 2, \ldots, n\,,$$

so that

$$\varphi(z) - \sum_{i=1}^{n} D_i \psi_i(z) \in \mathcal{B}$$

where $\mathcal{B}$ is the linear subspace defined by

$$\mathcal{B} := \bigoplus_{\lambda \in Q} \mathbf{C} z^{\lambda} .$$

Therefore, we obtain the following:

LEMMA 2.4. *The finite map*

$$\begin{aligned} \mathcal{B} \oplus \bigoplus_{i=1}^{n} \langle \mathcal{K}'_{r,i} \rangle &\longrightarrow \langle \mathcal{K}_r \rangle \\ (\varphi(z), \psi_1(z), \dots, \psi_n(z)) &\longmapsto \varphi(z) + D_1\psi_1(z) + \cdots + D_n\psi_n(z) \end{aligned}$$

is a surjection, provided a_k^{-1}, t_j^{-1} *all are sufficiently close to* 0, *a fortiori provided they are generic.*

Since $L = \bigcup_{r \geq 0} \mathcal{K}_r$, we have the following as an immediate consequence of Lemma 2.4.

LEMMA 2.5. *The map* $\mathcal{B} \oplus L^n \longrightarrow L$,

$$(\varphi(z), \psi_1(z), \dots, \psi_n(z)) \longmapsto \varphi(z) + D_1\psi_1(z) + \cdots + D_n\psi_n(z)$$

is surjective. In other words the following map is surjective:

$$\begin{aligned} \mathcal{N} : \ \mathcal{B} &\longrightarrow L/(D_1L + \cdots + D_nL) \\ \varphi(z) &\longmapsto \varphi(z) \bmod (D_1L + \cdots + D_nL) . \end{aligned}$$

In particular, $\quad \dim L/(D_1L + \cdots + D_nL) \leq \tilde{\kappa}$.

PROPOSITION 2.6. *The map* $\mathcal{N}$ *gives the canonical isomorphism*

$$\mathcal{B} \stackrel{\sim}{\rightarrow} L/(D_1L + \cdots + D_nL) .$$

PROOF. We want to prove that the map $\mathcal{N}$ is bijective. In order to prove it, it is sufficient to show $\dim L/(D_1L + \cdots + D_nL) = \tilde{\kappa}$. We consider the map $\mathcal{M}|_L : L \to \mathcal{H}$ by restricting the map $\mathcal{M}$ to L, where $\mathcal{M}$ is defined by (23). From $\dim \mathcal{H} = \tilde{\kappa}$ and Lemma 2.2, $\mathcal{M}|_L$ is surjective. From Lemma 1.5, the subspace $D_1L + \cdots + D_nL$ of L is included in the kernel of the map $\mathcal{M}|_L$, so that

$$\dim L/(D_1L + \cdots + D_nL) \geq \dim L/\ker \mathcal{M}|_L = \dim \mathcal{H} = \tilde{\kappa} .$$

This completes the proof. □

REMARK 2.6.1. By Proposition 2.6, we have $\dim \mathcal{B} = \dim \mathcal{H}$. This means that the composition of the surjective maps

$$\begin{array}{ccccc} \mathcal{B} & \longrightarrow & H & \xrightarrow{\overline{\mathcal{M}}} & \mathcal{H} \\ \varphi(z) & \longmapsto & \varphi(z) \bmod \nabla_q(R^n) & \longmapsto & \mathcal{M}\varphi(z) \end{array}$$

gives the canonical isomorphism $\mathcal{B} \xrightarrow{\sim} \mathcal{H}$, and $\mathcal{B}$ is embedded in H. We will see $\mathcal{B} \cong H$ later.

Since

$$\frac{T_{a_k}\Phi(z)}{\Phi(z)} = \prod_{i=1}^{n}(1-a_k^{-1}z_i)(1-a_kz_i)/z_i \ \in L\,, \tag{34}$$

$$\frac{T_{t_j}\Phi(z)}{\Phi(z)} = \prod_{1\le i<i'\le n}(1-t_j^{-1}z_i/z_{i'})(1-t_jz_i/z_{i'})(1-t_j^{-1}z_iz_{i'})(1-t_jz_iz_{i'})/z_i^2 \in L\,, \tag{35}$$

the q-shifts $a_k \to a_kq,\ t_j \to t_jq$ give rise to the following maps from L into L and hence from R into R:

$$\widehat{T}_{a_k}: \varphi(z) \longmapsto \frac{T_{a_k}\Phi(z)}{\Phi(z)}T_{a_k}\varphi(z)\,, \quad \widehat{T}_{t_j}: \varphi(z) \longmapsto \frac{T_{t_j}\Phi(z)}{\Phi(z)}T_{t_j}\varphi(z) \tag{36}$$

over the coefficients of rational functions of $a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_l$. Let us number, as a basis of $\mathcal{B}$, the set of monomials $z^\lambda,\ \lambda \in Q$. Then, from Proposition 2.6, we have the following unique expressions on $L/(D_1L + \cdots + D_nL)$.

$$\widehat{T}_{a_k}z^\lambda \equiv \sum_{\nu\in Q} z^\nu \eta_{\lambda,\nu}^{(a_k)} \bmod (D_1L+\cdots+D_nL) \quad 1\le k\le m\,, \tag{37}$$

$$\widehat{T}_{t_j}z^\lambda \equiv \sum_{\nu\in Q} z^\nu \eta_{\lambda,\nu}^{(t_j)} \bmod (D_1L+\cdots+D_nL) \quad 1\le j\le l\,, \tag{38}$$

where $\mathcal{Y}_{a_k} = (\eta_{\lambda,\nu}^{(a_k)}), \mathcal{Y}_{t_j} = (\eta_{\lambda,\nu}^{(t_j)})$ denote square matrices of degree $\tilde{\kappa}$ whose entries are rational functions of $a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_l$ respectively.

LEMMA 2.7. *$\mathcal{Y}_{a_k}, \mathcal{Y}_{t_j}$ are all invertible.*

PROOF. Consider the asymptotic behaviours of the matrices $\mathcal{Y}_{a_k}, \mathcal{Y}_{t_j}$ for $a_k, t_j \to +\infty$ for all k, j. Since

$$\frac{T_{a_k}\Phi(z)}{\Phi(z)} \sim (-1)^n a_k^n\,, \quad \frac{T_{t_j}\Phi(z)}{\Phi(z)} \sim t_j^{n(n-1)}\,,$$

from (34) and (35), we have

$$\eta_{\lambda,\nu}^{(a_k)} \sim (-1)^n a_k^n \delta_{\lambda\nu}\,, \quad \eta_{\lambda,\nu}^{(t_j)} \sim t_j^{n(n-1)}\delta_{\lambda\nu}$$

respectively where $\delta_{\lambda\nu}$ denotes the Kronecker delta. This implies that neither $\det\mathcal{Y}_{a_k}$ nor $\det\mathcal{Y}_{t_j}$ vanishes identically, i.e., the matrices $\mathcal{Y}_{a_k}, \mathcal{Y}_{t_j}$ are invertible. □

PROPOSITION 2.8. *R is written as $R = \mathcal{B} + \nabla_q(R^n)$. In particular,* $\dim H \le \tilde{\kappa}$.

PROOF. We can define $\widehat{T}_{a_k}^-, \widehat{T}_{t_j}^-$ from R into itself as

$$\widehat{T}_{a_k}^-\varphi(z) := \frac{T_{a_k}^-\Phi(z)}{\Phi(z)}T_{a_k}^-\varphi(z)\,, \quad \widehat{T}_{t_j}^-\varphi(z) := \frac{T_{t_j}^-\Phi(z)}{\Phi(z)}T_{t_j}^-\varphi(z)\,.$$

From Lemma 2.7 and the identities

$$\widehat{T}^-_{a_k}\nabla_q = \nabla_q \widehat{T}^-_{a_k}\,, \quad \widehat{T}^-_{t_j}\nabla_q = \nabla_q \widehat{T}^-_{t_j}\,,$$

$\widehat{T}^-_{a_k}$, $\widehat{T}^-_{t_j}$ are uniquely representable as

$$\widehat{T}^-_{a_k} z^\lambda \equiv \sum_{\nu\in Q} z^\nu \tilde{\eta}^{(a_k)}_{\lambda,\nu}\,, \tag{39}$$

$$\widehat{T}^-_{t_j} z^\lambda \equiv \sum_{\nu\in Q} z^\nu \tilde{\eta}^{(t_j)}_{\lambda,\nu} \tag{40}$$

respectively for certain rational functions $\tilde{\eta}^{(a_k)}_{\lambda,\nu}$, $\tilde{\eta}^{(t_j)}_{\lambda,\nu}$ of $a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_l$. Since we have

$$R = \bigcup_{h\ge 0} \left\{ \prod_{k=1}^{s} \widehat{T}^-_{a_k} \prod_{j=1}^{l} \widehat{T}^-_{t_j} \right\}^h L\,, \tag{41}$$

Proposition 2.8 holds. □

2.3. Proof of Theorems 1.6 and 1.8. Propositions 2.3 and 2.8 show that

$$\dim H = \tilde{\kappa}$$

and that $\{z^\lambda\,;\ \lambda \in Q\}$, which is a set of generators of $\mathcal{B}$, is a basis of H, i.e.,

$$\mathcal{B} \cong L/(D_1 L + \cdots + D_n L) \cong H\,.$$

Thus Theorem 1.6 is proved. Theorem 1.8 is an immediate consequence of (37) and (38) in view of Lemma 1.5.

REMARK. Moreover, (39) and (40) show that there exist matrices $\mathcal{Y}^-_{a_k} = (\tilde{\eta}^{(a_k)}_{\lambda,\nu})$, $\mathcal{Y}^-_{t_j} = (\tilde{\eta}^{(t_j)}_{\lambda,\nu})$ which represent the shift operators $T^-_{a_k} : a_k \to a_k q^{-1}$ and $T^-_{t_j} : t \to t q^{-1}$ respectively as

$$\mathcal{Y}^-_{a_k} = (T^-_{a_k}\mathcal{Y}_{a_k})^{-1} \quad \text{and} \quad \mathcal{Y}^-_{t_j} = (T^-_{t_j}\mathcal{Y}_{t_j})^{-1}\,. \tag{42}$$

3. The dimension of $H^n_{\mathrm{sym}}(X, \Phi, \nabla_q)$

In this section, we prove Theorems 1.7 and 1.9 for symmetric cohomology case. We first remark that

LEMMA 3.1. *Let $\mathcal{H}_{\mathrm{alt}}$ be the linear subspace of $\mathcal{H}$ consisting of the functions $f(z)$ satisfying $\sigma f(z) = \mathrm{sgn}(\sigma) f(z)$. Then $\mathcal{H}_{\mathrm{alt}}$ has dimension $\kappa = \binom{s+(n-1)l}{n}$ and has a basis consisting of*

$$\mathcal{A} \prod_{i=1}^{n} z_i^{\lambda_i} \vartheta\big(-(-1)^m z_i^{m+2(n-1)l} q^{\lambda_i + \frac{m+2(n-1)l}{2}}\,; q^{m+2(n-1)l}\big) \tag{43}$$

where λ runs over the set Q_{sym}.

PROOF. Since $\mathcal{A}$ is the projection from $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{alt}}$, $\mathcal{H}_{\mathrm{alt}}$ is identified with $\mathcal{AH}$, i.e., $\mathcal{H}_{\mathrm{alt}}$ coincides with the set of functions $\mathcal{A}f(z)$ where $f(z) \in \mathcal{H}$. Hence $\mathcal{H}_{\mathrm{alt}}$ is spanned by (43). On the other hand, for each $\lambda \in Q_{\mathrm{sym}}$, (43) contains the monomial term corresponding to λ

$$\prod_{i=1}^{n} z_i^{\lambda_i} \vartheta\big(-(-1)^m z_i^{m+2(n-1)l} q^{\lambda_i+\frac{m+2(n-1)l}{2}}; q^{m+2(n-1)l}\big)$$

but not any monomial term corresponding to other $\mu \in Q_{\mathrm{sym}}$. These monomials are linearly independent in view of Remark 2.1.2. This means (43) are also linearly independent. $\square$

The following is an immediate consequence of (4), (6), (7) and the definition of the map $\mathcal{M}$ in (23):

LEMMA 3.2. *If $\sigma \in W$, then $\sigma\mathcal{M}\varphi(z) = \mathcal{M}\sigma\varphi(z)$. In particular,*

$$\mathcal{AM}\varphi(z) = \mathcal{MA}\varphi(z)\,.$$

Moreover, if $\varphi(z) \in R_{\mathrm{alt}}$, then $\mathcal{M}\varphi(z) \in \mathcal{H}_{\mathrm{alt}}$.

3.1. Proof of Theorem 1.7. Let $\mathcal{M}|_{R_{\mathrm{alt}}}$ be the map restricting the domain R to R_{alt}. From Lemma 3.2, we regard $\mathcal{M}|_{R_{\mathrm{alt}}}$ as the map from R_{alt} to $\mathcal{H}_{\mathrm{alt}}$. The map $\mathcal{M}|_{R_{\mathrm{alt}}} : R_{\mathrm{alt}} \to \mathcal{H}_{\mathrm{alt}}$ is surjective, because the image of R_{alt} by $\mathcal{M}|_{R_{\mathrm{alt}}}$ coincides with $\mathcal{H}_{\mathrm{alt}}$:

$$\begin{aligned}\mathcal{M}R_{\mathrm{alt}} &= \mathcal{MA}R \quad \text{(by (10))}\\ &= \mathcal{AM}R \quad \text{(by Lemma 3.2)}\\ &= \mathcal{AH} \quad \text{(by Proposition 2.3)}\\ &= \mathcal{H}_{\mathrm{alt}}\,.\end{aligned}$$

From Remark 2.3.1 and (11), the kernel of $\mathcal{M}|_{R_{\mathrm{alt}}}$ coincides with $\mathcal{A}\nabla_q(R^n)$ as follows:

$$\ker\mathcal{M}|_{R_{\mathrm{alt}}} = \ker\mathcal{M} \cap R_{\mathrm{alt}} = \nabla_q(R^n) \cap R_{\mathrm{alt}} = \mathcal{A}\nabla_q(R^n)\,.$$

Thus we have the canonical isomorphism

$$H_{\mathrm{sym}} = R_{\mathrm{alt}}/\mathcal{A}\nabla_q(R^n) = R_{\mathrm{alt}}/\ker\mathcal{M}|_{R_{\mathrm{alt}}} \xrightarrow{\sim} \mathcal{M}R_{\mathrm{alt}} = \mathcal{H}_{\mathrm{alt}}$$

and we therefore obtain $\dim H_{\mathrm{sym}} = \dim \mathcal{H}_{\mathrm{alt}} = \kappa$ from Lemma 3.1.

Next we show that we can take the alternating sums $\mathcal{A}z^{\nu}$, $\nu \in Q_{\mathrm{sym}}$ as a basis of H_{sym}. Indeed, when $a_k^{-1}, t_j^{-1} \to 0$, from (33) we have by alternation,

$$\begin{aligned}\lim_{a_k^{-1}, t_j^{-1} \to 0} \mathcal{A}\, D_1 z^{\lambda} &= \mathcal{A}\big[z^{\lambda}\big(z_1^{-s-1-(n-1)l} - (-1)^m q^{\lambda_1} z_1^{s+1+(n-1)l}\big)\big]\\ &= \mathcal{A}z_1^{-s-1-(n-1)l+\lambda_1} z_2^{\lambda_2}\cdots z_n^{\lambda_n} - q^{\lambda_1}\mathcal{A}z_1^{s+1+(n-1)l+\lambda_1} z_2^{\lambda_2}\cdots z_n^{\lambda_n} \qquad (44)\end{aligned}$$

provided that $s+1+(n-1)l+\lambda_1, \lambda_2, \ldots, \lambda_n$ are different from each other and that they satisfy

$$|\lambda_2|, \ldots, |\lambda_n| \le |s+1+(n-1)l+\lambda_1| .$$

The highest terms on the right-hand side of (44) are

$$q^{\lambda_1}\mathcal{A}z_1^{s+1+(n-1)l+\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n}, \quad \mathcal{A}z_1^{s+1+(n-1)l-\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n}, \quad 2\mathcal{A}z_1^{s+1+(n-1)l}z_2^{\lambda_2}\cdots z_n^{\lambda_n},$$

according as $\lambda_1 > 0, \lambda_1 < 0, \lambda_1 = 0$ respectively. Hence an arbitrary skew-symmetric Laurent polynomial reduces to $\mathcal{A}z^{\nu}$, $\nu \in Q_{\mathrm{sym}}$ modulo $\mathcal{A}(D_1L+\cdots+D_nL)$. Thus Theorem 1.7 is proved.

3.2. Proof of Theorem 1.9. $\widehat{T}_{a_k}, \widehat{T}_{t_j}, \widehat{T}_{a_k}^{-}, \widehat{T}_{t_j}^{-}$ preserve R_{alt}, and we have the q-difference equations with respect to the basis $\mathcal{A}z^{\nu}$, $\nu \in Q_{\mathrm{sym}}$. Hence Theorem 1.9 holds.

REMARK 3.2.1. Y_{a_k}, Y_{t_j} being restrictions of $\mathcal{Y}_{a_k}, \mathcal{Y}_{t_j}$ to H_{sym} respectively, neither $\det Y_{a_k}$ nor $\det Y_{t_j}$ vanishes identically. In other words, Y_{a_k} and Y_{t_j} are all non-singular and $Y_{a_k}^{-}, Y_{t_j}^{-}$ can be defined similarly as in (39), (40) or (42).

4. Special symmetric cases

Let $\mathcal{H}_{\mathrm{sym}}$ be the linear space of holomorphic functions $f(z)$ on X satisfying $\sigma f(z) = f(z)$ and $T_{z_i}f(z) = (-1)^m q^{-\frac{m}{2}-(n-1)l+n+1} z_i^{-m-2(n-1)l+2(n+1)} f(z)$ for $1 \le i \le n$. Since an arbitrary $f(z) \in \mathcal{H}_{\mathrm{alt}}$ has the factor $\theta_{\mathrm{alt}}(z)$, $\mathcal{H}_{\mathrm{alt}}$ and $\mathcal{H}_{\mathrm{sym}}$ are isomorphic by the map $\theta_{\mathrm{alt}} : \mathcal{H}_{\mathrm{sym}} \to \mathcal{H}_{\mathrm{alt}}\,;\ f(x) \mapsto f(x)\theta_{\mathrm{alt}}(z)$. We consider the map

$$\begin{aligned} \mathcal{M}_{\mathrm{sym}} : R_{\mathrm{alt}} &\longrightarrow \quad \mathcal{H}_{\mathrm{sym}} \\ \varphi(z) &\longmapsto \mathcal{M}_{\mathrm{sym}}\varphi(z) := \frac{\langle \varphi, z\rangle}{\Theta(z)\theta_{\mathrm{alt}}(z)}, \end{aligned}$$

which is well-defined by (4), (7), (8) and Lemma 1.3. Since $\mathcal{M}_{\mathrm{sym}}$ is regarded as composition of the maps

$$R_{\mathrm{alt}} \xrightarrow{\mathcal{M}} \mathcal{H}_{\mathrm{alt}} \xrightarrow{\theta_{\mathrm{alt}}^{-1}} \mathcal{H}_{\mathrm{sym}},$$

we see that $\mathcal{M}_{\mathrm{sym}}$ naturally induces the isomorphism $H_{\mathrm{sym}} \xrightarrow{\sim} \mathcal{H}_{\mathrm{sym}}$.

Let $\phi_\lambda(z) := \mathcal{A}z^{\lambda}$ throughout this section. The set $\{\phi_\lambda(z)\,;\ \lambda \in Q_{\mathrm{sym}}\}$ is a basis of $H_{\mathrm{sym}} = H^n_{\mathrm{sym}}(X, \Phi, \nabla_q)$. Using the map $\mathcal{M}_{\mathrm{sym}}$, Eqs. (14) and (15) in Theorem 1.9 are rewritten as the equations in $\mathcal{H}_{\mathrm{sym}}$ as follows:

$$T_{a_k}\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\xi) = \sum_{\nu \in Q_{\mathrm{sym}}} \bar{y}^{(a_k)}_{\lambda,\nu}\mathcal{M}_{\mathrm{sym}}\phi_\nu(\xi),$$

$$T_{t_j}\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\xi)=\sum_{\nu\in Q_{\mathrm{sym}}}\bar{y}^{(t_j)}_{\lambda,\nu}\mathcal{M}_{\mathrm{sym}}\phi_\nu(\xi)$$

and $\overline{Y}_{a_k}:=\big(\bar{y}^{(a_k)}_{\lambda,\nu}\big)$, $\overline{Y}_{t_j}:=\big(\bar{y}^{(t_j)}_{\lambda,\nu}\big)$ denote square matrices of degree $\kappa=\binom{s+(n-1)l}{n}$ whose entries are rational functions of $a_1,a_2,\dots,a_m,t_1,t_2,\dots,t_l$ respectively.

As we have seen in the proof of the isomorphism $H\xrightarrow{\sim}\mathcal{H}$, the following two facts are also essential for the proof of isomorphism $H_{\mathrm{sym}}\xrightarrow{\sim}\mathcal{H}_{\mathrm{sym}}$. One is that $\overline{Y}_{a_k}$, $\overline{Y}_{t_j}$ are invertible, i.e., $\det\overline{Y}_{a_k}$, $\det\overline{Y}_{t_j}$ do not vanish identically. The other is that $\mathcal{M}_{\mathrm{sym}}\phi_\lambda(z)$, $\lambda\in Q_{\mathrm{sym}}$ are linearly independent in $\mathcal{H}_{\mathrm{sym}}$, i.e., there exist κ points $\zeta_{(\mu)}$ in X such that $\det\big(\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})\big)_{\lambda,\mu}$ does not vanish identically.

In this section, we mention more concrete results about them when $l=0$ and 1.

4.1. Symmetric case where $l=0$. In this case, $H^n_{\mathrm{sym}}(X,\Phi,\nabla_q)$ has dimension $\kappa=\binom{s}{n}$. We have already seen in Remark 3.2.1 that $\det Y_{a_k}$ does not vanish identically, and the explicit form of $\det Y_{a_1}$ is actually given in [7] as follows:

$$\det Y_{a_1}=(-a_1)^{-n\binom{s}{n}}\left(\frac{\prod_{k=2}^{2s+2}(1-a_1a_k)}{1-a_1a_2\cdots a_{2s+2}}\right)^{\binom{s-1}{n-1}}.$$

The parameters $a_1,a_2,\dots,a_{2s+2}$ can be replaced symmetrically in the above formula. According to the following theorem, we see directly that $\det\overline{Y}_{a_k}$ and $\det\big(\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})\big)_{\lambda,\mu}$ do not vanish identically:

THEOREM 4.1. *The explicit form of* $\det\overline{Y}_{a_1}$ *is given by*

$$\det\overline{Y}_{a_1}=\left(\frac{\prod_{k=2}^{2s+2}(1-a_1a_k)}{1-a_1a_2\cdots a_{2s+2}}\right)^{\binom{s-1}{n-1}}.$$

The parameters $a_1,a_2,\dots,a_{2s+2}$ can be replaced symmetrically in the above formula. Moreover, the $\kappa\times\kappa$ determinant with (λ,μ) entry $\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})$ is evaluated as

$$\det\big(\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})\big)_{\lambda,\mu}=(q;q)_\infty^{n\binom{s}{n}}\left(\frac{\prod_{1\le i<j\le 2s+2}(qa_i^{-1}a_j^{-1};q)_\infty}{(qa_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1};q)_\infty}\right)^{\binom{s-1}{n-1}}$$
$$\times\left(\prod_{1\le i<j\le s}\frac{\vartheta(a_i/a_j;q)\vartheta(a_ia_j;q)}{a_i\,(q;q)_\infty^2}\right)^{\binom{s-2}{n-1}}$$

where $\zeta_{(\mu)}:=(a_{\mu_1},a_{\mu_2},\dots,a_{\mu_n})\in X$ for $\mu=(\mu_1,\mu_2,\dots,\mu_n)\in Q_{\mathrm{sym}}$.

PROOF. See [7, 23]. See also [24] for another simple proof. □

REMARK 4.1.1. When $m=2n+2$, i.e., $s=n$, the above determinant, whose matrix size $\binom{s}{n}$ equals 1, becomes nothing but the formula investigated by Gustafson [15]. See also [22].

4.2. Symmetric case where $l = 1$. We shall simply write t in place of t_1. In this case, $H^n_{\mathrm{sym}}(X, \Phi, \nabla_q)$ has dimension $\kappa = \binom{s+n-1}{n}$. The explicit form of $\det Y_{a_1}$ is given in [8] as follows:

$$\det Y_{a_1} = (-a_1)^{-n\binom{s+n-1}{n}} \prod_{j=1}^{n} \left(\frac{\prod_{k=2}^{2s+2}(1 - t^{n-j}a_1a_k)}{1 - t^{n+j-2}a_1a_2\cdots a_{2s+2}} \right)^{\binom{s+j-2}{j-1}},$$

so that we have the following which implies that $\det \overline{Y}_{a_k}$ does not vanish identically:

THEOREM 4.2. *The explicit form of* $\det \overline{Y}_{a_k}$ *is given by*

$$\det \overline{Y}_{a_1} = \prod_{j=1}^{n} \left(\frac{\prod_{k=2}^{2s+2}(1 - t^{n-j}a_1a_k)}{1 - t^{n+j-2}a_1a_2\cdots a_{2s+2}} \right)^{\binom{s+j-2}{j-1}}.$$

The parameters $a_1, a_2, \ldots, a_{2s+2}$ *can be replaced symmetrically in the above formula.*

PROOF. See [8]. □

Next we show the explicit form of $\det\big(\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})\big)_{\lambda,\mu}$ for some κ points $\zeta_{(\mu)}$ in X. In order to explain this, we choose special critical points $\zeta_{(\mu)}$ for the Jackson integrals (1) in the following manner.

Let Z be the set of all compositions of n defined by

$$Z := \{(\mu_1, \mu_2, \ldots, \mu_s) \in \mathbf{Z}^s\ ;\ \mu_1 + \mu_2 + \cdots + \mu_s = n,\ \mu_1 \geq 0, \ldots, \mu_s \geq 0\}.$$

The number of elements in Z is equal to κ. For compositions $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_s) \in Z$, we define the ordering $\mu \prec_Z \nu$ on Z if there exists i such that

$$\mu_1 = \nu_1, \quad \mu_2 = \nu_2, \ldots, \mu_{i-1} = \nu_{i-1}, \quad \mu_i < \nu_i.$$

Corresponding to the composition $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \in Z$, we take the point $(\zeta_1, \zeta_2, \ldots, \zeta_n) \in X$ satisfying

$$\begin{cases} \zeta_i = a_i & \text{if } i \in \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_s\}, \\ \zeta_j/\zeta_{j+1} = t & \text{if } j \notin \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_s\} \end{cases}$$

or equivalently

$$\zeta_i = \begin{cases} a_1 t^{\mu_1 - i} & \text{if } 1 \leq i \leq \mu_1, \\ a_2 t^{\mu_1 + \mu_2 - i} & \text{if } \mu_1 + 1 \leq i \leq \mu_1 + \mu_2, \\ & \cdots \\ a_s t^{n-i} & \text{if } \mu_1 + \cdots + \mu_{s-1} + 1 \leq i \leq n. \end{cases}$$

We denote such a point by $\zeta_{(\mu)} = (\zeta_{(\mu)1}, \zeta_{(\mu)2}, \ldots, \zeta_{(\mu)n}) \in X$. For the point $\zeta_{(\mu)} \in X$, we denote by $\Lambda^+_{\zeta_{(\mu)}}$ the fan with the vertex $\zeta_{(\mu)}$ such that

$$\left\{ \zeta_{(\mu)} q^\nu \in X\ ;\ \begin{array}{ll} \nu_i > 0 & \text{if } i \in \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_s\}, \\ \nu_j - \nu_{j+1} > 0 & \text{if } j \notin \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_s\} \end{array} \right\}.$$

Since $\Phi(\xi) = 0$ if $\xi \in \Lambda_{\zeta(\mu)} \setminus \Lambda^+_{\zeta(\mu)}$, the Jackson integral (1) over $\Lambda_{\zeta(\mu)}$ is defined only over the fan $\Lambda^+_{\zeta(\mu)}$.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in Q_{\mathrm{sym}}$, we also define the reverse lexicographic ordering $\lambda \prec \nu$ on Q_{sym} if $\lambda_1 = \nu_1,\ \lambda_2 = \nu_2, \ldots, \lambda_{i-1} = \nu_{i-1},\ \lambda_i < \nu_i$ for some $i \in \{1, 2, \ldots, n\}$.

THEOREM 4.3. *The $\kappa \times \kappa$ determinant with (λ, μ) entry $\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})$ is evaluated as*

$$(q;q)_\infty^{n\binom{s+n-1}{n}} \prod_{k=1}^{n} \left(\frac{(qt^{-(n-k+1)};q)_\infty^s}{(qt^{-1};q)_\infty^s} \frac{\prod_{1\le i<j\le 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1};q)_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1};q)_\infty} \right)^{\binom{s+k-2}{k-1}}$$
$$\times \prod_{k=1}^{n} \left(\prod_{r=0}^{n-k} \prod_{1\le i<j\le s} \frac{\vartheta(t^{2r-(n-k)}a_i a_j^{-1};q)\vartheta(t^{n-k}a_i a_j;q)}{t^r a_i\,(q;q)_\infty^2} \right)^{\binom{s+k-3}{k-1}},$$

where the rows $\lambda \in Q_{\mathrm{sym}}$ and the columns $\mu \in Z$ of the matrix $\left(\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})\right)_{\lambda,\mu}$ are arranged in decreasing orders of $\prec$ and $\prec_Z$ respectively.

PROOF. See [8]. □

As a corollary, we see $\det\left(\mathcal{M}_{\mathrm{sym}}\phi_\lambda(\zeta_{(\mu)})\right)_{\lambda,\mu}$ does not vanish identically.

REMARK 4.3.1. In the special case where $(m, l) = (4, 1)$, i.e., $(s, l) = (1, 1)$, κ is equal to 1, and the determinant reduces to Jackson integral itself which is explicitly evaluated by van Diejen [13]. See also [11, 20, 21, etc].

4.3. q-difference equations in case where $n = 1$. In the special case where $n = 1$, we have $\kappa = s$. One can write explicitly q-difference equations (14) as follows. We denote by e_k the kth elementary symmetric polynomials in $a_1, \ldots, a_{2s+2}$. We simply write $z = z_1$ and $\lambda = \lambda_1$. The q-multiplicative function $\Phi(z)$ is written as

$$\Phi(z) = q^{s+1-\delta} \prod_{k=1}^{2s+2} \frac{(qa_k^{-1}z;q)_\infty}{(a_k z;q)_\infty}.$$

Further we put $u_\lambda = \langle z^\lambda - z^{-\lambda}, \xi \rangle$. Then (14) can be written as

$$T_{a_k} u_\lambda = u_{\lambda-1} - (a_k + 1/a_k)u_\lambda + u_{\lambda+1} \quad (1 \le \lambda \le s-1), \tag{45}$$

$$T_{a_k} u_s = u_{s-1} - (a_k + 1/a_k)u_s + \sum_{\mu=1}^{s} (-1)^{s-\mu} \frac{e_{s-\mu+1} - e_{s+\mu+1}}{1 - e_{2s+2}} u_\mu, \tag{46}$$

where $u_0 = 0$. In particular when $s = 1$, the Jackson integrals of $\Phi(z)$ give Askey–Wilson integrals (see [20]), while when $s = 2$ they give, as a special case, the Stieltjes transform of Askey–Wilson polynomials (the so-called 2nd solutions). See [6].

A. Appendix

In this appendix we prove the identity (17) by counting the number of graphs associated with $\{\mu^{(1)}, \dots, \mu^{(n)}\}$.

A.1. Admissible graphs. Let $\mathcal{R}$ be the set defined by (16) which consists of linear functions associated with the function $\Phi(z)$. We shall call a system of n-tuple of functions $\{\mu^{(1)}, \dots, \mu^{(n)}\} \subset \mathcal{R}$ *admissible* if their homogeneous parts $\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}$ are linearly independent.

An admissible system represents a graph $\mathcal{G}$ with $n+1$ vertices $\{0, 1, 2, 3, \dots, n\}$ and n coloured (white or red) edges in the following sense:

- The form $\langle \varepsilon_i, x\rangle - \alpha_k$ represents a white edge with the vertices $\{0, i\}$.
- The form $\langle \varepsilon_i - \varepsilon_j, x\rangle - \tau_k$ or $\langle \varepsilon_i + \varepsilon_j, x\rangle - \tau_k$ represents a white or a red edge with the vertices i, j.

In this case we call $\mathcal{G}$ admissible. We shall denote by $V(\mathcal{G})$, $E(\mathcal{G})$ the set of vertices and the set of edges of $\mathcal{G}$ respectively. Below we use the terminologies in [33].

PROPOSITION A.1. *An admissible graph $\mathcal{G}$ is characterized as having the following properties.*

(i) *$\mathcal{G}$ consists of two disjoint subgraphs $\mathcal{G}_+, \mathcal{G}_0$ such that $0 \in V(\mathcal{G}_+)$ and $0 \notin V(\mathcal{G}_0)$.*
(ii) *$\mathcal{G}_+$ is a routed tree at the root 0.*
(iii) *Each connected component of $\mathcal{G}_0$ has only one circuit.*
(iv) *Any circuit contains an odd number of red edges.*

We also call $\mathcal{G}_+, \mathcal{G}_0$ admissible. In this case, the absolute value of the determinant $\left|[\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]\right|$ equals 2^r where r denotes the number of connected components of $\mathcal{G}_0$.

Proposition A.1 is an immediate consequence of Lemmas A.2–A.5, which we present below. By assumption, all the homogeneous linear functions $\bar{\mu}$ corresponding to the edges in $E(\mathcal{G})$ are linearly independent and different from each other. Hence, by abuse of notation, we may identify an edge $\{0, i\}$ or $\{i, j\}$ with the corresponding homogeneous linear function ε_i or $\varepsilon_i \pm \varepsilon_j$. For the set of edges $E(S)$ of a subgraph S we denote by $\langle E(S)\rangle$ the linear space spanned by the edges in $E(S)$.

First note that $\mathcal{G}$ has no loop (with one vertex and one edge).

LEMMA A.2. *Every connected subgraph of $\mathcal{G}$ is admissible. Every connected graph S obtained from $\mathcal{G}$ is admissible by deleting some edges.*

LEMMA A.3. *Assume that S is a circuit in $\mathcal{G}$. S is admissible if and only if the number of red edges are odd.*

PROOF. Let S be a circuit with p vertices $i_1, i_2, \dots, i_p$ and p edges $\{i_1, i_2\}, \dots, \{i_{p-1}, i_p\}, \{i_p, i_1\} (p \geq 2)$. Then the homogeneous functions $\{i_1, i_2\}, \dots, \{i_{p-1}, i_p\}$ are linearly independent. For any point satisfying

$$\{i_k, i_{k+1}\} = 0 \quad (1 \leq k \leq p-1)$$

the function $\{i_p, i_1\}$ is equal to 0 or $\pm 2\varepsilon_{i_1}$ accordingly as the number of red edges being even or odd. The former case implies that the edges are linearly dependent. The latter case implies that the edges are linearly independent and ε_{i_k} $(1 \le k \le p)$ is a linear combination of $\{i_l, i_{l+1}\}$ $(1 \le l \le p-1)$ and $\{i_p, i_1\}$:

$$\varepsilon_{i_k} \equiv 0 \mod \langle E(S) \rangle . \tag{47}$$

Lemma A.3 is proved. □

LEMMA A.4. *If $\mathcal{G}_+$ is a connected component of $\mathcal{G}$ including the vertex* 0, *then $\mathcal{G}_+$ has no circuit.*

PROOF. Suppose on the contrary that $\mathcal{G}_+$ has a circuit S. From Lemma A.3 we may assume that $E(S)$ has an odd number of red edges. Then there exists a vertex i of S and a path S' connecting 0 and i such that $E(S) \cap E(S') = \emptyset$. We have

$$\varepsilon_i \equiv 0 \mod \langle E(S') \rangle .$$

From (47), we have

$$\varepsilon_i \equiv 0 \mod \langle E(S) \rangle .$$

These two equalities imply that the edges in $E(S \cup S')$ are linearly dependent. This is a contradiction. Lemma A.4 is proved. □

LEMMA A.5. *Every connected component of $\mathcal{G}_0$ has only one circuit.*

PROOF. Suppose that a connected component of $\mathcal{G}_0$ has no circuit, i.e., it is a tree. Then the number of vertices would be greater than the number of edges. Therefore the same would be true for $\mathcal{G}_0$, which is a contradiction, because the cardinality of $V(\mathcal{G}_0)$ is equal to the cardinality of $E(\mathcal{G}_0)$.

Next we suppose that $\mathcal{G}_0$ has two circuits, say S, S'. We may assume that each circuit has an odd number of red edges. We consider the following three cases.

Case 1: the case where the cardinality $|V(S) \cap V(S')| \ge 2$. Let i, j be two vertices in $V(S) \cap V(S')$. Both S, S' are divided into two disjoint paths S_1, S_2 and S'_1, S'_2 respectively which connect i and j. Then we may assume

$$\varepsilon_i + \varepsilon_j \equiv 0 \mod \langle E(S_1) \rangle , \quad \varepsilon_i - \varepsilon_j \equiv 0 \mod \langle E(S_2) \rangle ,$$

because one of S_1, S_2 has odd red edges and the other has even red edges. Likewise we have

$$\varepsilon_i + \varepsilon_j \equiv 0 \mod \langle E(S'_1) \rangle , \quad \varepsilon_i - \varepsilon_j \equiv 0 \mod \langle E(S'_2) \rangle .$$

This means that neither $E(S_1 \cup S'_1)$ nor $E(S_2 \cup S'_2)$ is admissible. This is a contradiction.

Case 2: the case where $|V(S) \cap V(S')| = 1$. Let $i \in V(S) \cap V(S')$. Then $E(S) \cap E(S') = \emptyset$ and

$$\varepsilon_i \equiv 0 \mod \langle E(S) \rangle , \quad \varepsilon_i \equiv 0 \mod \langle E(S') \rangle .$$

Hence $S \cup S'$ is not admissible, which is a contradiction.

Case 3: the case where $V(S) \cap V(S') = \emptyset$. There exist two vertices $i \in V(S)$, $j \in V(S')$ and a path S'' connecting i, j such that $E(S) \cap E(S'') = E(S') \cap E(S'') = \emptyset$. Then we have

$$\varepsilon_i \equiv 0 \mod \langle E(S) \rangle\,, \quad \varepsilon_j \equiv 0 \mod \langle E(S') \rangle\,, \quad \varepsilon_i + \varepsilon_j \text{ or } \varepsilon_i - \varepsilon_j \equiv 0 \mod \langle E(S'') \rangle$$

respectively. Then $S \cup S' \cup S''$ is not admissible. This is again a contradiction. Lemma A.5 is proved. □

We denote by $G(n)$ the number

$$G(n) := \sum_{\mathcal{G}:\text{admissible}} [\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2\,.$$

THEOREM A.6. $G(n) = \{m + 2(n-1)l\}^n\,, \quad n = 0, 1, 2, \dots$

We put $G(0) = 1$ and consider the generating function

$$\widehat{G}(t) = 1 + \sum_{n=1}^{\infty} \frac{G(n)}{n!} t^n\,.$$

Theorem A.6 is then equivalent to

$$\widehat{G}(t) = \sum_{n=0}^{\infty} \frac{\{m + 2(n-1)l\}^n}{n!} t^n\,.$$

A.2. Case where $m = 0, l = 1$**.** First we assume that $m = 0, l = 1$ so that $V(\mathcal{G}_+) = \{0\}$, $E(\mathcal{G}_+) = \emptyset$ and $E(\mathcal{G}) = E(\mathcal{G}_0)$. Remark that in this case $G(0) = 1$, $G(1) = 0$.

We denote by $g(n)$ the number of all connected admissible graphs $\mathcal{G}_0$ with n white or red edges and n vertices such that $\left|[\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]\right| = 2$. Then

LEMMA A.7.

$$g(n) = 2^{n-2}(n-1)! \sum_{k=0}^{n-2} \frac{n^k}{k!}\,.$$

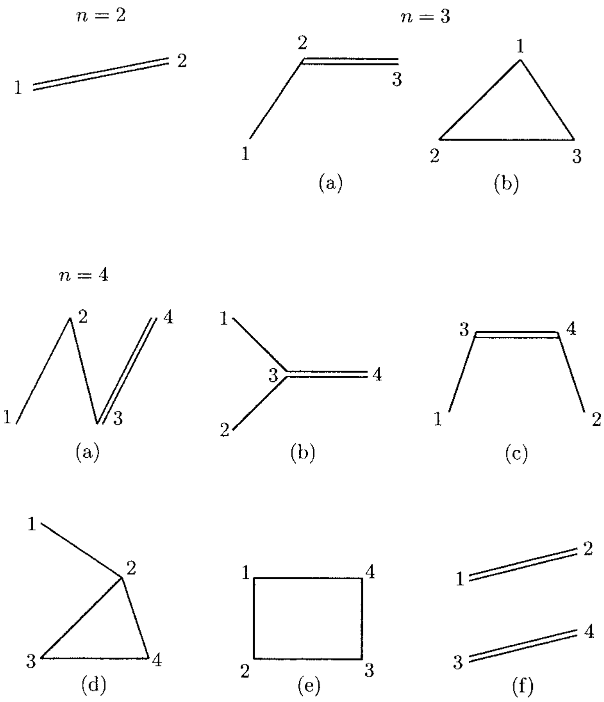
PROOF. We can apply to this case Cayley formula for counting trees with labelled vertices and prove that, for $n \geq r \geq 2$, the number of connected admissible graphs with one r-polygon and n vertices equals

$$2^{n-2}(n-1)(n-2)\cdots(n-r+1)n^{n-r}\,.$$

Hence we have

$$\sum_{\substack{\mathcal{G}_0:\,\text{connected}\\ \text{admissible}}} [\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2 = 2^2 g(n)\,.$$ □

REMARK A.7.1. The graphical figures $\mathcal{G}_0$ for small n are given below.



Theorem A.6 can be derived from the following proposition. Indeed, it is a standard fact that the generating function $\widehat{G}(t)$ of all admissible graphs is obtained from that of connected admissible graphs as follows (see [16, Theorem 1.2]):

$$\widehat{G}(t) = \exp\left[\sum_{k=2}^{\infty} \frac{g(k)}{k!} 2^2 t^k\right]. \tag{48}$$

PROPOSITION A.8.

$$\widehat{G}(t) = 1 + \sum_{n=2}^{\infty} \frac{2^n (n-1)^n}{n!} t^n .$$

PROOF. w satisfying the functional equation $w = ze^w$ has the power series expansion in z

$$w = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} z^n \tag{49}$$

or equivalently e^w has the expansion

$$e^w = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} z^k .$$

(See [30, Part 3, Problem 209].) Moreover we have

$$\begin{aligned} w^k &= k \sum_{n=k}^{\infty} \frac{n^{n-k-1}}{(n-k)!} z^n, \quad k = 1, 2, 3, \dots \\ e^{\lambda w} &= \sum_{n=0}^{\infty} \frac{\lambda (n+\lambda)^{n-1}}{n!} z^n \end{aligned} \tag{50}$$

for an arbitrary $\lambda \in \mathbf{R}$.

Let L be a differential operator of infinite order relative to λ, i.e., $L := \sum_{r=1}^{\infty} \frac{d^r}{d\lambda^r}$. Then from (50) we have

$$L\left[\frac{e^{\lambda w} - 1}{\lambda}\right]_{\lambda=0} = \sum_{n=2}^{\infty} \sum_{r=1}^{n-1} \frac{n^{n-r-1}}{n(n-r-1)!} z^n = \sum_{k=2}^{\infty} \frac{g(k)}{k!} 2^{2-k} z^k .$$

On the other hand, the left-hand side equals

$$-w - \log(1-w) = \sum_{n=2}^{\infty} \frac{w^n}{n} .$$

Hence we have

$$\exp\left[\sum_{k=2}^{\infty} \frac{g(k)}{k!} 2^{2-k} z^k\right] = e^{-w}(1-w)^{-1} .$$

One can see further that

$$\begin{aligned} e^{-w}(1-w)^{-1} = e^{-w}(1-ze^w)^{-1} &= \sum_{k=0}^{\infty} z^k e^{(k-1)w} \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{(k-1)(h+k-1)^{h-1}}{h!} z^{h+k} \\ &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \frac{(k-1)(n-1)^{n-k-1}}{(n-k)!} \\ &= \sum_{n=0}^{\infty} z^n \frac{(n-1)^n}{n!} . \end{aligned}$$

By the substitution $z = 2t$ and using (48), we conclude Proposition A.8. □

Thus Theorem A.6 is proved for the case $m = 0, l = 1$.

A.3. General case. Suppose first that $\mathcal{G}_+$ has $r+1$ vertices including $\{0\}$ and j adjacent vertices to 0 for $r \geq j \geq 1$ respectively so that $\mathcal{G}_0$ has $n-r$ vertices and edges. By applying Cayley counting formula for trees, the sum of $[\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2$ over the set of such graphs $\mathcal{G}_+$ equals

$$\sum_{j=1}^{r} m^j (2l)^{r-j} \sum_{\nu} \left(\prod_{i=1}^{j} \frac{\nu_i^{\nu_i - 1}}{\nu_i!} \right)$$

where the sum of $\nu = (\nu_1, \nu_2, \dots, \nu_j)$ is taken over the set

$$\nu_1 + \cdots + \nu_j = r\,, \quad r \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_j \geq 1\,.$$

This is equal to the coefficient of the term t^r in w^j by substitution $z = 2lt$

$$\sum_{j=1}^{r} m^j (2l)^{r-j} \frac{w^j}{j!}\,,$$

i.e., the coefficient of the term t^r in $e^{\frac{m}{2l}w}$ where w is defined as in (49).

On the other hand, as is obtained in the preceding section, we have

$$e^{-w}(1-w)^{-1} = \sum_{n=r}^{\infty} \frac{(n-r-1)^{n-r}}{(n-r)!} z^{n-r} = \sum_{n=r}^{\infty} \frac{\{2(n-r-1)l\}^{n-r}}{(n-r)!} t^{n-r}$$

if we put $z = 2lt$. Note that the sum of $[\bar{\mu}^{(1)}, \dots, \bar{\mu}^{(n)}]^2$ over the set of graphs $\mathcal{G}_0$ is equal to $\{2(n-r-1)l\}^{n-r}$, since each edge of $\mathcal{G}_0$ admits l choices of linear functions $\langle \varepsilon_i \pm \varepsilon_j, x\rangle - \tau_k$, $(1 \leq k \leq l)$.

Finally we have $\frac{n!}{r!(n-r)!}$ choices of vertices of $\mathcal{G}_+, \mathcal{G}_0$. Hence by putting $z = 2lt$, we have

$$\begin{aligned}
\widehat{G}(t) &= e^{(\frac{m}{2l}-1)w}(1-w)^{-1} = e^{(\frac{m}{2l}-1)w}(1-ze^{w})^{-1} = \sum_{r=0}^{\infty} z^r e^{(\frac{m}{2l}+r-1)w} \\
&= \sum_{r=0}^{\infty} z^r \sum_{j=0}^{r} \frac{(\frac{m}{2l}+j-1)(\frac{m}{2l}+r-1)^{r-j-1}}{(r-j)!} \\
&= \sum_{r=0}^{\infty} t^r \frac{\{m+2(r-1)l\}^r}{r!}\,.
\end{aligned}$$

Theorem A.6 is completely proved.

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