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AH-substitution and Markov Partition of a Group Automorphism on T^d

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Abstract. The existence of a Markov partition of a hyperbolic group automorphism generated by an integral matrix with determinant ± 1 is established by Sinai (see [22]). After that, there are many articles to construct Markov partitions of group automorphisms generated by non-negative matrices satisfying Pisot condition by the tiling method from substitutions (see [1], [7], [16], [19], [5]). One of the purpose of this paper is to establish the construction method of a Markov partition for a group automorphism generated by a non-positive matrix satisfying "negative Pisot" condition. An anti-homomorphic extension of a substitution, called AH-substitution, is introduced in the paper. Owing to this new substitution, the Markov partition of the group automorphism from the non-positive integral matrix is constructed.

1. Introduction

A substitution σ is a mapping from a (finite) alphabet $\mathcal{A}$ with d letters to the free monoid $\mathcal{A}^*$ which consists of finite words by the letters of $\mathcal{A}$. For each $j \in \mathcal{A}$, we note $\sigma(j) = W_1^{(j)} \cdots W_k^{(j)} \cdots W_{l_j}^{(j)}$ ($W_k^{(j)} \in \mathcal{A}$), where l_j (> 0) is the length of $\sigma(j)$. A substitution extends to mappings on $\mathcal{A}^*$ in two natural ways, that is, homomorphically and anti-homomorphically; its extensions are called H-substitution and AH-substitution, respectively in this paper. An H-substitution, that is, a substitution in the usual sense has been studied by many articles (see [3], [5], [9], [15], [17], [21], [23]), and many remarkable applications have been obtained for unimodular Pisot substitutions recently (see [2], [4], [13], [14], [16]). The following are the main parts of them.

(1) The existence of the set equations for the partial atomic surfaces, that is, there exists a collection of compact sets $\{X'_1, \ldots, X'_d\}$ with fractal boundaries and positive measure on the L_σ-invariant stable subspace W^s such that

$$L_\sigma^{-1} X'_i = \bigcup_{\binom{j}{k}: W_k^{(j)} = i} \left(X'_j + L_\sigma^{-1}(\pi_s(f(P_k^{(j)}))) \right),$$

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where L_σ, π_s, f and $P_k^{(j)}$ are the incidence matrix of σ, a projection of $\mathbf{R}^d$ to W^s, a canonical homomorphism and the prefix of $\sigma(j)$, respectively (see section 2 for detail).

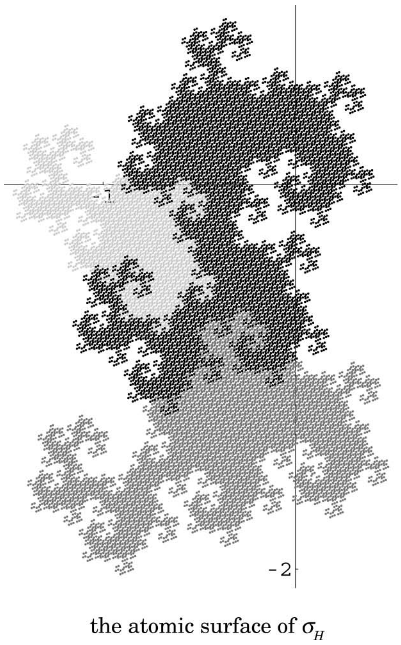
(2) The existence of a quasi-periodic tiling $\mathfrak{T}'$ of W^s with the protoset $\{X'_1, \ldots, X'_d\}$ and that of a tiling substitution $E_1(\sigma)^*$ on $\mathfrak{T}'$ given by

$$E_1(\sigma)^*(\pi_s(\boldsymbol{x}) + X'_i) = L_\sigma^{-1}\pi_s(\boldsymbol{x}) + \sum_{\binom{j}{k}:W_k^{(j)}=i} \left(X'_j + L_\sigma^{-1}\pi_s(f(P_k^{(j)})\right)$$

for $\boldsymbol{x} \in \mathbf{Z}^d$.

(3) The construction of a Markov partition of a group automorphism on d-dimensional torus generated by the non-negative matrix L_σ.

In the present paper, on the assumption that an AH-substitution is of unimodular irreducible negative Pisot type, we obtain a series of results for AH-substitution similar to (1), (2), and (3) for H-substitution. One of the reason why we study AH-substitution in detail is that a Markov partition of a group automorphism on d-dimensional torus which is determined by a *non-positive* matrix has not been constructed in general. We give an answer of this problem by using an AH-substitution. Another reason is that we expect an AH-substitution, which is a new substitution, can bring us new results.



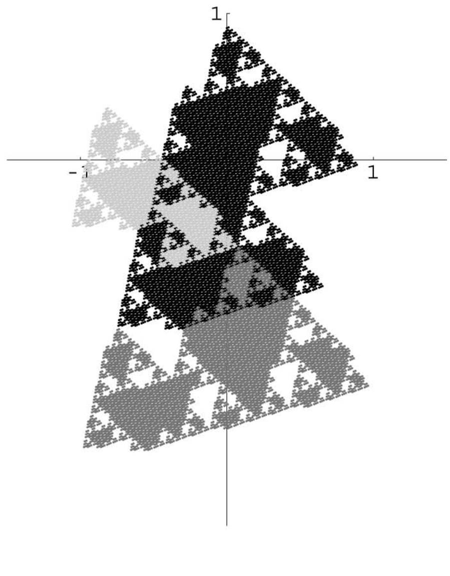


FIGURE 1. The figures above are the atomic surfaces of σ_H and σ_{AH} induced from the substitution $\sigma : 1 \mapsto 112,\ 2 \mapsto 32,\ 3 \mapsto 1$.

The outline of the paper is as follows. The basic concepts and definitions about an AH-substitution are introduced in section 2. In section 3, the Markov transformation and the natural extension for AH-substitution are discussed, and the existence of the partial atomic surfaces $\{X_1, \ldots, X_d\}$ of AH-substitution is showed. They are defined by projecting the geometrical fixed point of AH-substitution to W^s, and satisfy the set equation by the negative integral matrix $-L_\sigma$ as follows:

$$(-L_\sigma)^{-1} X_i = \bigcup_{\binom{j}{k}: W_k^{(j)}=i} (X_j + b_k^j)$$

where $b_k^j \in W^s$. In the last section, the existence of a quasi-periodic tiling $\mathfrak{T}$ on W^s with the protoset $\{X_1, \ldots, X_d\}$ under the some condition is showed and also the Markov partition of the group automorphism on d-dimensional torus which is determined by the *non-positive* matrix $-L_\sigma$ is constructed.

2. AH-substitution

2.1. AH-substitution. Let $\mathcal{A} = \{1, 2, \ldots, d\}$ be an alphabet with $d \geq 2$, and $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$. $\mathcal{A}^*$ is a free monoid, whose product is concatenation, with the empty word as the unit element denoted by ε.

If σ is a mapping from $\mathcal{A}$ to $\mathcal{A}^*$ satisfying the condition

$$\sigma(j) \neq \varepsilon \quad \text{for any } j \in \mathcal{A},$$

then σ is called a *substitution* on $\mathcal{A}$. We denote by $W_{\sigma,k}^{(j)}$ the letter at the position k in $\sigma(j)$, that is,

$$\sigma(j) = W_{\sigma,1}^{(j)} W_{\sigma,2}^{(j)} \cdots W_{\sigma,l_j}^{(j)} \ (W_{\sigma,k}^{(j)} \in \mathcal{A}),$$

and also denote

$$\sigma(j) = P_{\sigma,k}^{(j)} W_{\sigma,k}^{(j)} \cdots W_{\sigma,l_j}^{(j)}$$

by using the *k-th prefix* $P_{\sigma,k}^{(j)}$ of $\sigma(j)$

$$P_{\sigma,k}^{(j)} := \begin{cases} W_{\sigma,1}^{(j)} \cdots W_{\sigma,k-1}^{(j)} \ (2 \leq k \leq l_j), \\ \varepsilon \ (k = 1). \end{cases}$$

We omit the subscript σ in $W_{\sigma,k}^{(j)}$ and $P_{\sigma,k}^{(j)}$, and denote them by $W_k^{(j)}$ and $P_k^{(j)}$ as usual throughout this paper.

We can construct the extension of σ, whose domain is $\mathcal{A}^*$, in two natural ways. One way is to extend homomorphically, that is, its extension $\sigma_H : \mathcal{A}^* \to \mathcal{A}^*$ is defined by $\sigma_H(\varepsilon) := \varepsilon$, and for $a_1, a_2, \ldots, a_n \in \mathcal{A}$ ($n \in \mathbf{N}$),

$$\sigma_H(a_1 a_2 \cdots a_n) := \sigma(a_1)\sigma(a_2) \cdots \sigma(a_n).$$

We call σ_H a *homomorphic substitution* or an *H-substitution* on $\mathcal{A}^*$, which is well-known as substitution. Another is to extend anti-homomorphically.

DEFINITION 2.1. Let a transformation $\sigma_{AH} : \mathcal{A}^* \to \mathcal{A}^*$ be defined as follows: $\sigma_{AH}(\varepsilon) := \varepsilon$, and for $a_1, a_2, \ldots, a_n \in \mathcal{A}$ ($n \in \mathbf{N}$),

$$\sigma_{AH}(a_1a_2\cdots a_n) := \sigma(a_n)\cdots\sigma(a_2)\sigma(a_1)\,. \tag{2.1}$$

We call this transformation an *AH-substitution* or an *anti-homomorphic substitution* on $\mathcal{A}^*$.

It is evident that an *AH*-substitution σ_{AH} is anti-homomorphic, that is,

$$\sigma_{AH}(w_1w_2) = \sigma_{AH}(w_2)\sigma_{AH}(w_1) \quad \text{for any } w_1, w_2 \in \mathcal{A}^*\,. \tag{2.2}$$

The *incidence matrix* L_σ of a substitution σ (or an H-substitution σ_H) is defined as the $d \times d$ matrix, whose (i, j)-entry is the number of the occurrence of i in $\sigma(j)$. Since the matrix concerned with σ_{AH} intrinsically is $-L_\sigma$, we call $-L_\sigma$ the *incidence matrix* of σ_{AH}, which is an integral and non-positive matrix.

A mapping $f : \mathcal{A}^* \to \mathbf{Z}^d$ defined by $f(\varepsilon) := \mathbf{0}$ and

$$f(a_1a_2\cdots a_n) := \boldsymbol{e}_{a_1} + \boldsymbol{e}_{a_2} + \cdots + \boldsymbol{e}_{a_n} \quad \text{for any } a_1a_2\cdots a_n \in \mathcal{A}^*\backslash\{\varepsilon\}$$

is said to be a *canonical homomorphism* or a *homomorphism of abelianization*, where $(\boldsymbol{e}_j)_{j\in\mathcal{A}}$ is the canonical basis of $\mathbf{R}^d$. It is clear that f is homomorphic. The following properties are trivial from the definitions:

$$L_\sigma = (f(\sigma(1)), \ldots, f(\sigma(d)))\,, \tag{2.3}$$

$$f \circ \sigma_{AH} = f \circ \sigma_H = L_\sigma \circ f \text{ on } \mathcal{A}^*. \tag{2.4}$$

An algebraic integer α is called a *Pisot number* (and a *negative Pisot number*) if $\alpha > 1$ (and $\alpha < -1$) and all the conjugates except α are less than 1 in modulus, respectively.

DEFINITION 2.2. Let σ be a substitution on $\mathcal{A}$.

(1) σ (or σ_H) is of *Pisot type* if the Perron-Frobenius root λ of L_σ is a Pisot number,
(2) σ_{AH} is of *negative Pisot type* if the Perron-Frobenius root λ of L_σ is a Pisot number (by the fact $-L_\sigma$ is the incidence matrix of σ_{AH}).
(3) σ (σ_H or σ_{AH}) is *unimodular* if $\det L_\sigma = \pm 1$,
(4) σ (σ_H or σ_{AH}) is *irreducible* if the characteristic polynomial of L_σ is irreducible over $\mathbf{Q}$,
(5) σ (σ_H or σ_{AH}) satisfies the *fixed point condition* if there exists $j \in A$ such that $\sigma(j) = jw$ for some $w \in A^* \setminus \{\varepsilon\}$.
(6) σ (σ_H or σ_{AH}) is *primitive* if L_σ is primitive.

REMARK 2.1. We can set $j = 1$ without loss of generality in the definition of fixed point condition. The definition of the substitution of Pisot type in [8] is different from ours.

Since an AH-substitution σ_{AH} of irreducible negative Pisot type is primitive, we have the well-known proposition by Perron-Frobenius theorem.

PROPOSITION 2.1. *Given an AH-substitution σ_{AH} of irreducible negative Pisot type, then we have that $\mathbf{R}^d$ is decomposed into the direct sum*

$$\mathbf{R}^d = W^u \oplus W^s ,$$

where W^u is a 1-dimensional eigenspace of $-L_\sigma$ corresponding to $-\lambda$, and W^s is a $(d$-$1)$-dimensional contractive invariant subspace with respect to $-L_\sigma$. Moreover we can take a positive vector as an eigenvector of W^u.

We denote by π_u (and π_s) the projection to W^u (and W^s) with respect to this direct decomposition, respectively.

2.2. Fixed point of AH-substitution. In the rest of the paper, we shall consider the class of AH-substitutions on $\mathcal{A}^*$ satisfying the following conditions:

(NP) σ_{AH} is of negative Pisot type,

(UM) σ_{AH} is unimodular,

(IR) σ_{AH} is irreducible,

(FP) σ_{AH} satisfies the fixed point condition.

We set, for $k, l \in \mathbf{N} \cup \{0\}$

$$\tilde{\mathcal{A}}_{k,l} := \{a_{-k} \cdots a_{-1}.a_0 a_1 \cdots a_l \mid a_j \in \mathcal{A},\ j = -k, -k+1, \dots, l\} ,$$

which is the set of finite words of length $(k, l+1)$ with the decimal point, and

$$\tilde{\mathcal{A}} := \bigcup_{k,l \geq 0} \tilde{\mathcal{A}}_{k,l} .$$

Since $uwv \in \tilde{\mathcal{A}}$ for $u, v \in \mathcal{A}^*$, $w \in \tilde{\mathcal{A}}$, the free monoid $\mathcal{A}^*$ acts on the set $\tilde{\mathcal{A}}$ from the left and the right.

We shall also consider σ_{AH} as a transformation on $\tilde{\mathcal{A}}$ defined by

$$\sigma_{AH}(a_{-k} \cdots a_{-1}.a_0 a_1 \cdots a_l) := \sigma(a_l) \cdots \sigma(a_1).\sigma(a_0)\sigma(a_{-1}) \cdots \sigma(a_{-k}) ,$$

and we have that for $u, v \in \mathcal{A}^*$, $w \in \tilde{\mathcal{A}}$

$$\sigma_{AH}(uwv) = \sigma_{AH}(v)\sigma_{AH}(w)\sigma_{AH}(u) .$$

We can extend σ_{AH} to the transformation on $\mathcal{A}^{\mathbf{Z}}$ similarly. We define a relation $\preceq$ on $\tilde{\mathcal{A}}$ as follows. Let $w_1, w_2 \in \tilde{\mathcal{A}}$. Then we write $w_1 \preceq w_2$ if there exist $u, v \in \mathcal{A}^*$ such that $w_2 = uw_1v$, in addition we write $w_1 \prec w_2$ if $w_1 \neq w_2$ and $w_1 \preceq w_2$.

DEFINITION 2.3. If $s \in \mathcal{A}^{\mathbf{Z}}$ satisfies $\sigma_{AH}(s) = s$, then s is called *the fixed point of* σ_{AH}.

By Assumption (FP), we have $\sigma(1) = 1w$ ($w \in A^* \setminus \{\varepsilon\}$). Let us iterate $\sigma_{AH}^n(.1)$ ($n = 1, 2, 3, \dots$). Then we have

$$\sigma_{AH}(.1) = .\sigma(1) = .1w ,$$

$$\sigma_{AH}^2(.1) = \sigma_{AH}(w)\sigma_{AH}(.1) = \sigma_{AH}(w).1w\,,$$
$$\sigma_{AH}^3(.1) = \sigma_{AH}(w).1w\sigma_{AH}^2(w)\,,$$
$$\vdots$$

In general, we get

$$\sigma_{AH}^n(.1) = \begin{cases} \sigma_{AH}^{n-2}(w)\sigma_{AH}^{n-4}(w)\cdots\sigma_{AH}(w).1w\sigma_{AH}^2(w)\cdots\sigma_{AH}^{n-1}(w)\ (n : \text{odd})\,, \\ \sigma_{AH}^{n-1}(w)\sigma_{AH}^{n-3}(w)\cdots\sigma_{AH}(w).1w\sigma_{AH}^2(w)\cdots\sigma_{AH}^{n-2}(w)\ (n : \text{even})\,. \end{cases}$$

Therefore we have

$$\sigma_{AH}^n(.1) \prec \sigma_{AH}^{n+1}(.1)$$

for $n \in \mathbf{N}$, which induces a bi-infinite sequence s

$$s := \lim_{n\to\infty} \sigma_{AH}^n(.1) = \cdots s_{-2}s_{-1}.s_0s_1s_2\cdots$$

where $s_0 = 1$. It is clear that s is the fixed point of σ_{AH}.

2.3. Geometrical fixed point of AH-substitution. Next we shall give the geometrical representation of the fixed point s of σ_{AH}. We denote an oriented unit line segment with the base point $\boldsymbol{x}$ and the orientation $\boldsymbol{e}_j$ by

$$(\boldsymbol{x}, j) := \{\boldsymbol{x} + t\boldsymbol{e}_j \mid 0 \le t < 1\}$$

for $\boldsymbol{x} \in \mathbf{Z}^d$ and $j \in \mathcal{A}$. We set $\Lambda := \{(\boldsymbol{x}, j) \mid \boldsymbol{x} \in \mathbf{Z}^d,\ j \in \mathcal{A}\}$, and let $\mathcal{G}_1 = \mathcal{G}_1(\Lambda)$ be a $\mathbf{Z}$-free module generated by Λ. The action on $\mathcal{G}_1$ by $\mathbf{Z}^d$ (denote by $+$) is defined by

$$\boldsymbol{y} + (\boldsymbol{x}, j) := (\boldsymbol{y} + \boldsymbol{x}, j) \quad (\boldsymbol{y} \in \mathbf{Z}^d, (\boldsymbol{x}, j) \in \Lambda)\,.$$

A homomorphism $E_1(\sigma_{AH})$ on $\mathcal{G}_1$ is defined as follows:

$$E_1(\sigma_{AH})(\boldsymbol{x}, j) := (-L_\sigma)(\boldsymbol{x} + \boldsymbol{e}_j - \boldsymbol{e}_1) + \sum_{k=1}^{l_j}(f(P_k^{(j)}), W_k^{(j)})$$

for any generator $(\boldsymbol{x}, j)$ of $\mathcal{G}_1$.

By the calculation of $E_1(\sigma_{AH})^n(\mathbf{0}, j)$ $(n \in \mathbf{N})$, we obtain

$$E_1(\sigma_{AH})^n(\mathbf{0}, j) = \sum_{\binom{j_0\cdots j_{n-1}}{k_0\cdots k_{n-1}}:G_\sigma\text{-admissible},\ j_0=j} (s\tbinom{j_0\cdots j_{n-1}}{k_0\cdots k_{n-1}}, W_{k_{n-1}}^{(j_{n-1})})\,, \tag{2.5}$$

where

$$s\tbinom{j_0\cdots j_{n-1}}{k_0\cdots k_{n-1}} := \sum_{\alpha=1}^{n}(-L_\sigma)^{\alpha-1}\left((-L_\sigma)(\boldsymbol{e}_{j_{n-\alpha}} - \boldsymbol{e}_1) + f(P_{k_{n-\alpha}}^{(j_{n-\alpha})})\right), \tag{2.6}$$

and the summation of the right hand side of (2.5) means the sum of all G_σ-admissible sequences $\binom{j_0\cdots j_{n-1}}{k_0\cdots k_{n-1}}$ with $j_0 = j$. The term "G_σ-admissible" is defined in the next section.

If $\lambda \in \mathcal{G}_1$ is represented by

$$\lambda = (\boldsymbol{x}_1, j_1) + \cdots + (\boldsymbol{x}_k, j_k)\,, \quad ((\boldsymbol{x}_i, j_i) \in \Lambda,\ i = 1, 2, \ldots, k)\ ,$$

then we denote the union of $(\boldsymbol{x}_i, j_i)$'s by $|\lambda|$, that is, $|\lambda| := \bigcup_{i=1}^{k} (\boldsymbol{x}_i, j_i)$. Let $\bar{\mathcal{G}}_1$ be a set of the finite or countable unions of the elements of Λ. Since for any $L \in \bar{\mathcal{G}}_1$, there exist some finite or countable index set A and $(\boldsymbol{x}_\alpha, j_\alpha) \in \Lambda$ $(\alpha \in A)$ such that

$$L = \bigcup_{\alpha \in A} (\boldsymbol{x}_\alpha, j_\alpha)\,,$$

a mapping $\bar{E}_1(\sigma_{AH}) : \bar{\mathcal{G}}_1 \to \bar{\mathcal{G}}_1$ can be defined by

$$\bar{E}_1(\sigma_{AH})(L) := \bigcup_{\alpha \in A} |E_1(\sigma_{AH})(\boldsymbol{x}_\alpha, j_\alpha)|\,.$$

Thus we have for any $n \in \mathbf{N}$ and $j \in \mathcal{A}$

$$\bar{E}_1(\sigma_{AH})^n(\mathbf{0}, j) = |E_1(\sigma_{AH})^n(\mathbf{0}, j)|\,.$$

From (2.5) and (2.6), we have that $\bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1)$ is a connected oriented broken line segment through the origin satisfying

$$\bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1) \subset \bar{E}_1(\sigma_{AH})^{n+1}(\mathbf{0}, 1)\ (n \in \mathbf{N})\,.$$

Therefore there exists the limit $\bar{s}$ of the sequence $(\bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1))_{n\in\mathbf{N}}$:

$$\bar{s} := \lim_{n\to\infty} \bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1) = \bigcup_{n=0}^{\infty} \bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1)\,.$$

It holds that $\bar{E}_1(\sigma_{AH})(\bar{s}) = \bar{s}$. The limit $\bar{s}$ is called the *geometrical fixed point* of σ_{AH}.

EXAMPLE 2.1. Let σ be a substitution as follows:

$$\sigma : \begin{cases} 1 \ \mapsto\ 112 \\ 2 \ \mapsto\ 12\,. \end{cases}$$

Then the incidence matrix of σ is

$$L_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\,,$$

the characteristic polynomial of L_σ is $t^2 - 3t + 1$, and σ_{AH} satisfies the conditions (NP), (UM), (IR) and (FP). The words with decimal point $\sigma_{AH}^n(.1)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\sigma_{AH}(.1) = .112$$

$$\sigma_{AH}^2(.1) = \sigma(2)\sigma(1).\sigma(1) = 12112.112$$
$$\sigma_{AH}^3(.1) = 12112.1121211211212112$$
$$\vdots$$

By the definition of $E_1(\sigma_{AH})$, we obtain

$$E_1(\sigma_{AH})(\mathbf{0}, 1) = (\mathbf{0}, 1) + (\boldsymbol{e}_1, 1) + (2\boldsymbol{e}_1, 2)$$
$$E_1(\sigma_{AH})(\mathbf{0}, 2) = (\boldsymbol{e}_1, 1) + (2\boldsymbol{e}_1, 2)\,,$$

therefore we have

$$\begin{aligned} E_1(\sigma_{AH})^2(\mathbf{0}, 1) = {} & (\mathbf{0}, 1) + (\boldsymbol{e}_1, 1) + (2\boldsymbol{e}_1, 2) \\ & + (\tbinom{-2}{-1}, 1) + (\tbinom{-1}{-1}, 1) + (\tbinom{0}{-1}, 2) \\ & + (\tbinom{-3}{-2}, 1) + (\tbinom{-2}{-2}, 2)\,, \end{aligned}$$
$$\vdots$$

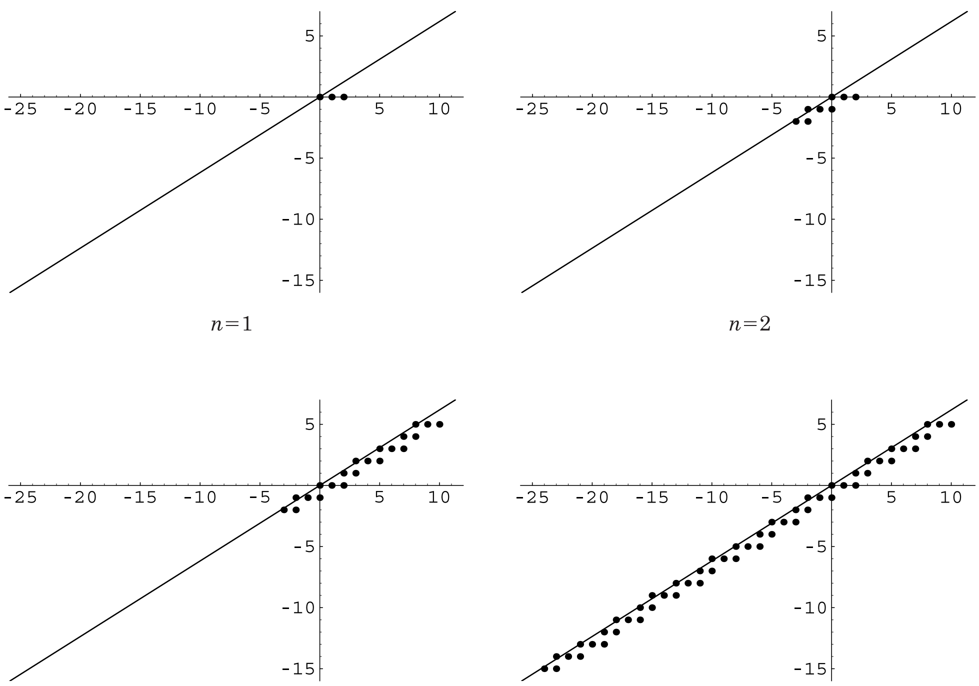


FIGURE 2. $\bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1)$ $(n = 1, 2, 3, 4)$.

The geometric representations of $\sigma_{AH}^n(.1)$, that is, $\bar{E}_1(\sigma_{AH})^n(\mathbf{0},1)$ $(n = 1,2,3,4)$ is given by Figure 2 which draw only base points.

3. Markov transformation

3.1. Markov transformation on J. Associated to a substitution σ on $\mathcal{A}$, a directed graph $G_\sigma = (\mathcal{V}_\sigma, \mathcal{E}_\sigma, \partial_\sigma^+, \partial_\sigma^-)$ is defined in the following way. The vertex set $\mathcal{V}_\sigma$ of G_σ is the alphabet $\mathcal{A}$. The following is the edge set $\mathcal{E}_\sigma$ of G_σ:

$$\mathcal{E}_\sigma := \left\{ \binom{j}{k} \;\middle|\; 1 \le j \le d,\ 1 \le k \le l_j \right\}.$$

Two mappings $\partial_\sigma^+ \colon \mathcal{E}_\sigma \to \mathcal{V}_\sigma$ and $\partial_\sigma^- \colon \mathcal{E}_\sigma \to \mathcal{V}_\sigma$ are defined by $\partial_\sigma^+(\binom{j}{k}) := j$ and $\partial_\sigma^-(\binom{j}{k}) := W_k^{(j)}$, respectively.

The graph G_σ is primitive, and the adjacency matrix of G_σ is ${}^tL_\sigma$. The edge shift space $\hat{\Omega}_\sigma$ of G_σ is defined by

$$\hat{\Omega}_\sigma := \left\{ \binom{\cdots j_{-1} j_0 j_1 \cdots j_n \cdots}{\cdots k_{-1} k_0 k_1 \cdots k_n \cdots} \in \mathcal{E}_\sigma^{\mathbf{Z}} \;\middle|\; W_{k_n}^{(j_n)} = j_{n+1},\ n \in \mathbf{Z} \right\},$$

and its shift map S_σ on $\hat{\Omega}_\sigma$

$$S_\sigma \colon \hat{\Omega}_\sigma \to \hat{\Omega}_\sigma, \quad S_\sigma\left(\binom{j_n}{k_n}_{n\in\mathbf{Z}} \right) := \binom{j'_n}{k'_n}_{n\in\mathbf{Z}},$$

where $\binom{j'_n}{k'_n} = \binom{j_{n+1}}{k_{n+1}}$. It is clear that $(\hat{\Omega}_\sigma, S_\sigma)$ is a shift of finite type (see [18]) and it is well-known that there exists an S_σ-invariant maximal measure μ_σ on $\hat{\Omega}_\sigma$ and that $(\hat{\Omega}_\sigma, S_\sigma, \mu_\sigma)$ is the measure preserving dynamical system (see [12]). Similarly we define the one-sided shift space of G_σ by

$$\Omega_\sigma := \left\{ \binom{j_0 j_1 \cdots j_n \cdots}{k_0 k_1 \cdots k_n \cdots} \in \mathcal{E}_\sigma^{\mathbf{N}\cup\{0\}} \;\middle|\; W_{k_n}^{(j_n)} = j_{n+1},\ n \in \mathbf{N}\cup\{0\} \right\},$$

and denote its shift map by the same letter S_σ. A finite or infinite sequence of $\mathcal{E}_\sigma$ which occurs some element of $\hat{\Omega}_\sigma$ is called *G_σ-admissible* .

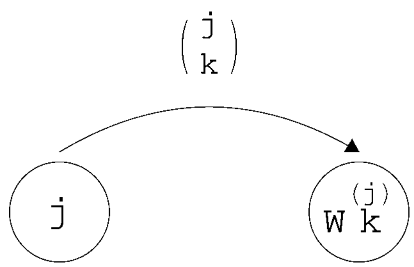


FIGURE 3. A directed graph G_σ associated to a substitution σ.

Let $\boldsymbol{v} \in W^u$ be the Perron-Frobenius vector, which can be taken as the positive and unit eigenvector of L_σ corresponding to λ. We define the closed intervals of W^u as follows: for $\boldsymbol{a}, \boldsymbol{b} \in W^u$ with $\boldsymbol{a} = a\boldsymbol{v}, \boldsymbol{b} = b\boldsymbol{v}$ $(a, b \in \mathbf{R})$, if $a \le b$, we set

$$[\boldsymbol{a}, \boldsymbol{b}] := \left\{ \boldsymbol{x} \mid \boldsymbol{x} = t\boldsymbol{v},\ a \le t \le b \right\}.$$

Let J_j $(j \in \mathcal{A})$ be the following closed intervals on W^u:

$$J_j := \left[-\frac{\lambda}{1+\lambda}\pi_u \boldsymbol{e}_1, -\frac{\lambda}{1+\lambda}\pi_u \boldsymbol{e}_1 + \pi_u \boldsymbol{e}_j \right].$$

Then we have

$$(-\lambda)J_j = \left[-\frac{\lambda}{1+\lambda}\pi_u \boldsymbol{e}_1 - \lambda \pi_u(\boldsymbol{e}_j - \boldsymbol{e}_1),\ \frac{\lambda^2}{1+\lambda}\pi_u \boldsymbol{e}_1 \right].$$

From (2.4), we can obtain the following decomposition of $(-\lambda)J_j$

$$(-\lambda)J_j = \bigcup_{k=1}^{l_j} (J_{W_k^{(j)}} + a_k^j) \tag{3.1}$$

where $a_k^j := (-\lambda)\pi_u(\boldsymbol{e}_j - \boldsymbol{e}_1) + \pi_u(f(P_k^{(j)}))$. Let J be the direct sum of J_j $(j \in \mathcal{A})$, that is, $J := \coprod_{j \in \mathcal{A}} J_j$. By giving the 1-dimensional Lebesgue measure $|\cdot|$ to J, we can consider J as a measure space.

By (3.1), a Markov transformation $T_{\sigma_{AH}}$: $J \to J$ is defined for almost all points of J as follows:

$$T_{\sigma_{AH}} \boldsymbol{x} := (-\lambda)\boldsymbol{x} - a_k^j \in J_{W_k^{(j)}} \quad (\boldsymbol{x} \in J_j).$$

Thus we have the following theorem.

THEOREM 3.1. *$T_{\sigma_{AH}}$ has the following properties*:

(1) *There exists a $T_{\sigma_{AH}}$-invariant probability measure μ on J which is absolutely continuous with respect to the Lebesgue measure $|\cdot|$,*

(2) *For almost all $\boldsymbol{x} \in J_j$ $(j \in \mathcal{A})$, there exists a G_σ-admissible sequence $\binom{j_0 j_1 \cdots j_n \cdots}{k_0 k_1 \cdots k_n \cdots} \in \Omega_\sigma$ with $j_0 = j$ such that*

$$\boldsymbol{x} = \sum_{n=0}^{\infty} \frac{a_{k_n}^{j_n}}{(-\lambda)^{n+1}}.$$

PROOF. (1) is trivial from the Markov structure. We shall show (2) next. Let $\boldsymbol{x} \in J_j$, and put $j_0 := j$. Then there exists a k_0 such that

$$T_{\sigma_{AH}} \boldsymbol{x} := (-\lambda)\boldsymbol{x} - a_{k_0}^{j_0} \in J_{W_{k_0}^{(j_0)}}$$

Set $j_1 := W_{k_0}^{(j_0)}$. Similarly for some k_1, it holds

$$T_{\sigma_{AH}}^2 \boldsymbol{x} := (-\lambda) T_{\sigma_{AH}} \boldsymbol{x} - a_{k_1}^{j_1} \in J_{W_{k_1}^{(j_1)}} .$$

Repeatedly we get

$$T_{\sigma_{AH}}^n \boldsymbol{x} := (-\lambda) T_{\sigma_{AH}}^{n-1} \boldsymbol{x} - a_{k_{n-1}}^{j_{n-1}} \in J_{W_{k_{n-1}}^{(j_{n-1})}}$$

for any $n \in \mathbf{N}$. Therefore we have

$$\boldsymbol{x} = \sum_{s=0}^{n-1} \frac{a_{k_s}^{j_s}}{(-\lambda)^{s+1}} + \frac{1}{(-\lambda)^n} T_{\sigma_{AH}}^n \boldsymbol{x} ,$$

which means that the conclusion holds when n tends to ∞. □

We can define φ: $\Omega_\sigma \to J$ by

$$\varphi \left(\binom{j_0 j_1 \cdots j_n \cdots}{k_0 k_1 \cdots k_n \cdots} \right) := \sum_{n=0}^{\infty} \frac{a_{k_n}^{j_n}}{(-\lambda)^{n+1}} ,$$

and see that φ: $\Omega_\sigma \to J$ is bijective almost everywhere. Therefore we have the following theorem.

THEOREM 3.2. *Dynamical system* $(J, T_{\sigma_{AH}}, \mu)$ *is isomorphic* (*a.e.*) *to Markov shift* $(\Omega_\sigma, S, \mu_\sigma)$ *by the isomorphism* φ.

3.2. Atomic surface of AH-substitution. We shall recall the basic concepts. The closure and the interior of a subset A in W^s are denoted by $\bar{A}$ and by $\operatorname{Int} A$ respectively. $| \cdot |$ denotes Lebesgue measure on W^s . If $A_1, \ldots, A_n$ are Lebesgue measurable sets with $|A_i \cap A_j| = 0$ $(i \neq j)$, then we call $\cup_{j=1}^n A_j$ a *non-overlapping* union of $A_1, \ldots, A_n$ and denote it by $\cup_{j=1}^n A_j$ (non-overlapping).

Setting Y_j $(j \in \mathcal{A})$, Y by

$$Y_j := \left\{ \boldsymbol{y} \in \mathbf{Z}^d \;\middle|\; (\boldsymbol{y}, j) \subset \bar{s} \right\} ,$$
$$Y := \left\{ \boldsymbol{y} \in \mathbf{Z}^d \;\middle|\; (\boldsymbol{y}, j) \subset \bar{s},\ j \in \mathcal{A} \right\} ,$$

we shall define the *partial atomic surfaces* X_j $(j \in \mathcal{A})$ and the *atomic surface* X of AH-substitution σ_{AH} as follows:

$$X_j := \overline{\pi_s Y_j} , \quad X := \overline{\pi_s Y} .$$

Then we have $X = \bigcup_{j=1}^d X_j$.

THEOREM 3.3. *We have the following properties about the atomic surfaces.*

(1) *$X,\ X_j\ (j \in \mathcal{A})$ are compact,*

(2) *For each $i \in \mathcal{A}$,*

$$(-L_\sigma)^{-1}X_i = \bigcup_{\binom{j}{k}:W_k^{(j)}=i} (X_j + b_k^j)\,, \tag{3.2}$$

where $b_k^j = \pi_s(\boldsymbol{e}_j - \boldsymbol{e}_1) + (-L_\sigma)^{-1}(\pi_s(f(P_k^{(j)}))$,

(3) $\mathrm{Int}\, X_j \neq \phi\ (j \in \mathcal{A})$,

(4) $\overline{\mathrm{Int}\, X_j} = X_j\ (j \in \mathcal{A})$.

REMARK 3.1. (3.2) is called *the set equation* with respect to the partial atomic surfaces of σ_{AH}. The right hand side of (3.2) means the union of $X_j + b_k^j$'s with respect to all $\binom{j}{k} \in \mathcal{E}_\sigma$ such that $W_k^{(j)} = i$.

PROOF. The proof can be obtained by the analogy of [11], [15]. Therefore we will give a sketch of the proof here. We show (1) first. Set for each $n \in \mathbf{N}$,

$$Y^{(n)} := \left\{ \boldsymbol{y} \in \mathbf{Z}^d \;\middle|\; (\boldsymbol{y}, j) \subset \bar{E}_1(\sigma_{AH})^n(\mathbf{0}, 1),\ j \in \mathcal{A} \right\},$$

then we get from (2.5) and (2.6)

$$Y^{(n)} = \left\{ s\binom{j_0 \cdots j_{n-1}}{k_0 \cdots k_{n-1}} \;\middle|\; \binom{j_0 \cdots j_{n-1}}{k_0 \cdots k_{n-1}} : G_\sigma\text{-admissible},\ j_0 = 1 \right\}$$

Let I be

$$\left\{ (-L_\sigma)(e_j - e_1) + f(P_k^{(j)}) \;\middle|\; \binom{j}{k} \in \mathcal{E} \right\}$$

and θ the maximum of the eigenvalues of L_σ except λ in modulus, then there exists a constant $C > 0$ for any $\boldsymbol{y} \in I$ and any $n \in \mathbf{N}$

$$|(-L_\sigma)^n(\pi_s \boldsymbol{y})| \le C\theta^n.$$

Therefore we have

$$\left| \pi_s \left(s\binom{j_0 \cdots j_{n-1}}{k_0 \cdots k_{n-1}} \right) \right| < \frac{C}{1-\theta}\,.$$

This means that the X is compact and that so are $X_i\ (i \in \mathcal{A})$.

We show (2) next. Let (Y_i, i) denote the set $\left\{ (\boldsymbol{y}, i) \;\middle|\; (\boldsymbol{y}, i) \subset \bar{s} \right\}$. Then we have $\bar{s} = \bigcup_{j=1}^d (Y_j, j)$, and

$$\begin{aligned}
(Y_i, i) &= \left\{ (\boldsymbol{y}, i) \;\middle|\; (\boldsymbol{y}, i) \subset \bar{s} \right\} \\
&= \left\{ (\boldsymbol{y}, i) \mid (\boldsymbol{y}, i) \subset \bar{E}_1(\sigma_{AH})\left(\bigcup_{j=1}^d (Y_j, j) \right) \right\} \\
&= \bigcup_{j=1}^d \left\{ (\boldsymbol{y}, i) \;\middle|\; (\boldsymbol{y}, i) \subset \bar{E}_1(\sigma_{AH})(Y_j, j) \right\}
\end{aligned}$$

$$= \bigcup_{\binom{j}{k}:W_k^{(j)}=i} \left\{(-L_\sigma)(\boldsymbol{y}) + (\boldsymbol{e}_j - \boldsymbol{e}_1) + (-L_\sigma)^{-1}(f(P_k^{(j)})), i) \;\middle|\; \boldsymbol{y} \in Y_j\right\},$$

which shows the set equation (3.2).

We show (3). Let ${}^t(1, v_1, \dots, v_{d-1})$ be the positive eigenvector of λ. Since the characteristic polynomial of L_σ is the minimal polynomial of the algebraic integer λ, $\{1, v_1, \dots, v_{d-1}\}$ is a $\mathbf{Q}$-basis of the algebraic number field $\mathbf{Q}(\lambda)$, which means $\overline{\pi_s \mathbf{Z}^d} = W^s$. We shall consider the following hyperplane:

$$P := \left\{\boldsymbol{x} \in \mathbf{R}^d \;\middle|\; \langle \boldsymbol{x}, {}^t(1, 1, \dots, 1)\rangle = 0\right\},$$

where $\langle\ ,\ \rangle$ is the inner product of $\mathbf{R}^d$. By considering the lattice

$$L := \left\{\sum_{j=2}^{d} n_j(\boldsymbol{e}_j - \boldsymbol{e}_1) \mid n_i \in \mathbf{Z}\right\}$$

on P, we get $\mathbf{Z}^d = \bigcup_{n=1}^{\infty}(Y^{(n)} + L) = Y + L$, which shows $W^s = X + \pi_s L$. From Baire category theorem, we have $\mathrm{Int}\, X \neq \phi$. By the set equation and the primitivity of L_σ, we have $\mathrm{Int}\, X_j \neq \phi$ $(j \in \mathcal{A})$.

(4) is clear from the set equation and (3). □

The following lemma can be found in [5], [20].

LEMMA 3.4. *Let A be a primitive matrix with the largest eigenvalue λ. Suppose that $\boldsymbol{v}$ is a positive vector such that $A\boldsymbol{v} \geq \lambda\boldsymbol{v}$. Then the inequality becomes equality and $\boldsymbol{v}$ is the eigenvector of A corresponding to λ.*

LEMMA 3.5. ${}^t(|X_1|, |X_2|, \dots, |X_d|)$ *is the positive eigenvector of L_σ corresponding to λ, where $|\cdot|$ is Lebesgue measure on W^s.*

PROOF. From the set equations (3.2) and the fact that the determinant of $L_\sigma^{-1}|_{W^s}$ is λ,

$$\lambda|X_i| = |(-L_\sigma)^{-1}X_i| \leq \sum_{j=1}^{d} l_{ij}|X_j|\ (i \in \mathcal{A}), \tag{3.3}$$

where l_{ij} is (i, j)-entry of L_σ. By the previous lemma, the equality of (3.3) holds. □

REMARK 3.2. The set equations (3.2) are non-overlapping unions by Lemma 3.5.

DEFINITION 3.1. *The coincidence condition of AH-substitution* means that we can take $j_0 \in \mathcal{A}$, $n \geq 1$ such that for any $i \in \mathcal{A}$, there exist d different G^*-admissible sequences $\binom{j_{-1}^{(i)} \cdots j_{-n}^{(i)}}{k_{-1}^{(i)} \cdots k_{-n}^{(i)}}$ satisfying

$$j_{-n}^{(i)} = i\,, \quad W_{k_{-1}^{(i)}}^{(j_{-1}^{(i)})} = j_0\,,$$

and $\sum_{\alpha=1}^{n}(-L_\sigma)^{-(n-\alpha)} b_{k_{-\alpha}^{(i)}}^{j_{-\alpha}^{(i)}}$ are constant vector for all $i \in \mathcal{A}$.

THEOREM 3.6. *Under the coincidence condition of AH-substitution σ_{AH}, we have*

$$X = \bigcup_{j \in \mathcal{A}} X_j \text{ (non - overlapping)}$$

PROOF. By the set equations (3.2), we have

$$(-L_\sigma)^n X_{j_0} = \bigcup_{\substack{\binom{j_{-1}\cdots j_{-n}}{k_{-1}\cdots k_{-n}}: G_\sigma^*\text{-admissible} \\ j_0 = W_{k_{-1}}^{(j_{-1})}}} \left(X_{j_{-n}} + \sum_{\alpha=1}^{n} (-L_\sigma)^{-(n-\alpha)} b_{k_{-\alpha}}^{j_{-\alpha}} \right) . \tag{3.4}$$

By Remark 3.2, we have that the set equations are non-overlapping, and so are (3.4). On the other hand, we know that $\bigcup_{i=1}^{d} \left(X_{j_{-n}^{(i)}} + \sum_{\alpha=1}^{n}(-L_\sigma)^{-(n-\alpha)} b_{k_{-\alpha}^{(i)}}^{j_{-\alpha}^{(i)}} \right)$ is a subpatch of $(-L_\sigma)^n X_{j_0}$ from the coincidence condition. Therefore we have the conclusion. □

EXAMPLE 3.1. Figure 1 and Figure 4 are the atomic surfaces of the H-substitution and the AH-substitution induced by the following substitution:

$$\sigma : 1 \mapsto 112,\ 2 \mapsto 32,\ 3 \mapsto 1\,.$$

The set equations (3.2) say that for $\boldsymbol{x} \in X_i$, there exist $\binom{j}{k} \in \mathcal{E}_\sigma$ such that $W_k^{(j)} = i$ and $(-L_\sigma)^{-1}\boldsymbol{x} \in X_j + b_k^j$. Therefore a Markov transformation $T_{\sigma_{AH}}^*: X \to X$ is defined for almost all points of X as follows:

$$T_{\sigma_{AH}}^* \boldsymbol{x} := (-L_\sigma)^{-1}\boldsymbol{x} - b_k^j \ \in X_j \ (\boldsymbol{x} \in X_i)\,.$$

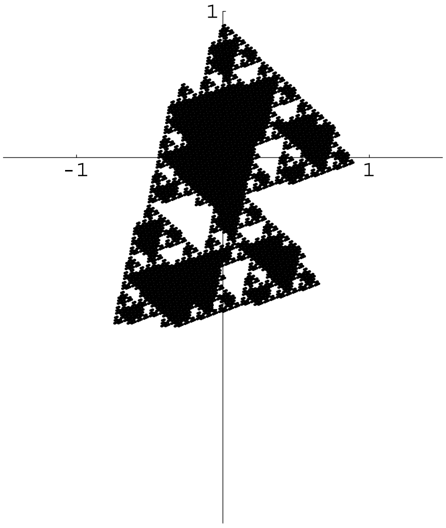
Let the (backward) one-sided shift space of G_σ be

$$\Omega_\sigma^* := \left\{ \binom{j_{-1} j_{-2} \cdots}{k_{-1} k_{-2} \cdots} \in \mathcal{E}_\sigma^{\mathbf{N}} \;\middle|\; W_{k_{-(n+1)}}^{j_{-(n+1)}} = j_{-n},\ n \in \mathbf{N} \right\},$$

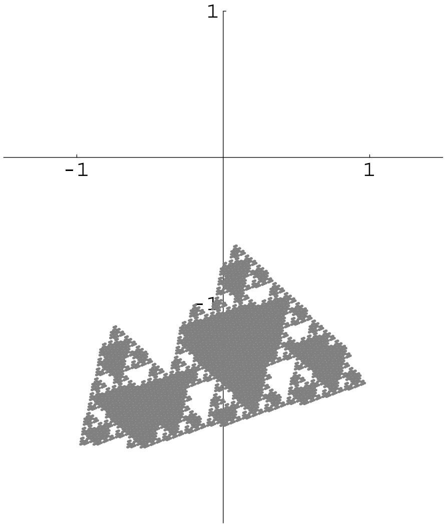
and its shift map S_σ^* on Ω_σ^*

$$S_\sigma^*: \Omega_\sigma^* \to \Omega_\sigma^*,\ S_\sigma^*\left(\binom{j_{-1} j_{-2} j_{-3} \cdots}{k_{-1} k_{-2} k_{-3} \cdots}\right) := \binom{j_{-2} j_{-3} j_{-4} \cdots}{k_{-2} k_{-3} k_{-4} \cdots}\,.$$

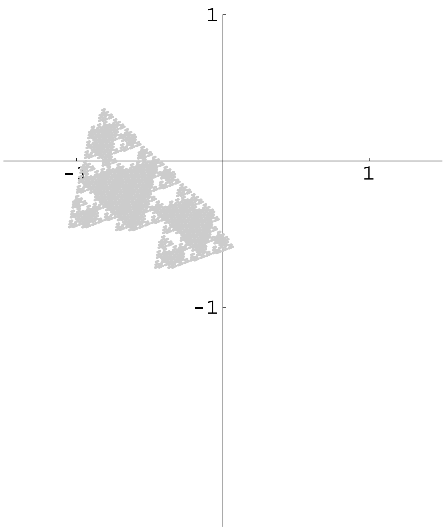
A finite or infinite sequence of $\mathcal{E}_\sigma$ which occurs some element of Ω_σ^* is called *G_σ^*-admissible sequence*.



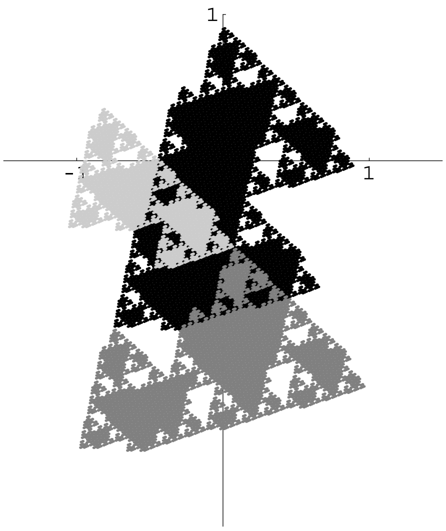
the partial atomic surface X_1 of σ_{AH} in Ex 3.1



the partial atomic surface X_2 of σ_{AH} in Ex 3.1



the partial atomic surface X_3 of σ_{AH} in Ex 3.1



the atomic surface X of σ_{AH} in Ex 3.1

FIGURE 4. X_1, X_2, X_3, X.

THEOREM 3.7. *For almost all $\boldsymbol{x} \in X_{j_0}$ ($j_0 \in \mathcal{A}$), there exists some G_σ^*-admissible sequence $\binom{j_{-1} j_{-2} \dots}{k_{-1} k_{-2} \dots} \in \Omega_\sigma^*$ with $W_{k_{-1}}^{(j_{-1})} = j_0$ such that*

$$\boldsymbol{x} = \sum_{n=1}^{\infty} (-L_\sigma)^n b_{k_{-n}}^{j_{-n}} .$$

PROOF. In the same way as Theorem 3.1, for each $n \in \mathbf{N}$, there exists G_σ^*-admissible sequence $\binom{j_{-1} j_{-2} \cdots j_{-n}}{k_{-1} k_{-2} \cdots k_{-n}} \in \Omega_\sigma^*$ such that

$$\boldsymbol{x} = \sum_{\alpha=1}^{n} (-L_\sigma)^\alpha b_{k_{-\alpha}}^{j_{-\alpha}} + (-L_\sigma)^n \left(T_{\sigma_{AH}}^{*n} \boldsymbol{x} \right) .$$

As L_σ is contractive on W^s, the second term converges to $\mathbf{0}$ ($n \to \infty$). □

We can define φ^*: $\Omega_\sigma^* \to X$ by

$$\varphi^* \left(\binom{j_{-1} j_{-2} \cdots j_{-n} \cdots}{k_{-1} k_{-2} \cdots k_{-n} \cdots} \right) := \sum_{n=1}^{\infty} (-L_\sigma)^n b_{k_{-n}}^{j_{-n}} ,$$

and see that φ^*: $\Omega_\sigma^* \to X$ is bijective almost everywhere.

THEOREM 3.8. *The transformations $T_{\sigma_{AH}}^*$ and S_σ^* are isomorphic by φ^* almost everywhere.*

PROOF. This is clear from the definitions. □

3.3. Natural Extension of Markov transformation. We set

$$\hat{X}_i := X_i - J_i = \left\{ \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{x} \in X_i,\ \boldsymbol{y} \in J_i \right\} \quad (i \in \mathcal{A}) ,$$

and

$$\hat{X} := \bigcup_{i \in \mathcal{A}} \hat{X}_i \quad \text{(non-overlapping)} .$$

Let us define a realization map $\hat{\varphi}$: $\hat{\Omega}_\sigma \to \hat{X}$ by

$$\begin{aligned} \hat{\varphi}\left(\binom{\cdots j_{-1} j_0 j_1 \cdots}{\cdots k_{-1} k_0 k_1 \cdots}\right) &: = \varphi^*\left(\binom{j_{-1} j_{-2} \cdots}{k_{-1} k_{-2} \cdots}\right) - \varphi\left(\binom{j_0 j_1 \cdots}{k_0 k_1 \cdots}\right) \\ &= \sum_{n=1}^{\infty} (-L_\sigma)^n b_{k_{-n}}^{j_{-n}} - \sum_{n=0}^{\infty} \frac{a_{k_n}^{j_n}}{(-\lambda)^{n+1}} , \end{aligned}$$

then we see that $\hat{\varphi}$ is bijective for almost all points in $\hat{\Omega}_\sigma$.

By the set equations (3.2), we have for each $i \in \mathcal{A}$

$$X_i = \bigcup_{\binom{j}{k} : W_k^{(j)} = i} (-L_\sigma)\,(X_j + b_k^j) .$$

We define

$$X_{\binom{j}{k}} := (-L_\sigma)(X_j + b_k^j) , \quad \hat{X}_{\binom{j}{k}} := X_{\binom{j}{k}} - J_i \ \binom{j}{k}$$

for every $\binom{j}{k}$ such that $W_k^{(j)} = i$. Then we have the non-overlapping decompositions:

$$X_i = \bigcup_{\binom{j}{k}: W_k^{(j)}=i} X_{\binom{j}{k}}, \quad \hat{X}_i = \bigcup_{\binom{j}{k}: W_k^{(j)}=i} \hat{X}_{\binom{j}{k}}.$$

Set

$$\hat{\Omega}_\sigma(i) := \left\{ \left(\begin{smallmatrix} \cdots j_{-1} j_0 j_1 \cdots \\ \cdots k_{-1} k_0 k_1 \cdots \end{smallmatrix}\right) \in \hat{\Omega}_\sigma \;\middle|\; j_0 = i \right\},$$

$$\hat{\Omega}_\sigma\binom{j}{k} := \left\{ \left(\begin{smallmatrix} \cdots j_{-1} j_0 j_1 \cdots \\ \cdots k_{-1} k_0 k_1 \cdots \end{smallmatrix}\right) \in \hat{\Omega}_\sigma \;\middle|\; \binom{j_{-1}}{k_{-1}} = \binom{j}{k} \right\}$$

for every $\binom{j}{k}$ such that $W_k^{(j)} = i$, then $\hat{\Omega}_\sigma$ is decomposed as follows:

$$\hat{\Omega}_\sigma = \bigcup_{i \in \mathcal{A}} \hat{\Omega}_\sigma(i), \quad \hat{\Omega}_\sigma(i) = \bigcup_{\binom{j}{k}: W_k^j = i} \hat{\Omega}_\sigma\binom{j}{k}.$$

Theorems 3.1 and 3.7 give us the following equalities for almost all points:

$$\hat{X}_i = \hat{\varphi}\left(\hat{\Omega}_\sigma(i)\right), \; \hat{X}_{\binom{j}{k}} = \hat{\varphi}\left(\hat{\Omega}_\sigma\binom{j}{k}\right). \tag{3.5}$$

For almost all $\boldsymbol{x} \in \hat{X}_j$, there exists $\left(\begin{smallmatrix} \cdots j_{-1} j_0 j_1 \cdots \\ \cdots k_{-1} k_0 k_1 \cdots \end{smallmatrix}\right) \in \hat{\Omega}_\sigma$ with $j_0 = j$ such that

$$\boldsymbol{x} = \sum_{n=1}^{\infty} (-L_\sigma)^n \, b_{k_{-n}}^{j_{-n}} - \sum_{n=0}^{\infty} \frac{a_{k_n}^{j_n}}{(-\lambda)^{n+1}}.$$

Then we have

$$\begin{aligned} (-L_\sigma)\,\boldsymbol{x} = &\sum_{n=1}^{\infty} (-L_\sigma)^n \, b_{k_{-(n-1)}}^{j_{-(n-1)}} - \sum_{n=0}^{\infty} \frac{a_{k_{n+1}}^{j_{n+1}}}{(-\lambda)^{n+1}} \\ &- \left\{ (-L_\sigma)\,(\boldsymbol{e}_{j_0} - \boldsymbol{e}_1) + f(P_{k_0}^{(j_0)}) \right\}. \end{aligned}$$

Therefore by (3.5), we get

$$(-L_\sigma)\,\hat{X}_j = \bigcup_{1 \le k \le l_j} \left(\hat{X}_{\binom{j}{k}} + c_k^j \right),$$

where $c_k^j := (-L_\sigma)\,(\boldsymbol{e}_{j_0} - \boldsymbol{e}_1) + f(P_{k_0}^{(j_0)})$, hence we obtain a Markov transformation $\hat{T}_{\sigma_{AH}}$: $\hat{X} \to \hat{X}$ with a Markov partition $\left\{ \hat{X}_{\binom{j}{k}} \;\middle|\; \binom{j}{k} \in \mathcal{E}_\sigma \right\}$ by

$$\hat{T}_{\sigma_{AH}}\boldsymbol{x} := (-L_\sigma)\boldsymbol{x} - c_k^j \; (\boldsymbol{x} \in \hat{X}_j)$$

Thus we have the following theorem.

THEOREM 3.9. *The transformations $\hat{T}_{\sigma_{AH}}$ and S_σ are isomorphic by $\hat{\varphi}$ almost everywhere, that is, the following diagram is commutative almost everywhere.*

$$\begin{array}{ccc} \hat{\Omega}_\sigma & \xrightarrow{S_\sigma} & \hat{\Omega}_\sigma \\ \hat{\varphi}\big\downarrow & & \big\downarrow\hat{\varphi} \\ \hat{X} & \xrightarrow[\hat{T}_{\sigma_{AH}}]{} & \hat{X} \end{array}$$

4. Tiling by atomic surfaces

4.1. Stepped surface induced by AH-substitution. For $\boldsymbol{x} \in \mathbf{Z}^d$, $j \in \mathcal{A}$, we define $(\boldsymbol{x}, j^*)$ by

$$(\boldsymbol{x}, j^*) := \left\{ \boldsymbol{x} + \sum_{k\in\mathcal{A},\ k\neq j} t_k \boldsymbol{e_k} \ \middle|\ 0 \le t_k < 1 \right\},$$

that is, $(d-1)$-dimensional oriented unit square with the base point $\boldsymbol{x}$ generated by $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{j-1}, \boldsymbol{e}_{j+1}, \dots, \boldsymbol{e}_d\}$ ([21]). We set

$$\Lambda^* := \left\{ (\boldsymbol{x}, j^*) \ \middle|\ \boldsymbol{x} \in \mathbf{Z}^d, j \in \mathcal{A} \right\},$$

and let $\mathcal{G}_1^*$ be a $\mathbf{Z}$-free module generated by Λ^*. Let us define the action on $\mathcal{G}_1^*$ by $\mathbf{Z}^d$ (denote by $+$) as

$$\boldsymbol{y} + (\boldsymbol{x}, j^*) := (\boldsymbol{y} + \boldsymbol{x}, j^*) \quad (y \in \mathbf{Z}^d, (\boldsymbol{x}, j^*) \in \Lambda^*).$$

and a homomorphism $E_1(\sigma)^*$: $\mathcal{G}_1^* \to \mathcal{G}_1^*$ as

$$E_1(\sigma_{AH})^*(\boldsymbol{x}, i^*) := \sum_{\binom{j}{k}: W_k^{(j)} = i} \left((\boldsymbol{e}_j - \boldsymbol{e}_1) + (-L_\sigma)^{-1}(\boldsymbol{x} + f(P_k^{(j)})),\ j^* \right) \tag{4.1}$$

for any generator $(\boldsymbol{x}, i^*)$ of $\mathcal{G}_1^*$.

Let us calculate

$$E_1(\sigma_{AH})^{*n}(\boldsymbol{0}, i^*) = \sum_{\substack{\binom{j_{-1}\cdots j_{-n}}{k_{-1}\cdots k_{-n}}: G_\sigma^*\text{-admissible} \\ i = W_{k_{-1}}^{(j_{-1})}}} \left(s^*\binom{j_{-1}\cdots j_{-n}}{k_{-1}\cdots k_{-n}}, W_{k_{-n}}^{(j_{-n})*} \right) \tag{4.2}$$

$(n = 1, 2, 3, \dots)$, where

$$s^*\binom{j_{-1}\cdots j_{-n}}{k_{-1}\cdots k_{-n}} := \sum_{\alpha=0}^{n-1} (-L_\sigma)^{-\alpha} \left((\boldsymbol{e}_{j_{-n+a}} - \boldsymbol{e}_1) + (-L_\sigma)^{-1}(f(P_{k_{-n+a}}^{(j_{-n+a})})) \right). \tag{4.3}$$

If $\lambda^* \in \mathcal{G}_1^*$ is represented by

$$\lambda^* = (\boldsymbol{x}_1, j_1^*) + \cdots + (\boldsymbol{x}_k, j_k^*) \ \big((\boldsymbol{x}_i, j_i^*) \in \Lambda,\ i = 1, 2, \dots, k\big)\ ,$$

then we denote the union of $(\boldsymbol{x}_i, j_i^*)$'s by $|\lambda^*|$, that is, $|\lambda^*| := \cup_{i=1}^{k}(\boldsymbol{x}_i, j_i^*)$. Let $\bar{\mathcal{G}}_1^*$ be a set of the finite or countable unions of the elements of Λ^*. Since for any $C \in \bar{\mathcal{G}}_1^*$, there exist some finite or countable index set A and $(\boldsymbol{x}_\alpha, j_\alpha^*) \in \Lambda^*$ $(\alpha \in A)$ such that

$$C = \bigcup_{\alpha \in A} (\boldsymbol{x}_\alpha, j_\alpha^*)\ ,$$

a mapping $\bar{E}_1(\sigma_{AH})^* : \bar{\mathcal{G}}_1^* \to \bar{\mathcal{G}}_1^*$ can be defined by

$$\bar{E}_1(\sigma_{AH})^*(C) := \bigcup_{\alpha \in A} |E_1(\sigma_{AH})^*(\boldsymbol{x}_\alpha, j_\alpha^*)|\ .$$

Thus we have for any $n \in \mathbf{N}$ and $j \in \mathcal{A}$

$$\bar{E}_1(\sigma_{AH})^{*n}(\mathbf{0}, j^*) = |E_1(\sigma_{AH})^{*n}(\mathbf{0}, j^*)|\ .$$

Therefore we have the following proposition from (4.2) and (4.3).

PROPOSITION 4.1. *The partial atomic surfaces X_j $(j \in \mathcal{A})$ can be given by*

$$X_j = \lim_{n \to \infty} (-L_\sigma)^n \pi_s \bar{E}_1(\sigma_{AH})^{*n}(\mathbf{0}, j^*) \quad (j \in \mathcal{A})\ ,$$

where the right hand side means the convergence with respect to Hausdorff distance.

PROOF. From (4.3), we have

$$(-L_\sigma)^n \pi_s s^*\binom{j_{-1} \cdots j_{-n}}{k_{-1} \cdots k_{-n}} = \sum_{\alpha=1}^{n} (-L_\sigma)^\alpha b\binom{j_{-\alpha}}{k_{-\alpha}}\ .$$

Thus we obtain the conclusion by Theorem 3.2. □

EXAMPLE 4.1. Figure 5 depicts $\bar{E}_1(\sigma_{AH})^{*n}(\mathbf{0}, i^*)$ $(n = 1, 2, 3, \dots)$ of the substitution given by $\sigma : 1 \mapsto 112, 2 \mapsto 32, 3 \mapsto 1$

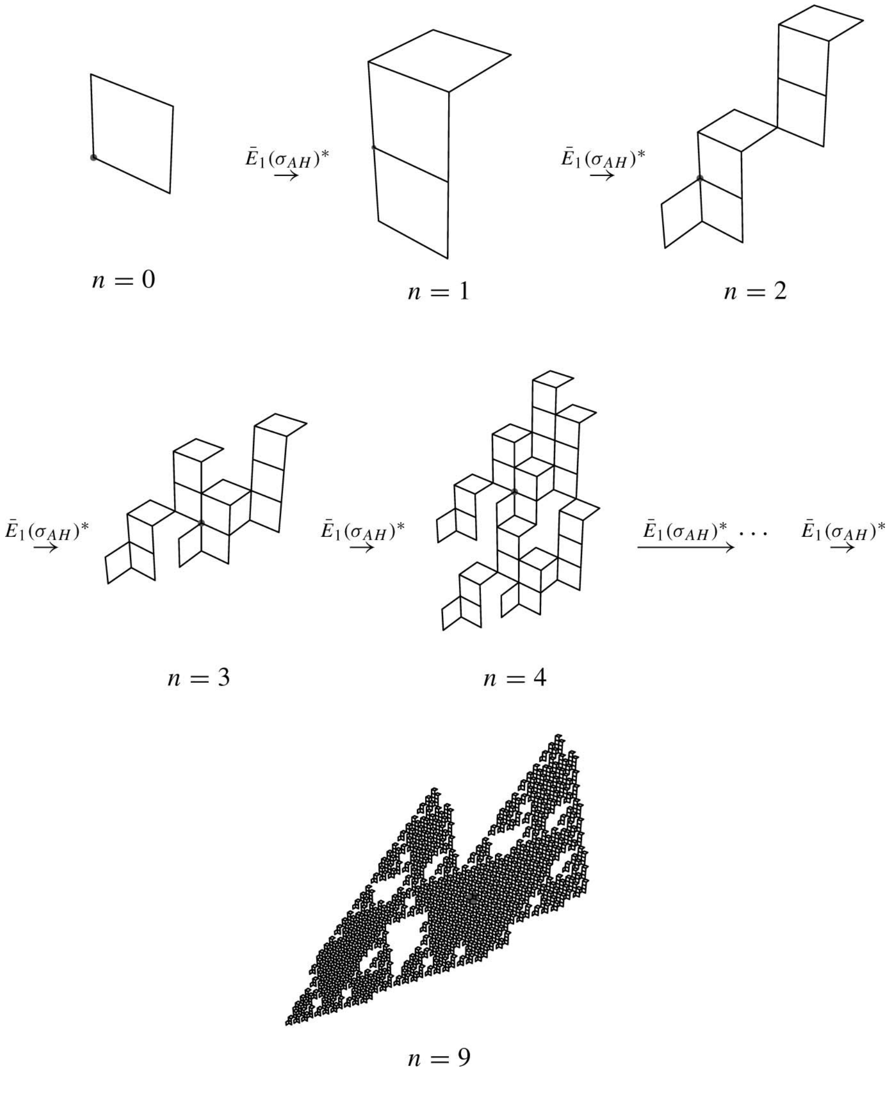


FIGURE 5. $\bar{E}_1(\sigma_{AH})^{*n}\,(\mathbf{0}, 1^*)$, $n = 0, 1, \ldots, 4, 9$.

Setting

$$I_j := \left[-\frac{\lambda}{1+\lambda}\pi_u \boldsymbol{e}_1,\ -\frac{\lambda}{1+\lambda}\pi_u \boldsymbol{e}_1 + \pi_u \boldsymbol{e}_j\right) \quad (j \in \mathcal{A})$$

$$P := W^s - \frac{\lambda}{1+\lambda}\pi_u \boldsymbol{e}_1 ,$$

we define *the stepped-surface* of P by

$$\mathcal{S} := \left\{(\boldsymbol{x}, j^*) \mid \pi_u \boldsymbol{x} \in I_j,\ j \in \mathcal{A}\right\}.$$

We also call $S := \bigcup_{(\boldsymbol{x}, j^*) \in \mathcal{S}} (\boldsymbol{x}, j^*)$ *the stepped-surface* of P. Let us define an affine transformation:

$$\pi_s' : \mathbf{R}^d \to P \ : \ \pi_s'(\boldsymbol{x}) := \pi_s(\boldsymbol{x}) - \frac{\lambda}{1+\lambda} \pi_u \boldsymbol{e}_1 .$$

Then we have the following lemma:

LEMMA 4.2 (see [5], [10]). *$\bar{E}_1(\sigma_{AH})^*$ has the following properties*:

(1) $\bar{E}_1(\sigma_{AH})^*(S) = S$.

(2) *If* $(\boldsymbol{x}, i^*)$, $(\boldsymbol{x}', i'^*) \in \mathcal{S}$ *with* $\mathrm{Int}(\boldsymbol{x}, i^*) \cap \mathrm{Int}(\boldsymbol{x}', i'^*) = \phi$, *then* $\mathrm{Int}\, \bar{E}_1(\sigma_{AH})^*(\boldsymbol{x}, i^*) \bigcap \mathrm{Int}\, \bar{E}_1(\sigma_{AH})^*(\boldsymbol{x}', i'^*) = \phi$,

where $\mathrm{Int}\, A$ *is the interior of* A *for* $A \subset \mathbf{R}^d$.

PROOF. As (2) is clear, we shall show (1). Setting

$$\boldsymbol{y} := (\boldsymbol{e}_j - \boldsymbol{e}_1) + (-L_\sigma)^{-1}(\boldsymbol{x} + f(P_k^{(j)}))$$

for any $(\boldsymbol{x}, i^*) \in \mathcal{G}_1^*$ and any $\binom{j}{k}$ with $W_k^{(j)} = i$, we have $\pi_u \boldsymbol{x} = (-\lambda)\pi_u \boldsymbol{y} - a_k^j$. Take any $(\boldsymbol{x}, i^*) \subset S$. By the definition of the stepped surface, $\pi_u \boldsymbol{x} \in I_i$. Then we get

$$(-\lambda)\pi_u \boldsymbol{y} \in I_{W_k^{(j)}} + a_k^j \subset (-\lambda) I_j .$$

Therefore we obtain $(\boldsymbol{y}, j^*) \subset S$ for any $\binom{j}{k}$ with $W_k^{(j)} = i$, which means $\bar{E}_1(\sigma_{AH})^*(\boldsymbol{x}, i^*) \subset S$. The converse is also similar. □

4.2. Tiling by atomic surfaces. Projecting the stepped surface $\mathcal{S}$ to P by π_s', we obtain a parallelogram tiling of P:

$$\mathcal{T}' := \left\{\pi_s'(\boldsymbol{x}, j^*) \mid (\boldsymbol{x}, j^*) \in \mathcal{S}\right\},$$

which has $\left\{\pi_s'(\mathbf{0}, j^*) \mid j \in \mathcal{A}\right\}$ as a protoset.

We set the parallelogram $\hat{C}(\mathbf{0}, j^*)$ by

$$\hat{C}(\mathbf{0}, j^*) := \pi_s(\mathbf{0}, j^*) - J_j ,$$

and denote these union $\bigcup_{i=1}^d \hat{C}(\mathbf{0}, j^*)$ by $\hat{C}(\mathbf{0})$.

PROPOSITION 4.3 (see [6]). *The set $\hat{C}(\mathbf{0})$ has the following properties*:

(1) $|\hat{C}(\mathbf{0})| = 1$

(2) $\mathcal{T}'' = \left\{\boldsymbol{x} + \hat{C}(\mathbf{0}) \mid \boldsymbol{x} \in \mathbf{Z}^d\right\} = \hat{C}(\mathbf{0}) + \mathbf{Z}^d$ *is a tiling on* $\mathbf{R}^d$.

PROOF. The assertion (1) is clear. We prove (2). We consider the intersection of P and $\hat{C}(\mathbf{0}) + \mathbf{Z}^d$. By the fact $\hat{C}(\mathbf{0}, j^*)$ is between P and $P + \boldsymbol{e_j}$ for each $j \in \mathcal{A}$, $(\boldsymbol{x} +$

$\hat{C}(\mathbf{0}, j^*)) \cap P \neq \phi$ means $(\boldsymbol{x}, j^*) \in \mathcal{S}$ for any $\boldsymbol{x} \in \mathbf{Z}^d$. Since $\mathcal{T}'$ is a tiling of P, we get $\cup_{\boldsymbol{x} \in \mathbf{Z}^d} (\boldsymbol{x} + \hat{C}(\mathbf{0})) \cap P = P$. Therefore we have

$$\cup_{\boldsymbol{x} \in \mathbf{Z}^d} (\boldsymbol{x} + \hat{C}(\mathbf{0})) \cap (P + \boldsymbol{y}) = P + \boldsymbol{y} \ (\boldsymbol{y} \in \mathbf{Z}^d)\,.$$

Hence we know $\mathbf{R}^d = \hat{C}(\mathbf{0}) + \mathbf{Z}^d$. From (1), we see that $\mathcal{T}''$ is a tiling on $\mathbf{R}^d$. □

Replacing parallelograms $\pi_s'(\boldsymbol{x}, j^*)$ by the fractal sets $\pi_s'(\boldsymbol{x}) + \pi_s'(X_i)$, we define a collection:

$$\mathfrak{T} := \left\{ \pi_s'(\boldsymbol{x}) + \pi_s'(X_i) \;\middle|\; \boldsymbol{x} \in \mathcal{S} \right\}.$$

The following theorem is obtained by slight modification of the proof of Theorem 3.3 in [15].

THEOREM 4.4. *The following statements are equivalent*:

(1) $\mathfrak{T}$ *is a tiling of* P,

(2) $|X_i| = |\pi_s(\mathbf{0}, i^*)|$

(3) *For some* i, *the radius of the largest ball contained* $\bar{E}_1(\sigma_{AH}^*)^n(\mathbf{0}, i^*)$ *diverges as* $n \to \infty$.

(4) *For some* i, $\lim_{n\to\infty} \partial(-L_\sigma)^n \pi_s \bar{E}_1(\sigma_{AH}^*)^n(\mathbf{0}, i^*) = \partial X_i$.

The following Corollary is proved in the same manner as Proposition 4.3.

COROLLARY 4.5. *Under the coincidence condition*, *if one of the statements in Theorem* 4.4 *holds*, *then* $\hat{\mathfrak{T}} := \left\{ \boldsymbol{x} + \hat{X} \;\middle|\; \boldsymbol{x} \in \mathbf{Z}^d \right\}$ *is a tilling on* $\mathbf{R}^d$

The following is the main theorem.

THEOREM 4.6. *Assume that an AH-substitution* σ_{AH} *satisfies* (NP), (UM), (IR) *and* (FP), *that* σ_{AH} *satisfies the coincidence condition for AH-substitution*, *and that one of the statements in Theorem* 4.4 *holds. Then we have the following statements*:

(1) $\hat{X}$ *is a torus.*

(2) $\hat{T}_{\sigma_{AH}} : \hat{X} \to \hat{X}$ *is the group automorphism with a Markov partition* $\left\{ \hat{X}_{\binom{j}{k}} \mid \binom{j}{k} \in \mathcal{E} \right\}$.

(3) *The following commutative relation holds*:

$$\begin{array}{ccc} \mathbf{R}^d & \xrightarrow{-L_\sigma} & \mathbf{R}^d \\ {\scriptstyle pr}\big\downarrow & & \big\downarrow{\scriptstyle pr} \\ \hat{X} & \xrightarrow[\hat{T}_{\sigma_{AH}}]{} & \hat{X}, \end{array}$$

where pr *means a natural projection of* $\mathbf{R}^d$ *to the torus* $\hat{X}$.

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