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Radius Sphere Theorems for Compact Manifolds with Radial Curvature Bounded Below

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Abstract. We show a radius sphere theorem for a compact Riemannian manifold whose radial curvature at the base point is bounded from below by that of a 2-sphere of revolution. The diameter sphere theorem is expanded to a wide class of metrics.

1. Introduction

H. Hopf asked the following very natural question, that is, *noting the standard sphere is the only simply connected manifold of constant positive sectional curvature*, *we can hope to be able to prove if the sectional curvature is close to a positive constant*, *the underlying manifold will still be the sphere* (see [B2] for more details). For the first time, Rauch [R] gave the answer with a pinching constant of roughly $3/4$. By the race of a sharp estimate of a pinching constant between Klingenberg [Kl1], [Kl2] and Berger [B1], Hopf's question became a well-known theorem, so-called the *Classical Sphere Theorem*, with the pinching constant of $1/4$. We will emphasize that the standard sphere is employed as a reference space in comparison theorems to obtain that theorem.

In 1977, Grove and Shiohama have proved the following theorem, in which the hypothesis of *pinching* in between $(1/4, 1]$ is replaced by the hypothesis of the *diameter*:

THEOREM 1.1 (Diameter sphere theorem, [GS]). *Let X be a compact Riemannian n-manifold whose sectional curvature is bounded from below by* 1. *If the diameter is larger than* $\pi/2$, *then X is homeomorphic to the sphere* $\mathbf{S}^n$.

Here the reference space is the standard sphere again.

Recently, the author and Ohta [KO, Theorem A] have generalized the Diameter Sphere Theorem to compact Riemannian manifolds whose radial curvature at the base point are bounded from below by that of a von Mangoldt surface of revolution, which have a wider class of metrics than those described in [GS]. One purpose of this article is *to show the radius*

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sphere theorem to an another class of metrics, *which is different from the class described in* [KO] *and also extend* Theorem 1.1. Before stating our results, we will mention the definition of manifolds whose radial curvature at the base point are bounded from below by that of a model surface of revolution in the next subsection.

1.1. Manifolds with Radial Curvature Bounded From Below. Let (M, p) be a complete Riemannian n-manifold with a base point $p \in M$. We say (M, p) has *radial curvature at p bounded from below by* $K : [0, \ell) \to \mathbf{R}$ if, along every unit speed minimal geodesic $\gamma : [0, a) \to M$ with $\gamma(0) = p$, its sectional curvature K_M satisfies

$$K_M(\gamma'(t), X) \geq K(t)$$

for all $t \in [0, a)$ and $X \in T_{\gamma(t)}M$ with $X \perp \gamma'(t)$. Here $0 < \ell \leq \infty$ and $0 < a \leq \infty$ are constant. The function K is called the *radial curvature function* of a model surface $(\tilde{M}, \tilde{p})$ such that its metric $d\tilde{s}^2$ is expressed by, in terms of the geodesic polar coordinates around a base point $\tilde{p} \in \tilde{M}$,

$$d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2\,, \quad (t, \theta) \in (0, \ell) \times \mathbf{S}^1_{\tilde{p}}\,.$$

Here $f : (0, \ell) \to \mathbf{R}$ is a positive smooth function satisfying the Jacobi equation

$$f'' + Kf = 0\,, \quad f(0) = 0\,, \quad f'(0) = 1\,.$$

In the following Theorem 1.3, let $(\tilde{M}, \tilde{p})$ be a von Mangoldt surface of revolution (cf. [SST, Chapter 7], [T]). Namely, the radial curvature function $K : [0, \ell) \to \mathbf{R}$ of $(\tilde{M}, \tilde{p})$ is assumed to be monotone non-increasing on $(0, \ell)$. A round sphere is the only compact 'smooth' (i.e., $\lim_{t \to \ell} f'(t) = -1$) von Mangoldt surface of revolution. If a von Mangoldt surface of revolution $(\tilde{M}, \tilde{p})$ has the property $\ell < \infty$ and if it is not a round sphere, then $\lim_{t \to \ell} f(t) = 0$ and $\lim_{t \to \ell} f'(t) > -1$. Therefore $(\tilde{M}, \tilde{p})$ has a singular point, say $\tilde{q} \in \tilde{M}$, at the maximal distance from $\tilde{p} \in \tilde{M}$ such that $d(\tilde{p}, \tilde{q}) = \ell$. Its shape can be understood as a 'balloon'. We will emphasize that the radial curvature function of $(\tilde{M}, \tilde{p})$ may change signs, that is, does not always have positive. For example,

EXAMPLE 1.2 ([SiT2]).

(E–1) If $f(t) = \dfrac{t(1-t)(1+t)}{11t^4 - 25t^2 + 18}$, then we see $K' < 0$ on $(0, 1)$ and $-\infty < \lim_{t \to 1} K(t) < 0$. In particular, This compact von Mangoldt surface of revolution has a singular point at $t = 1$.

(E–2) If $f(t) = \sin t - \sin^3 t (= \sin t \cos^2 t)$, then we see $K' < 0$ on $(0, \frac{\pi}{2})$ and $\lim_{t \to \frac{\pi}{2}} K(t) = -\infty$.

Define $\mathrm{rad}_p := \sup_{x \in M} d(p, x)$ and fix a point $p^* \in M$ satisfying $d(p, p^*) = \mathrm{rad}_p$ (Remark that such a point is unique, see Proposition 3.3 in [KO]). Then, the author and Ohta have generalized Theorem 1.1 to compact Riemannian manifolds whose radial curvature at the base point are bounded from below by that of a von Mangoldt surface of revolution, kind like this:

THEOREM 1.3 ([KO, Theorem A]). *Let (M, p) be a compact Riemannian n-manifold whose radial curvature at p is bounded from below by $K : [0, \ell) \to \mathbf{R}$ for $\ell < \infty$, and let $\rho \in (0, \ell)$ be the zero of f' on $(0, \ell)$. If $\mathrm{rad}_p > \rho$ and if p is a critical point for some $z \in M \setminus \overline{B_\rho(p)}$, then M is homeomorphic to a sphere $\mathbf{S}^n$.*

Theorem 1.3 provides a sphere theorem for a new class of metrics, for the radial curvature may change signs (see Example 1.2). By GTCT-II (see [IMS, Theorem 1.3]) and the Clairaut relation, we see $\mathrm{rad}_p = \mathrm{rad}_{p^*}$ if and only if $d(p, p^*) = \mathrm{diam}(M)$ (see Appendix), thus the condition "*p is a critical point for some $z \in M \setminus \overline{B_\rho(p)}$*" in Theorem 1.3 is weaker condition than the "$\mathrm{rad}_p = \mathrm{rad}_{p^*}$". Therefore, Theorem 1.3 containes Theorem 1.1 as a special case, that is, p and p^* are points satisfying $d(p, p^*) = \mathrm{diam}(M)$, $f(t) = \sin t$, $\rho = \pi/2$, and, moreover, all sectional curvatures are bounded.

1.2. The 2-Sphere of Revolution. Now we will consider another class of compact models so called a 2-sphere of revolution.

A compact Riemannian manifold $\tilde{V}$ homeomorphic to a 2-sphere is called a *2-sphere of revolution* if $\tilde{V}$ admits a point $\tilde{p}$ such that for any two points $\tilde{q}_1, \tilde{q}_2$ on $\tilde{V}$ with $d(\tilde{p}, \tilde{q}_1) = d(\tilde{p}, \tilde{q}_2)$, there exists an isometry φ on $\tilde{V}$ satisfying $\varphi(\tilde{q}_1) = \tilde{q}_2$ and $\varphi(\tilde{p}) = \tilde{p}$. The point $\tilde{p}$ is called a *pole* of $\tilde{V}$. Let $(r, \theta) \in (0, \ell) \times \mathbf{S}^1_{\tilde{p}}$ denote geodesic polar coordinates around a pole $\tilde{p}$ of $\tilde{V}$. Then, we may give $\tilde{V}$ the Riemannian metric $\tilde{g} = dr^2 + m(r)^2 d\theta^2$ on $\tilde{V} \setminus \{\tilde{p}, \tilde{q}\}$, where $\tilde{q}$ denotes the unique cut point of $\tilde{p}$ with $d(\tilde{p}, \tilde{q}) = \ell$, and $m(r(\tilde{x})) := \sqrt{\tilde{g}\left(\left(\frac{\partial}{\partial\theta}\right)_{\tilde{x}}, \left(\frac{\partial}{\partial\theta}\right)_{\tilde{x}}\right)}$. Sinclair and Tanaka [SiT1, Lemma 2.1] have proved that *each pole of a 2-sphere of revolution $\tilde{V}$ has a unique cut point.* A pole $\tilde{p}$ and its unique cut point $\tilde{q}$ are called a *pair of poles*. Each geodesic emanating from a pole is a periodic geodesic through its cut point. Each periodic geodesic through a pair of poles is called a *meridian*. Throughout this article, our 2-sphere of revolution $(\tilde{V}, \tilde{p}) := (\tilde{V}, \tilde{g})$ with a pair of poles $\tilde{p}, \tilde{q}$ is *symmetric with respect to the reflection fixing the equator $r = \ell/2$* (this implies $m(r) = m(\ell - r)$ for any $r \in (0, \ell)$, in particular $m'(\ell/2) = 0$), and *the Gaussian curvature $K_{\tilde{V}}(\tilde{x}) = -\frac{m''(r(\tilde{x}))}{m(r(\tilde{x}))}$ of $(\tilde{V}, \tilde{p})$ for each $\tilde{x} \in \tilde{V} \setminus \{\tilde{p}, \tilde{q}\}$ is monotone non-increasing along a meridian from the point $\tilde{p}$ to the point on the equator $r = \ell/2$.* Now, we will note that a 2-sphere of revolution does not always have positive Gaussian curvature. The following example is due to Sinclair-Tanaka [SiT1]: Set $(m(r), 0, z(r))$ such that

$$m(r) := \frac{\sqrt{3}}{10}\left(9 \sin \frac{\sqrt{3}}{9} r + 7 \sin \frac{\sqrt{3}}{3} r\right), \quad z(r) := \int_0^r \sqrt{1 - m'(r)^2}\, dr$$

Then, $(m(r), 0, z(r))$ is a 2-sphere of revolution, and its Gaussian curvature is monotone non-increasing on $[0, 3\sqrt{3}\pi/2]$ and takes -1 on the equator $r = 3\sqrt{3}\pi/2$.

Being based on this example, we will consider a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $K_{\tilde{V}}(\ell/2) < 0$. Since $K_{\tilde{V}}$ along a meridian from $\tilde{p}$ is monotone non-increasing on $[0, \ell/2]$, $m'(r_0)$ is negative for some $r_0 \in (0, \ell/2)$, and there exist two numbers $\rho_1 \leq \rho_2$ in $(0, \ell/2)$ such that $m' > 0$ on $[0, \rho_1) \cup (\ell/2, \ell - \rho_2)$, $m' = 0$ on $[\rho_1, \rho_2] \cup [\ell - \rho_2, \ell - \rho_1]$, and $m' < 0$ on $(\rho_2, \ell/2) \cup (\ell - \rho_1, \ell]$. In particular, $m(\rho_1) = m(\rho_2)$ is the maximum of $m[0, \ell/2]$ which is greater than $m(\ell/2)$ (see [SiT1, Lemma 2.4] for the proof of these).

1.3. Main Theorems. Now, we will present sphere theorems, as our main theorems, for a compact Riemannian n-manifold (V, p) whose radial curvature at p is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $K_{\tilde{V}}(\ell/2) < 0$. Let $q \in V$ be the point such that $d(p, q) = \mathrm{rad}_p := \sup_{x \in V} d(p, x)$. In the following sphere theorems for these manifolds, let $\rho_1 \leq \rho_2$ in $(0, \ell/2)$ be the numbers in subsection 1.2, and set $\rho_3 := \ell - \rho_2$ and $\rho_4 := \ell - \rho_1$, so that $m'(\rho_i) = 0,\ i = 1, 2, 3, 4$ and $m(\rho_1) = m(\rho_2) = m(\rho_3) = m(\rho_4)$ is the maximum of $m[0, \ell]$ which is greater than $m(\ell/2)$. Then, we have the following:

THEOREM A. *Let (V, p) be a compact Riemannian n-manifold whose radial curvature at p is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $K_{\tilde{V}}(\ell/2) < 0$. Then, (V, p) is homeomorphic to a sphere $\mathbf{S}^n$, if one of the following conditions is satisfied*:

(A–1) $\ell/2 > \mathrm{rad}_p > \rho_2$ *and p is a critical point for some point in* $V \setminus \overline{B_{\rho_2}(p)}$.

(A–2) $\mathrm{rad}_p > \rho_4$ *and p is a critical point for some point in* $V \setminus \overline{B_{\rho_4}(p)}$.

Theorem A provides a sphere theorem for a new class of metrics, for the radial curvature may change signs (see Subsection 1.2).

On the other hand, if one replaces the $K_{\tilde{V}}(\ell/2) < 0$ by $m' \geq 0$ on $[0, \ell/2]$, then $m' > 0$ on $(0, \rho_5)$, where ρ_5 denotes the minimum of $m^{-1}(m(\ell/2))$. Furthermore, m attains the maximum $m(\ell/2)$ of $m[0, \ell]$ at each pint of $[\rho_5, \ell - \rho_5]$ (see [SiT1, Lemma 2.4]). Remark $m' = 0$ on $[\rho_5, \ell - \rho_5]$. In this situation, we define $\rho_6 := \ell - \rho_5$. Then, we have the following Corollary to Theorem A;

COROLLARY A. *Let (V, p) be a compact Riemannian n-manifold whose radial curvature at p is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $m' \geq 0$ on $[0, \ell/2]$. If $\mathrm{rad}_p > \rho_6$ and p is a critical point for some point in $V \setminus \overline{B_{\rho_6}(p)}$, then, (V, p) is homeomorphic to a sphere $\mathbf{S}^n$.*

Therefore, Theorem A containes Theorem 1.1 as a special case, that is, p and q are points satisfying $d(p, q) = \mathrm{diam}(V)$, $m(r) = \sin r$, $\rho_6 = \pi/2$, and, moreover, all sectional curvatures are bounded.

We denote by $\mathrm{vol}(V)$ the volume of V. Then, we also obtain the following another kind of sphere theorem;

THEOREM B. *Let (V, p) be a compact Riemannian n-manifold whose radial curvature at p is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $K_{\tilde{V}}(\ell/2) < 0$. Then (V, p) is homeomorphic to a sphere $\mathbf{S}^n$, if the following condition is satisfied*:

$$\mathrm{vol}(V) > \frac{1}{2}\{\mathrm{vol}(B^n_{\rho_4}(\tilde{p})) + \mathrm{vol}(\tilde{V}^n)\}\,.$$

Here $\tilde{V}^n$ is an n-model of a 2-sphere type, $B^n_{\rho_4}(\tilde{p}) \subset \tilde{V}^n$ is the distance ρ_4-ball around the base point $\tilde{p} \in \tilde{V}^n$.

Theorem B provides a sphere theorem for a new class of metrics, for the radial curvature may change signs (see Subsection 1.2).

As well as Theorem A, if one replaces the $K_{\tilde{V}}(\ell/2) < 0$ by $m' \geq 0$ on $[0, \ell/2]$, then we have the following Corollary to Theorem B;

COROLLARY B. *Let (V, p) be a compact Riemannian n-manifold whose radial curvature at p is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $m' \geq 0$ on $[0, \ell/2]$. Then (V, p) is homeomorphic to a sphere $\mathbf{S}^n$, if the following condition is satisfied*:

$$\mathrm{vol}(V) > \frac{1}{2}\{\mathrm{vol}(B^n_{\rho_6}(\tilde{p})) + \mathrm{vol}(\tilde{V}^n)\}\,.$$

2. Preliminaries

Let (V, p) be a compact Riemannian n-manifold whose radial curvature is bounded from below by the Gaussian curvature $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$. Sinclair and Tanaka have proved the following structure theorem of the cut locus of $\tilde{x} \in \tilde{V} \setminus \{\tilde{p}, \tilde{q}\}$.

THEOREM 2.1 ([SiT1, Main Theorem]). *Let $(\tilde{V}, \tilde{p})$ be a 2-sphere of revolution with a pair of poles $\tilde{p}, \tilde{q}$. Then, the cut locus of a point $\tilde{x} \in \tilde{V} \setminus \{\tilde{p}, \tilde{q}\}$ with $\theta(\tilde{x}) = 0$ is a single point or a subarc of the opposite half meridian $\theta = \pi$ (respectively the parallel $r = \ell - r(\tilde{x})$) when $K_{\tilde{V}}$ is montone non-increasing (respectively non-decreasing) along a meridian from $\tilde{p}$ to the point on $r = \ell/2$. Furthermore if the cut locus of a point $\tilde{x} \in \tilde{V} \setminus \{\tilde{p}, \tilde{q}\}$ is a single point, the Gaussian curvature is constant.*

By Theorem 2.1, it is possible to find a geodesic triangle $\tilde{\triangle}(pxy) := \triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \tilde{V}$ corresponding to an arbitrarily given geodesic triangle $\triangle(pxy) \subset V$. Then, Sinclair and Tanaka have also proved the following Toponogov comparison theorem, which is the basic tool used in this article.

THEOREM 2.2 ([SiT1, Theorem 6.1]). *Let (V, p) be a compact Riemannian n-manifold whose radial curvature is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$. Suppose the cut locus of any point on $(\tilde{V}, \tilde{p})$ distinct from its two poles $\tilde{p}, \tilde{q}$ is a subarc of the opposite meridian to the point. Then, for every geodesic triangle $\triangle(pxy) \subset V$, there exists a geodesic triangle $\tilde{\triangle}(pxy) = \triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \tilde{V}$ such that*

$$d(\tilde{p}, \tilde{x}) = d(p, x), \quad d(\tilde{p}, \tilde{y}) = d(p, y), \quad d(\tilde{x}, \tilde{y}) = d(x, y)$$

and that

$$\angle(pxy) \geq \angle(\tilde{p}\tilde{x}\tilde{y}), \quad \angle(pyx) \geq \angle(\tilde{p}\tilde{y}\tilde{x}), \quad \angle(xpy) \geq \angle(\tilde{x}\tilde{p}\tilde{y}).$$

Here we denote by $\angle(pxy)$ the angle between the geodesics from x to p and y forming the triangle $\triangle(pxy)$.

From this theorem, we may have the following Alexandrov Convexity property:

COROLLARY 2.3. *Under the same assumption as in Theorem 2.2, let $\gamma : [0, a] \to V$ and $\tilde{\gamma} : [0, a] \to \tilde{V}$ be the edges of $\triangle(pxy)$ and $\tilde{\triangle}(pxy)$ from x and $\tilde{x}$ to y and $\tilde{y}$, respectively. Then we have, for all $s \in [0, a]$,*

$$d(p, \gamma(s)) \geq d(\tilde{p}, \tilde{\gamma}(s)).$$

REMARK 2.4. We refer [IMS], the author and Ohta' [KO], and [SiT1] for the history of comparison theorems for radial curvature sort.

Let V be an arbitrary complete Riemannian manifold. Then, recall, for a fixed point $q \in V$, a point $x \in V \setminus \{q\}$ is called a *critical point* for q if, for every nonzero tangent vector $v \in T_x V$, we find a minimal geodesic γ from x to q satisfying $\angle(v, \gamma'(0)) \leq \pi/2$ (see [Gv]). Then, we have the following theorem:

THEOREM 2.5 ([Gv, Isotopy Lemma]). *Let V be a complete Riemannian manifold. If $0 < R_1 < R_2 \leq \infty$, and if $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ has no critical point for p, then $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ is homeomorphic to $\partial B_{R_1}(p) \times [R_1, R_2]$.*

3. Proof of Theorem A

In this section, let (V, p) be a compact Riemannian n-manifold V whose radial curvature at p is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $K_{\tilde{V}}(\ell/2) < 0$. We only prove the case of (A–2) in Theorem A, for one may show the case of (A–1) in Theorem A by the same way of proof of the case of (A–2). Thus, (V, p) satisfies $\mathrm{rad}_p > \rho_4$.

By Theorem 2.2 and the assumption in the case of (A–2), we have

PROPOSITION 3.1. *Assume p is a critical point for some point in $V \setminus \overline{B_{\rho_4}(p)}$. Then there is no critical point for p in $\overline{B_{\rho_2}(p)} \setminus \{p\}$ and $\overline{B_{\rho_4}(p)} \setminus B_{\ell/2}(p)$ respectively.*

One may show Proposition 3.1 by the same way of the proof of [KO, Proposition 3.4], so we omit the proof in this article.

PROPOSITION 3.2. *Assume p is a critical point for some point in $V \setminus \overline{B_{\rho_4}(p)}$. There is no critical point for p in $\overline{B_{\ell/2}(p)} \setminus B_{\rho_2}(p)$ and $(V \setminus \{q\}) \setminus B_{\rho_4}(p)$ respectively. Here, q is the point in V such that $d(p, q) = \mathrm{rad}_p$. In particular, $d(p, \cdot)$ attains its maximum at a unique point $q \in V$.*

REMARK 3.3. This proposition will be also obtained by the same way of the proofs of [KO, Lemma 3.1], [KO, Lemma 3.2], and [KO, Proposition 3.3], but we will give another proof in the following.

PROOF. We only prove that $(V \setminus \{q\}) \setminus B_{\rho_4}(p)$ has no critical point of $d(p, \cdot)$, for one may prove that $\overline{B_{\ell/2}(p)} \setminus B_{\rho_2}(p)$ has no critical point of $d(p, \cdot)$ by the same way.

We suppose that there exists a critical point $x \in (V \setminus \{q\}) \setminus B_{\rho_4}(p)$. Fix a minimal geodesic $\tau : [0, 1] \to V$ from z to x. Here, $z \in V \setminus B_{\rho_4}(p)$ is the point which p is a critical point for. As x is a critical point for p, we find a minimal geodesic $\gamma : [0, 1] \to V$ from p to x for which $\angle(\tau'(1), \gamma'(1)) \le \pi/2$ holds. Furthermore, since p is a critical point for z, there also exists a minimal geodesic $\sigma : [0, 1] \to V$ from p to z satisfying $\angle(\sigma'(0), \gamma'(0)) \le \pi/2$. Consider a comparison triangle $\tilde{\triangle}(pzx) \subset \tilde{V}$ corresponding to the triangle $\triangle(pzx)$ consisting of γ, τ, and σ, and denote by $\tilde{\gamma}$, $\tilde{\tau}$, and $\tilde{\sigma}$ the edges corresponding to γ, τ, and σ, respectively. It follows from Theorem 2.2 that we have

$$\angle(\tilde{\tau}'(1), \tilde{\gamma}'(1)) \le \angle(\tau'(1), \gamma'(1)) \le \frac{\pi}{2}, \tag{3.1}$$

$$\angle(\tilde{\sigma}'(0), \tilde{\gamma}'(0)) \le \angle(\sigma'(0), \gamma'(0)) \le \frac{\pi}{2}. \tag{3.2}$$

Then, by (3.1), we have the following two possibilities;

(P–1) there exist two numbers $0 < s_- < s_+ < 1$ such that $\tilde{\tau}((s_-, s_+)) \subset B_{\rho_4}(\tilde{p})$ with $\tilde{\tau}(s_-), \tilde{\tau}(s_+) \in \partial B_{\rho_4}(\tilde{p})$.

(P–2) $\tilde{\tau}(s) \subset \tilde{V} \setminus B_{\rho_4}(\tilde{p})$ for any $s \in [0, 1]$.

In the case of (P–1), by (3.2), we have

$$\angle(\tilde{\tau}(s_-)\tilde{p}\tilde{\tau}(s_+)) < \frac{\pi}{2}. \tag{3.3}$$

Since $\partial B_{\rho_4}(\tilde{p})$ is a simple closed geodesic by $m'(\rho_4) = 0$, it follows from (3.3) that there is an another minimal geodesic between $\tilde{\tau}(s_-)$ and $\tilde{\tau}(s_+)$ contained in $\partial B_{\rho_4}(\tilde{p})$, and hence $\tilde{\tau}(s_+) \in \mathrm{Cut}(\tilde{\tau}(s_-))$. This contradicts to the Sinclair and Tanaka' structure theorem of the cut locus (see Theorem 2.1 in Section 2). Therefore, $(V \setminus \{q\}) \setminus B_{\rho_4}(p)$ has no critical point of $d(p, \cdot)$.

In the case of (P–2), by (3.1), there exists $s_0 \in (0, 1]$ such that $\angle(-\nabla r(\tilde{\tau}(s_0)), \tilde{\tau}'(s_0)) = \pi/2$, where $\nabla r := \frac{\partial}{\partial r}$ is the gradient vector field of the distance function to $\tilde{p}$. Remark we have $d(\tilde{p}, \tilde{\tau}(s_0)) < d(\tilde{p}, \tilde{z})$. By the Clairaut relation, we have

$$m(d(\tilde{p}, \tilde{\tau}(s_0))) = m(d(\tilde{p}, \tilde{\tau}(s_0))) \sin \frac{\pi}{2} = m(d(\tilde{p}, \tilde{z})) \sin \angle(-\tilde{\gamma}'(1), \tilde{\tau}'(0)) . \tag{3.4}$$

The relation (3.4) implies $m(d(\tilde{p}, \tilde{\tau}(s_0))) < m(d(\tilde{p}, \tilde{z}))$. Since $m' < 0$ on (ρ_4, ℓ), we have $d(\tilde{p}, \tilde{\tau}(s_0)) > d(\tilde{p}, \tilde{z})$, which contradicts to the relation $d(\tilde{p}, \tilde{\tau}(s_0)) < d(\tilde{p}, \tilde{z})$. Therefore, $(V \setminus \{q\}) \setminus B_{\rho_4}(p)$ has no critical point of $d(p, \cdot)$.

Finally, we will prove that $d(p, \cdot)$ attains its maximum at a unique point $q \in V$. Suppose that there exists a point $q^* \in \partial B_{\mathrm{rad}_p}(p)$ such that $q \neq q^*$. Take a comparison triangle $\tilde{\triangle}(pqq^*) \subset \tilde{V}$ corresponding to the triangle $\triangle(pqq^*)$, and let $\tilde{\tau} : [0, 1] \to \tilde{V}$ and $\tau : [0, 1] \to V$ be minimal geodesics joining $\tilde{q}$ and $\tilde{q^*}$, q and q^*, respectively. By the above consideration in (P–1) and (P–2) (more precisely by the Clairaut relation for a minimal geodesic in $\tilde{V} \setminus B_{\rho_4}(\tilde{p})$), we see, for every $s \in (0, 1)$,

$$\tilde{\tau}(s) \subset \tilde{V} \setminus \overline{B_{\mathrm{rad}_p}(\tilde{p})} . \tag{3.5}$$

Thus, by Corollary 2.3 and (3.5), for every $s \in (0, 1)$, we have

$$d(p, \eta(s)) \geq d(\tilde{p}, \tilde{\eta}(s)) > \mathrm{rad}_p .$$

This contradicts to the definition of rad_p, so that $d(p, \cdot)$ attains its maximun at a unique point q. □

Thus, by Proposition 3.1 and 3.2, $d(p, \cdot)$ has only two critical point p, q. Therefore, it follows from [Gv, Isotopy Lemma] (see Theorem 2.5 in Section 2) that (V, p) is homeomorphic to a sphere $\mathbf{S}^n$. □

4. Proof of Theorem B

Let (V, p) be a compact Riemannian n-manifold whose radial curvature is bounded from below by $K_{\tilde{V}} : [0, \ell) \to \mathbf{R}$ of a 2-sphere of revolution $(\tilde{V}, \tilde{p})$ with $K_{\tilde{V}}(\ell/2) < 0$, and (V, p) satisfies

$$\mathrm{vol}(V) > \frac{1}{2}\{\mathrm{vol}(B^n_{\rho_4}(\tilde{p})) + \mathrm{vol}(\tilde{V}^n)\} . \tag{4.1}$$

We denote by $S_pV \subset T_pV$ the unit tangent sphere at p, and set

$$D(p) := \{rv \mid v \in S_pV,\ r \geq 0,\ \exp_p([0, r]v) \cap \mathrm{Cut}(p) = \emptyset\} .$$

Define the map $\Pi : T_pV \setminus \{0\} \to S_pV$ by $\Pi(v) := v/\|v\|$. For each $t > 0$, we put

$$\Omega_t := \Pi(\exp_p^{-1}[V \setminus B_t(p)] \cap D(p)) \subset S_pV .$$

By (4.1), we find

$$\mathrm{vol}(V) > \frac{1}{2}\,\mathrm{vol}(\tilde{V}) + \frac{1}{2}\,\mathrm{vol}(B_{\rho_4}(\tilde{p})) = \frac{1}{2}\,\mathrm{vol}(\tilde{V} \setminus B_{\rho_4}(\tilde{p})) + \mathrm{vol}(B_{\rho_4}(\tilde{p}))$$

$$\geq \frac{1}{2}\operatorname{vol}(\tilde{V} \setminus B_{\rho_4}(\tilde{p})) + \operatorname{vol}(B_{\rho_4}(p)),$$

and hence, we get

$$\operatorname{vol}(V \setminus B_{\rho_4}(p)) > \frac{1}{2}\operatorname{vol}(\tilde{V} \setminus B_{\rho_4}(\tilde{p})). \tag{4.2}$$

This implies that we can choose $\varepsilon > 0$ and $t > \rho_4$ such that Ω_t is $(\pi/2 - \varepsilon)$-dense in $(S_pV, \angle)$, where we denote by $\angle$ the angle distance on S_pV. Then, let γ be a minimizing geodesic emanating from p to any point $x \in V$. By the denseness of $\Omega_t \subset S_pV$,

(B–1) there exist a point $y \in V \setminus B_t(p)$ and a minimizing geodesic σ emanating from p to y such that $\angle(\sigma'(0), \gamma'(0)) \leq \frac{\pi}{2} - \varepsilon$.

Moreover, it follows from (4.2) that we have

$$\operatorname{rad}_p > \rho_4 \tag{4.3}$$

which implies $V \setminus B_{\rho_4}(p)$ is not empty. The (B–1) and (4.3) are the essentially same situations as Theorem A. Thus, by the same way of proofs of propositions in Theorem A, we have the following two propositions;

PROPOSITION 4.1. *There is no critical point for p in $\overline{B_{\rho_2}(p)} \setminus \{p\}$ and $\overline{B_{\rho_4}(p)} \setminus B_{\ell/2}(p)$ respectively.*

PROPOSITION 4.2. *There is no critical point for p in $\overline{B_{\ell/2}(p)} \setminus B_{\rho_2}(p)$ and $(V \setminus \{q\}) \setminus B_{\rho_4}(p)$ respectively. Here, q is the point in V such that $d(p, q) = \operatorname{rad}_p$. In particular, $d(p, \cdot)$ attains its maximum at a unique point $q \in V$.*

Thus, by Proposition 4.1 and 4.2, $d(p, \cdot)$ has only two critical point p, q. Therefore, it follows from [Gv, Isotopy Lemma] (see Theorem 2.5 in Section 2) that (V, p) is homeomorphic to a sphere $\mathbf{S}^n$. □

5. Appendix

In this appendix, we show the following lemma.

LEMMA 5.1. *Let (M, p) be a compact Riemannian n-manifold whose radial curvature at p is bounded from below by the radial curvature function $K : [0, \ell) \to \mathbf{R}$ of a von Mangoldt surface of revolution $(\tilde{M}, \tilde{p})$ for $\ell < \infty$, and let $\rho \in (0, \ell)$ be the zero of f' on $(0, \ell)$ and satisfy $\operatorname{rad}_p := d(p, p^*) > \rho$. Then, $\operatorname{rad}_p = \operatorname{rad}_{p^*}$ if and only if $d(p, p^*) = \operatorname{diam}(M)$.*

PROOF. If $d(p, p^*) = \operatorname{diam}(M)$ holds, then we have $\operatorname{rad}_p = \operatorname{rad}_{p^*}$. So, in the following we will show that if $\operatorname{rad}_p = \operatorname{rad}_{p^*}$ holds, then we have $d(p, p^*) = \operatorname{diam}(M)$. We first remark that p^* is a unique point satisfying $\operatorname{rad}_p = d(p, p^*)$ (see Proposition 3.3 in [KO]). Suppose that there exist a point $p^{**} \in M$ such that $p^{**} \neq p$ and $d(p^*, p^{**}) = \operatorname{rad}_{p^*}$. Fix a minimal geodesic $\gamma : [0, 1] \to M$ from p to p^{**}. Since $\sup_{x \in M} d(p^*, x) = \operatorname{rad}_{p^*}$, p^{**} is a critical point for p^* so that there exists a minimal geodesic $\tau : [0, 1] \to M$ from p^* to p^{**} for which $\angle(\gamma'(1), \tau'(1)) \leq \pi/2$ holds. On the other hand, since $d(p^*, p) = \mathrm{rad}_p = \mathrm{rad}_{p^*} = \sup_{x\in M} d(p^*, x)$, p is also a critical point for p^* so that there exists a minimal geodesic $\sigma : [0, 1] \to M$ from p to p^* satisfying $\angle(\sigma'(0), \gamma'(0)) \leq \pi/2$. Consider a comparison triangle $\tilde{\triangle}(pp^*p^{**}) \subset \tilde{M}$ corresponding to the triangle $\triangle(pp^*p^{**}) \subset M$ consisting of γ, τ, and σ, and denote by $\tilde{\gamma}$, $\tilde{\tau}$, and $\tilde{\sigma}$ the edges corresponding to γ, τ, and σ, respectively. Then it follows from GTCT-II (see [IMS, Theorem 1.3]) that

$$\angle(\tilde{\gamma}'(1), \tilde{\tau}'(1)) \leq \angle(\gamma'(1), \tau'(1)) \leq \frac{\pi}{2}, \tag{5.1}$$

$$\angle(\tilde{\sigma}'(0), \tilde{\gamma}'(0)) \leq \angle(\sigma'(0), \gamma'(0)) \leq \frac{\pi}{2}. \tag{5.2}$$

Then, we see $\tilde{p}^{**} \in B_\rho(\tilde{p})$. Indeed, by the inequality (5.1), there exists $s_0 \in (0, 1]$ such that $\angle(-\nabla t(\tilde{\tau}(s_0)), \tilde{\tau}'(s_0)) = \pi/2$ where $\nabla t := \frac{\partial}{\partial t}$ is the gradient vector field of the distance function to $\tilde{p}$. By the Clairaut relation, we have

$$f(d(\tilde{p}, \tilde{\tau}(s_0))) = f(d(\tilde{p}, \tilde{\tau}(s_0))) \sin\frac{\pi}{2} = f(d(\tilde{p}, \tilde{\tau}(1))) \sin\angle(-\tilde{\gamma}'(1), \tilde{\tau}'(1)). \tag{5.3}$$

The relation (5.3) implies $f(d(\tilde{p}, \tilde{\tau}(s_0))) \leq f(d(\tilde{p}, \tilde{\tau}(1)))$. Since $f' > 0$ on $(0, \rho)$, we have $d(\tilde{p}, \tilde{\tau}(s_0)) \leq d(\tilde{p}, \tilde{\tau}(1)) = d(\tilde{p}, \tilde{p}^{**}) < \rho$, that is $\tilde{p}^{**} \in B_\rho(\tilde{p})$.

By the assumption $\mathrm{rad}_p > \rho$ and $\tilde{p}^{**} \in B_\rho(\tilde{p})$, we can take $s_- \in (0, s_0)$ with $\tilde{\tau}(s_-) \in \partial B_\rho(\tilde{p})$. Note, if we extend $\tilde{\tau}$, then $\tilde{\tau}(s_+) \in \partial B_\rho(\tilde{p})$, where we set $s_+ := 2s_0 - s_-$. It follows from (5.2) that $\angle(\tilde{\tau}(s_-)\tilde{p}\tilde{\tau}(s_+)) < 2\angle(\tilde{p}^*\tilde{p}\tilde{p}^{**}) \leq \pi$, and hence $\tilde{\tau}$ is minimal on $[s_-, s_+]$. However, there is an another minimal geodesic between $\tilde{\tau}(s_-)$ and $\tilde{\tau}(s_+)$ contained in $\partial B_\rho(\tilde{p})$, and hence $\tilde{\tau}(s_+) \in \mathrm{Cut}(\tilde{\tau}(s_-))$. This contradicts to Tanaka' structure theorem of the cut locus $\mathrm{Cut}(\tilde{\tau}(s_-))$ (see [T, Main Theorem]). Thus, $p \in M$ is a unique point satisfying $d(p^*, p) = \mathrm{rad}_{p^*} = \sup_{x\in M} d(p^*, x)$. Therefore we have $d(p, p^*) = \mathrm{diam}(M)$. □

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