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On a Characterization of Compact Hausdorff Space X for Which Certain Algebraic Equations Are Solvable in $C(X)$

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Abstract. Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex-valued continuous functions on X. We consider the following property of $C(X)$: for each $f \in C(X)$ there exist a $g \in C(X)$ and positive integers p and q such that p does not divide q and $f^q = g^p$. When X is locally connected, we give a necessary and sufficient condition for $C(X)$ to have this property. We also give a characterization of a first-countable compact Hausdorff space X for which $C(X)$ has the property above. As a corollary, we prove that if X is locally connected, or first-countable, then $C(X)$ has the property above if and only if $C(X)$ is algebraically closed.

1. Introduction and the statement of results

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex-valued continuous functions on X with respect to the pointwise operations and the supremum norm $\|\cdot\|_\infty$. Suppose that X is locally connected and A is a uniform algebra on X. Čirka [2] proved that if to each $f \in A$ there corresponds a $g \in A$ such that $f = g^2$, then $A = C(X)$. On the other hand, there is no continuous function on the unit circle S^1 in the complex plane $\mathbf{C}$, whose square is the identity function on S^1. Hatori and Miura [8, Theorem 2.2] gave a characterization in order for $C(X)$ to be *square root closed*, that is, to each $f \in C(X)$ there corresponds a $g \in C(X)$ such that $f = g^2$. To be more explicit, $C(X)$ is square root closed if and only if the covering dimension of X is less than or equal to 1 and the first Čech cohomology group with integer coefficient is trivial.

Let $P(x, z)$ be a monic polynomial over $C(X)$: for a positive integer n and $a_0, a_1, \dots, a_{n-1} \in C(X)$, $P(x, z) = z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x)$ for $x \in X$. We say that $C(X)$ is *algebraically closed* if for each monic polynomial $P(x, z)$ over $C(X)$ there exists an $f \in C(X)$ such that $P(x, f(x)) = 0$ for every $x \in X$. By definition, $C(X)$ is square root closed if $C(X)$ is algebraically closed. Deckard and Pearcy [4, 5] proved that $C(X)$ is algebraically closed if X is a Stonian space, or a totally disconnected compact Hausdorff space, or a linearly ordered and order-complete topological space. They also remarked that if X is the closure of the graph of the function $y = \sin 1/x$, $0 < x \le 1$, then there exists

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a continuous function f of X into $\mathbf{C}$ such that $f \neq g^2$ for any $g \in C(X)$. Countryman, Jr. [3] gave some necessary and sufficient conditions for a first-countable compact Hausdorff space X in order that $C(X)$ is algebraically closed. For example, $C(X)$ is algebraically closed if and only if $C(X)$ is square root closed. Moreover, for every first-countable space, these are also equivalent to the condition that X is hereditarily unicoherent and almost locally connected. Miura and Niijima [13] gave some necessary and sufficient condition for a locally connected compact Hausdorff space X in order for $C(X)$ be algebraically closed.

It seems that Gorin and Karahanjan [7] strengthened the above result of Čirka as follows: If A is a uniform algebra on a locally connected compact Hausdorff space X with the property that for each $f \in A$ there exist a $g \in A$ and a $p \in \mathbf{N}$, $p \geq 2$ such that $f = g^p$, then $A = C(X)$. Furthermore, Karahanjan (cf. [9, Theorem 1]) weakened the hypothesis in the following way and proved that $A = C(X)$ whenever X is locally connected:

($*$) For every $f \in A$ there exist a $g \in A$ and $p, q \in \mathbf{N}$ such that $q/p \notin \mathbf{N}$ and $f^q = g^p$.

Note that if we replace "$q/p \notin \mathbf{N}$" with "$q/p \in \mathbf{N}$" in ($*$), then the condition ($*$) obviously holds for every A.

In this paper, we give a necessary and sufficient condition for a locally connected compact Hausdorff space X in order that $C(X)$ satisfies the condition ($*$). As a corollary, we also prove that if X is locally connected, or first-countable, then the condition ($*$) holds for $C(X)$ if and only if $C(X)$ is algebraically closed; In this case, ($*$) for $C(X)$ is equivalent to the square root closedness of $C(X)$.

We say that a topological space T is *almost locally connected* if T contains no mutually disjoint connected closed subsets C_n ($n \in \mathbf{N}$), which are open in the closure of $\cup_{n\in\mathbf{N}} C_n$ in T, with the following property: There exist $x_n, y_n \in C_n$ such that $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ converge to distinct points. For example, the closure of the graph of the function $y = \sin 1/x$, $0 < x \leq 1$ is *not* almost locally connected.

We say that a topological space T is *hereditarily unicoherent* if $M \cap N$ is connected for every pair of closed connected subsets M and N of T. For example, the unit circle S^1 is *not* hereditarily unicoherent.

Let Y be a normal space and n a non-negative integer. The covering dimension $\dim Y$ of Y is less than or equal to n if for every finite open covering $\mathfrak{A}$ of Y there exists a refinement $\mathfrak{B}$ of $\mathfrak{A}$ such that each $y \in Y$ belongs to at most $(n+1)$ elements of $\mathfrak{B}$. It is well-known that $\dim Y \leq n$ if and only if for every closed subset F of Y and every S^n-valued continuous function f on F, there exists an S^n-valued continuous function $\tilde{f}$ on Y such that $\tilde{f}|_F = f$, where S^n is the n-sphere (cf. [14]).

Let X be a compact Hausdorff space. Then $\check{H}^1(X; \mathbf{Z})$ denotes the first Čech cohomology group of X with integer coefficients. Let $C(X)^{-1}$ be the multiplicative group of all invertible elements of $C(X)$ and $\exp C(X) = \{e^f : f \in C(X)\}$. It is well-known that $\check{H}^1(X; \mathbf{Z})$ is isomorphic to the quotient group $C(X)^{-1}/\exp C(X)$, by a theorem of Arens and Royden [6]. In particular, $\check{H}^1(X; \mathbf{Z})$ is *trivial* if and only if $C(X)^{-1} = \exp C(X)$.

Now we are ready to state our main result. The main result of this paper is as follows:

THEOREM 1.1. *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

(a) *For each $f \in C(X)$ there exist $p, q \in \mathbf{N}$ and $g \in C(X)$ such that $q/p \notin \mathbf{N}$ and $f^q = g^p$.*

(b) *X is hereditarily unicoherent.*

(c) $\dim X \leq 1$ *and* $\check{H}^1(X; \mathbf{Z})$ *is trivial.*

(d) $\{g^p : g \in C(X)\}$ *is uniformly dense in $C(X)$ for every $p \in \mathbf{N}$.*

(e) *For each $f \in C(X)$ and $p \in \mathbf{N}$ there exists a $g \in C(X)$ such that $f = g^p$.*

COROLLARY 1.2. *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

(a) *For each $f \in C(X)$ there exist $p, q \in \mathbf{N}$ and $g \in C(X)$ such that $q/p \notin \mathbf{N}$ and $f^q = g^p$.*

(b) $\{g^p : g \in C(X)\}$ *is uniformly dense in $C(X)$ for* every $p \in \mathbf{N}$.

(c) *For each $f \in C(X)$ and $p \in \mathbf{N}$ there exists a $g \in C(X)$ such that $f = g^p$.*

(d) *$C(X)$ is algebraically closed.*

(e) *$C(X)$ is square-root closed.*

(f) *X is hereditarily unicoherent.*

(g) $\dim X \leq 1$ *and* $\check{H}^1(X; \mathbf{Z})$ *is trivial.*

COROLLARY 1.3. *Let X be a first-countable compact Hausdorff space. Then each of the following conditions implies the other.*

(a) *For each $f \in C(X)$ there exist $p, q \in \mathbf{N}$ and $g \in C(X)$ such that $q/p \notin \mathbf{N}$ and $f^q = g^p$.*

(b) *$C(X)$ is algebraically closed.*

(c) *$C(X)$ is square-root closed.*

(d) *X is hereditarily unicoherent and almost locally connected.*

(e) *X is almost locally connected*, $\dim X \leq 1$ *and* $\check{H}^1(X; \mathbf{Z})$ *is trivial.*

2. Lemmas

We require some lemmas before proving Theorem 1.1. To prove Lemmas 2.1 and 2.2, we use ideas by Countryman, Jr. [3, Lemma 2.1, Lemma 2.3].

LEMMA 2.1. *Let X be a compact Hausdorff space. If the condition* (a) *of Theorem* 1.1 *holds, then X is hereditarily unicoherent.*

PROOF. Assume that the condition (a) holds. We will show that X is hereditarily unicoherent. Suppose not. Then, by definition, there exist non-empty closed connected subsets M and N of X such that $M \cap N$ is disconnected. So, there are non-empty closed subsets A and B such that $M \cap N = A \cup B$ and $A \cap B = \emptyset$. Let f be a continuous mapping of X into the closed unit interval [0, 1] such that $f(x) = 0$ on A and $f(x) = 1$ on B. Put

$$h(x) = \begin{cases} \exp(i\pi f(x)) & x \in M \\ \exp(-i\pi f(x)) & x \in N \setminus M\,. \end{cases}$$

Then we see that h is continuous on $M \cup N$. Let $\tilde{h} \in C(X)$ be a mapping so that $\tilde{h}|_{M\cup N} = h$. By the condition (a), there exist positive integers p, q and an element $\tilde{g}$ in $C(X)$ such that p does not divide q and $\tilde{h}^q = \tilde{g}^p$. Put $q = sp + r$, where s and r are integers with $1 \le r \le p-1$ (note $q/p \notin \mathbf{N}$). Since h does not vanish on $M \cup N$, the function $g = \tilde{g}|_{M\cup N}/h^s$ is a well-defined continuous mapping of $M \cup N$ into $\mathbf{C}$. Since $\tilde{h}^q = \tilde{g}^p$, for each $x \in M \cup N$ we obtain

$$g^p(x) = \left(\frac{\tilde{g}(x)}{h^s(x)}\right)^p = \frac{\tilde{h}^q(x)}{h^{sp}(x)} = h^{q-sp}(x) = h^r(x)\,,$$

and so $h^r = g^p$ on $M \cup N$. Since

$$g^p(x) = h^r(x) = \exp(i\pi r f(x))$$

for $x \in M$, we get

$$g(x) = \omega(x)\exp\left(\frac{i\pi r f(x)}{p}\right)$$

for every $x \in M$, where $\omega(x)$ is one of the p-th roots of 1. The above equation and the continuity of f and g imply that $\omega(x)$ is a continuous mapping of M into the set of all p-th roots of 1. Since M is connected, ω must be constant. So there is a p-th root ω_0 of 1 such that

$$g(x) = \omega_0 \exp\left(\frac{i\pi r f(x)}{p}\right) \tag{1}$$

for each x in M. In a way similar to the above, we see that there exists a p-th root γ_0 of 1 such that

$$g(x) = \gamma_0 \exp\left(-\frac{i\pi r f(x)}{p}\right) \tag{2}$$

for each x in N.

Pick an $x_0 \in A$ arbitrarily. Since $x_0 \in A \subset M \cap N$, the equations (1) and (2) imply that

$$\omega_0 \exp\left(\frac{i\pi r f(x_0)}{p}\right) = g(x_0) = \gamma_0 \exp\left(-\frac{i\pi r f(x_0)}{p}\right).$$

Recall that $f = 0$ on A and $f = 1$ on B, and so $f(x_0) = 0$. We thus obtain $\omega_0 = \gamma_0$. For $y \in B$, it follows from (1), (2) and $\omega_0 = \gamma_0$ that

$$\omega_0 \exp\left(\frac{i\pi r}{p}\right) = g(y) = \omega_0 \exp\left(-\frac{i\pi r}{p}\right),$$

because $B \subset M \cap N$. Thus we have $r/p \in \mathbf{N}$, which contradicts $1 \le r < p-1$. We conclude that X is hereditarily unicoherent. □

LEMMA 2.2. *Let X be a compact Hausdorff space. If the condition* (a) *of Theorem* 1.1 *holds, then X is almost locally connected.*

PROOF. Assume that (a) holds and suppose that X is not almost locally connected. By definition, X contains mutually disjoint connected closed subsets C_n ($n \in \mathbf{N}$), which are open in $\overline{\cup_{n\in\mathbf{N}} C_n}$, the closure of $\cup_{n\in\mathbf{N}} C_n$ in X, with the following property: to each $n \in \mathbf{N}$ there correspond $x_n, y_n \in C_n$ such that $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ converge to distinct points, say x_0 and y_0. Put $F = \cup_{n\in\mathbf{N}} C_n$. Since X is a compact Hausdorff space, there exist open neighborhoods A and B of x_0 and y_0 respectively such that $\bar{A} \cap \bar{B} = \emptyset$. Let f be a continuous mapping of X into the interval $[-1, 1]$ such that $f(x) = 1$ on $\bar{A}$ and $f(x) = -1$ on $\bar{B}$. We consider the following mapping h of $\bar{F}$ into $\mathbf{C}$:

$$h(x) = \begin{cases} f(x) + \dfrac{i}{n}(1 - f^2(x)) & x \in C_n;\ n \text{ is even} \\ f(x) - \dfrac{i}{n}(1 - f^2(x)) & x \in C_n;\ n \text{ is odd} \\ f(x) & x \in \bar{F} \setminus F\,. \end{cases}$$

We see that $h \in C(\bar{F})$. Let $\tilde{h} \in C(X)$ be a mapping with $\tilde{h}|_{\bar{F}} = h$. Since the condition (a) of Theorem 1.1 is assumed to hold, there exist a continuous mapping $\tilde{g} \in C(X)$ and $p, q \in \mathbf{N}$ with $q/p \notin \mathbf{N}$ such that $\tilde{h}^q = \tilde{g}^p$ on X. Put $q = sp + r$, where s and r are integers with $1 \le r \le p-1$ (note $q/p \notin \mathbf{N}$). Now we define the mapping g of $\bar{F}$ into $\mathbf{C}$ as follows:

$$g(x) = \begin{cases} \dfrac{\tilde{g}(x)}{h^s(x)} & x \in \bar{F},\ h(x) \neq 0 \\ 0 & x \in \bar{F}\,, h(x) = 0\,. \end{cases}$$

Recall that $\tilde{h}|_{\bar{F}} = h$. Since $\tilde{h}^q = \tilde{g}^p$ on X, for each $x \in \bar{F}$ with $h(x) \neq 0$ we obtain

$$g^p(x) = \left(\frac{\tilde{g}(x)}{h^s(x)}\right)^p = \frac{\tilde{h}^q(x)}{h^{sp}(x)} = h^{q-sp}(x) = h^r(x)\,,$$

and so $h^r(x) = g^p(x)$ whenever $x \in \bar{F}$, $h(x) \neq 0$. It follows that $g \in C(\bar{F})$ such that $h^r = g^p$ on $\bar{F}$.

Pick an $n \in \mathbf{N}$ arbitrarily. By the definition of h, there is a continuous mapping θ_n of C_n such that $h(x) = |h(x)| \exp(i\theta_n(x))$ for every $x \in C_n$ and that $\theta_n(C_n) \subset [0, \pi]$ if n is even and $\theta_n(C_n) \subset [-\pi, 0]$ if n is odd. Since $h^r = g^p$ on $\bar{F}$, for each $x \in C_n$

$$g^p(x) = |h(x)|^r \exp(ir\theta_n(x))\,,$$

and so there is a p-th root $\omega_n(x)$ of 1 such that

$$g(x) = \omega_n(x)|h(x)|^{r/p} \exp\left(\frac{ir\theta_n(x)}{p}\right).$$

Since h, g and θ_n are continuous, $\omega_n(x)$ is a continuous mapping of C_n into the set of all p-th roots of 1. Furthermore, since C_n is connected, $\omega_n(x)$ must be constant, say ω_n. So,

$$g(x) = \omega_n|h(x)|^{r/p} \exp\left(\frac{ir\theta_n(x)}{p}\right) \qquad (x \in C_n)\,. \tag{3}$$

Since $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ converge to $x_0 \in A$ and $y_0 \in B$, respectively, we may assume that $\{x_n\}_{n\in\mathbf{N}} \subset A$ and $\{y_n\}_{n\in\mathbf{N}} \subset B$. Recall that $f = 1$ on $\bar{A}$ and $f = -1$ on $\bar{B}$. So, we get $h(x_n) = 1$ and $h(y_n) = -1$ for every $n \in \mathbf{N}$. Since $\theta_{2n}(C_{2n}) \subset [0, \pi]$ and $\theta_{2n-1}(C_{2n-1}) \subset [-\pi, 0]$ for every $n \in \mathbf{N}$, it follows from the equation $h(x) = |h(x)| \exp(i\theta_n(x))$ that $\theta_n(x_n) = 0$, $\theta_{2n}(y_{2n}) = \pi$ and $\theta_{2n-1}(y_{2n-1}) = -\pi$ for every $n \in \mathbf{N}$. It follows from (3) that $g(x_n) = \omega_n$ converges to $g(x_0)$. On the other hand, since $g(y_n)$ converges to $g(y_0)$, we see from (3) that both $g(y_{2n}) = \omega_{2n} \exp(ir\pi/p)$ and $g(y_{2n-1}) = \omega_{2n-1} \exp(-ir\pi/p)$ converge to $g(y_0)$. That is,

$$g(x_0) \exp\left(\frac{ir\pi}{p}\right) = g(y_0) = g(x_0) \exp\left(\frac{-ir\pi}{p}\right).$$

Since $|g(x_0)| = |h(x_0)|^{r/p} = |f(x_0)|^{r/p} = 1$, we see that $\exp(ir\pi/p) = \exp(-ir\pi/p)$. In other words, $r/p \in \mathbf{N}$, which contradicts $1 \leq r \leq p - 1$. We thus conclude that X is almost locally connected. □

The following results, Lemma 2.3 and 2.4 are deduced from [13, Theorem 3.3]; Moreover, Lemma 2.4 is well-known (cf. [11, Chap.VIII §57 Section III, Theorem 3, p.438]). Here we give a proof for the sake of completeness.

LEMMA 2.3. *Let X be a locally connected compact Hausdorff space. If X is hereditarily unicoherent, then* $\dim X \leq 1$.

PROOF. Let $\mathfrak{A} = \{O_k\}_{k=1}^n$ be a finite open covering of X. We show that there is an open refinement $\mathfrak{B}$ for $\mathfrak{A}$ such that every $x \in X$ is in at most two elements of $\mathfrak{B}$. Since X is assumed to be locally connected, it follows from [13, Lemma 3.2] that X is an A-space, that is, the class of all open sets whose boundaries are finite sets forms an open base. Without loss of generality we may assume that each O_k has at most finitely many boundary points. Put $B = \bigcup_{k=1}^n (\overline{O_k} \setminus O_k)$, where $\bar{\cdot}$ denotes the closure in X. We define mutually disjoint open family $\{V_k\}_{k=1}^n$ as follows:

$$V_1 = O_1 \setminus B \ \text{ and } \ V_k = O_k \setminus \left(B \cup \bigcup_{j=1}^{k-1} \overline{O_j}\right) \ \text{ for } \ k = 2, 3, \ldots, n\,.$$

Since $\{O_k\}_{k=1}^n$ is an open covering of X, we see that $\bigcup_{k=1}^n V_k = X \setminus B$.

Since B consists of at most finitely many points, to each $x \in B$ there corresponds an open neighborhood U_x of x with the following property: $U_x \subset O_k$ for some k and $U_x \cap U_y = \emptyset$ whenever $x, y \in B$, $x \neq y$. Put $\mathfrak{B} = \{V_k\}_{k=1}^{n} \cup \{U_x : x \in B\}$. We see that $\mathfrak{B}$ is an open covering of X. Recall that both $\{V_k\}_{k=1}^{n}$ and $\{U_x : x \in B\}$ are mutually disjoint. This implies that if $x \in X$, then at most two elements of $\mathfrak{B}$ contain x. So, we get $\dim X \leq 1$. □

LEMMA 2.4. *Let X be a locally connected compact Hausdorff space. If X is hereditarily unicoherent, then $\check{H}^1(X, \mathbf{Z})$ is trivial.*

PROOF. Assume that X is hereditarily unicoherent. By a theorem of Arens and Royden, it is enough to show that the equality $C(X)^{-1} = \exp C(X)$ holds: Since $\exp C(X) \subset C(X)^{-1}$, it suffices to prove that $C(X)^{-1} \subset \exp C(X)$. To do this, pick $f \in C(X)^{-1}$ arbitrarily. Since X is locally connected, each connected component of X is open. It follows that X has at most finitely many connected components. Without loss of generality, we may assume that X is connected. Recall that $f \in C(X)^{-1}$, and so f vanishes nowhere. Since X is locally connected, for each x in X there exists a connected open neighborhood V_x of x and a continuous mapping g_x of the closure $\overline{V_x}$ of V_x into $\mathbf{C}$ such that $f = e^{g_x}$ on $\overline{V_x}$. Since X is compact, there are finite number of points $x_1, x_2, \dots, x_{n+1}$ such that $\cup_{k=1}^{n+1} V_{x_k} = X$. For simplicity, we denote $g_k = g_{x_k}$ and $V_k = V_{x_k}$ for $k = 1, 2, \dots, n+1$. Note that $\{\overline{V_k}\}_{k=1}^{n+1}$ is a class of non-empty connected closed sets with $\cup_{k=1}^{n+1} \overline{V_k} = X$. Since X is connected, $\overline{V_1}$ intersects at least one of $\overline{V_2}, \overline{V_3}, \dots, \overline{V_{n+1}}$; we may assume that $\overline{V_1}$ meets $\overline{V_2}$. Then $e^{g_1} = f = e^{g_2}$ on $\overline{V_1} \cap \overline{V_2}$, and so we have $e^{g_1 - g_2} = 1$ on $\overline{V_1} \cap \overline{V_2}$. Since X is hereditarily unicoherent, $\overline{V_1} \cap \overline{V_2}$ is connected. Hence by the continuity of $g_1 - g_2$, the equation $e^{g_1 - g_2} = 1$ implies the existence of an integer k_1 such that

$$g_1 - g_2 = 2k_1 \pi i \quad \text{on} \quad \overline{V_1} \cap \overline{V_2}\,.$$

We define a mapping $\widetilde{g_1}$ of $\overline{V_1} \cup \overline{V_2}$ into $\mathbf{C}$ as follows:

$$\widetilde{g_1}(x) = \begin{cases} g_1(x) & x \in \overline{V_1} \\ g_2(x) + 2k_1 \pi i & x \in \overline{V_2} \setminus \overline{V_1}\,. \end{cases}$$

It is easy to see that $\widetilde{g_1}$ is continuous on $\overline{V_1} \cup \overline{V_2}$ and

$$f = e^{\widetilde{g_1}} \quad \text{on} \quad \overline{V_1} \cup \overline{V_2}\,.$$

In the same way, $\overline{V_1} \cup \overline{V_2}$ intersects at least one of $\overline{V_3}, \overline{V_4}, \dots, \overline{V_{n+1}}$. We may assume that $\overline{V_1} \cup \overline{V_2}$ meets $\overline{V_3}$. The equation $e^{\widetilde{g_1}} = f = e^{g_3}$ holds on $(\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}$, and so $e^{\widetilde{g_1} - g_3} = 1$ on $(\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}$. Since X is hereditarily unicoherent, $(\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}$ is connected. Hence by the continuity of $\widetilde{g_1} - g_3$, there exists an integer k_2 such that

$$\widetilde{g_1} - g_3 = 2k_2 \pi i \quad \text{on} \quad (\overline{V_1} \cup \overline{V_2}) \cap \overline{V_3}\,.$$

We define a mapping $\widetilde{g_2}$ of $(\overline{V_1}\cup\overline{V_2})\cup\overline{V_3}$ into $\mathbf{C}$ as follows: If x is in $\overline{V_1}\cup\overline{V_2}$, let $\widetilde{g_2}(x)=\widetilde{g_1}(x)$, and let $\widetilde{g_2}(x)=g_3(x)+2k_2\pi i$ otherwise. It is easy to see that $\widetilde{g_2}$ is continuous on $\overline{V_1}\cup\overline{V_2}\cup\overline{V_3}$ and

$$f=e^{\widetilde{g_2}} \quad \text{on} \quad \overline{V_1}\cup\overline{V_2}\cup\overline{V_3}\,.$$

Continuing this process, we have a continuous mapping $\widetilde{g_n}$ of $\cup_{k=1}^{n+1}\overline{V_k}$ such that

$$f=e^{\widetilde{g_n}} \quad \text{on} \quad \bigcup_{k=1}^{n+1}\overline{V_k}$$

Since $\cup_{k=1}^{n+1}\overline{V_k}=X$, we have that $f\in\exp C(X)$. Since $f\in C(X)^{-1}$ was arbitrary, we conclude that $C(X)^{-1}\subset\exp C(X)$ and the proof is complete. □

LEMMA 2.5. *Let X be a compact Hausdorff space. If* $\dim X\le 1$ *and* $\check{H}^1(X;\mathbf{Z})$ *is trivial, then* $\{g^p : g\in C(X)\}$ *is uniformly dense in* $C(X)$ *for every* $p\in\mathbf{N}$.

PROOF. Pick $p\in\mathbf{N}$ and $f\in C(X)$ arbitrarily. We show that for every $\varepsilon>0$ there exists a $g\in C(X)$ such that $\|f-g^p\|_\infty<\varepsilon$. Without loss of generality we may assume that $\|f\|_\infty\le 1$. Choose a $k\in\mathbf{N}$ so that $2^p/\varepsilon^p<k$. Then put

$$E_k=\left\{x\in X : |f(x)|\ge\frac{1}{k}\right\}.$$

Since $\dim X\le 1$, there exists a $u\in C(X)^{-1}$ with $|u|=1$ on X such that $u=f/|f|$ on E_k. Then $\tilde{u}(x)=\max\{|f(x)|,1/k\}u(x)$ is in $C(X)^{-1}$ with $\tilde{u}|=f$ on E_k. Since $\check{H}^1(X;\mathbf{Z})$ is trivial, by a theorem of Arens and Royden there exists a $v\in\exp C(X)$ such that $\tilde{u}=v^p$. We define mappings g and h as follows:

$$g(x)=\frac{\sqrt[p]{|f(x)|}\,v(x)}{|v(x)|} \qquad (x\in X)\,,$$

$$h(x)=\begin{cases}0 & f(x)=0\\ \dfrac{f(x)}{g(x)^{p-1}} & f(x)\neq 0\,.\end{cases}$$

Then we see that $g,h\in C(X)$, $\|g\|_\infty\le 1$ and $f=g^{p-1}h$. Since $f\,(=\tilde{u})=v^p$ on E_k, we see that $g=v=h$ on E_k. Therefore

$$\begin{aligned}\|g-h\|_\infty &= \sup\{|g(x)-h(x)| : x\in X\setminus E_k\}\\ &\le 2\sup\left\{\sqrt[p]{|f(x)|} : x\in X\setminus E_k\right\}\le 2\left(\frac{1}{k}\right)^{1/p}<\varepsilon\,.\end{aligned}$$

Since $f=g^{p-1}h$ and $\|g\|_\infty\le 1$, it follows that

$$\|f-g^p\|_\infty=\|g^{p-1}h-g^p\|_\infty\le\|g^{p-1}\|_\infty\,\|h-g\|_\infty<\varepsilon\,.$$

This completes the proof. □

The case where $p = 2$ in Lemma 2.6 was essentially proved in [1, Corollary 5.9]. Here, we generalize the result to the case where $p \geq 2$.

LEMMA 2.6. *Let X be a locally connected compact Hausdorff space and $p \in \mathbf{N}$ with $p \geq 2$. If $\{f_n{}^p\}_{n\in\mathbf{N}} \subset C(X)$ converges uniformly to $f \in C(X)$, then there is a Cauchy subsequence of $\{f_n\}_{n\in\mathbf{N}}$.*

PROOF. For each $k \in \mathbf{N}$, set

$$E(k) = \left\{ x \in X : |f(x)| > \frac{1}{k} \right\} .$$

Note that the closure $\overline{E(k)}$ of $E(k)$ in X is a compact subset of $E(2k)$. Since X is locally connected, each connected component of $E(2k)$ is open. So, there are finitely many connected components $C(k, 1), C(k, 2), \dots, C(k, N_k)$ such that $C(k, j) \cap E(k) \neq \emptyset$ for each j, $1 \leq j \leq N_k$ and that

$$E(k) \subset \bigcup_{j=1}^{N_k} C(k, j) \subset E(2k) . \tag{4}$$

Pick $x_{k,j} \in C(k, j) \cap E(k)$ for each $k \in \mathbf{N}$ and j, $1 \leq j \leq N_k$. By a diagonal argument, we obtain a subsequence of $\{f_n\}_{n\in\mathbf{N}}$ converging at each point $x_{k,j}$, which we denote by the same letter $\{f_n\}_{n\in\mathbf{N}}$. We show that $\{f_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C(X)$. Put $\omega_l = \exp(2l\pi i/p)$ for $l = 0, 1, 2, \dots, p-1$. Fix $k \in \mathbf{N}$ arbitrarily. We define $\varepsilon(k)$ as follows:

$$\varepsilon(k) = \min \left\{ \frac{1}{2k} - \left(\frac{1}{2k}\right)^p , \left(\frac{1}{4k} |\omega_1 - 1|\right)^p \right\} . \tag{5}$$

Since $\lim_{n\to\infty} \|f_n{}^p - f\|_\infty = 0$ and since $\{f_n\}$ converges at each point $x_{k,j}$, we have, for a sufficiently large $n(k) \in \mathbf{N}$,

$$\|f_n{}^p - f_m{}^p\|_\infty < \varepsilon(k) , \tag{6}$$

$$\|f_n{}^p - f\|_\infty < \varepsilon(k) , \tag{7}$$

$$|f_n(x_{k,j}) - f_m(x_{k,j})| < \varepsilon(k)^{1/p} \tag{8}$$

for $n, m \geq n(k)$ and $j = 1, 2, \dots, N_k$. Fix $n, m \geq n(k)$ and $x \in E(2k)$ arbitrarily. Since

$$f_n{}^p(x) - f_m{}^p(x) = \prod_{l=0}^{p-1} (f_n(x) - \omega_l f_m(x)) ,$$

it follows from (6) that there exists an l with $0 \leq l \leq p-1$ such that the inequality

$$|f_n(x) - \omega_l f_m(x)| < \varepsilon(k)^{1/p} \tag{9}$$

holds. To prove the uniqueness of such l, suppose that there exists another l', $l \neq l'$ such that the equation (9) is valid for l' in place of l. We get

$$|\omega_l - \omega_{l'}|\,|f_m(x)| \leq |\omega_l f_m(x) - f_n(x)| + |f_n(x) - \omega_{l'} f_m(x)|$$
$$< 2\varepsilon(k)^{1/p} \leq \frac{1}{2k}|\omega_1 - 1|\,,$$

and so

$$|\omega_l - \omega_{l'}|\,|f_m(x)| < \frac{1}{2k}|\omega_1 - 1|\,. \tag{10}$$

On the other hand, since $x \in E(2k)$, the inequality (7) implies that

$$|f_m(x)|^p \geq |f(x)| - |f(x) - f_m{}^p(x)| > \frac{1}{2k} - \varepsilon(k) \geq \left(\frac{1}{2k}\right)^p\,.$$

It follows that

$$|\omega_l - \omega_{l'}|\,|f_m(x)| \geq |\omega_1 - 1|\,|f_m(x)| \geq \frac{1}{2k}|\omega_1 - 1|\,,$$

which contradicts (10). Hence the uniqueness is proved.

Since $x \in E(2k)$ was arbitrary, we have proved that to each $x \in E(2k)$ there corresponds a unique l such that (9) holds. This implies that if we define

$$G_l(k) = \{x \in E(2k) : |f_n(x) - \omega_l f_m(x)| < \varepsilon(k)^{1/p}\}$$

for $l = 0, 1, \ldots, p-1$, then $\{G_l(k)\}_{l=0}^{p-1}$ is a mutually disjoint family with $E(2k) = \cup_{l=0}^{p-1} G_l(k)$. Since $G_l(k)$ is open for $l = 0, 1, 2, \ldots, p-1$, each connected component of $E(2k)$ is contained in a unique $G_l(k)$. By the inequality (8), we get $x_{k,j} \in G_0(k)$ for $j = 1, 2, \ldots, N_k$. Hence $C(k, j) \subset G_0(k)$ for $j = 1, 2, \ldots, N_k$. By the definition of $G_l(k)$, it follows from (4) that

$$|f_n(x) - f_m(x)| < \varepsilon(k)^{1/p} \tag{11}$$

for every $x \in E(k)$. If $x \in X \setminus E(k)$, then we see from (7) that

$$|f_n(x)|^p \leq |f(x)| + \varepsilon(k) < \frac{1}{k} + \frac{1}{2k} < \frac{2}{k}\,.$$

Thus, we have that

$$|f_n(x) - f_m(x)| \leq |f_n(x)| + |f_m(x)| < 2\left(\frac{2}{k}\right)^{1/p} \tag{12}$$

for every $x \in X \setminus E(k)$. It follows from (5), (11) and (12) that

$$\|f_n - f_m\|_\infty \leq 2\left(\frac{2}{k}\right)^{1/p}\,.$$

Since $k \in \mathbf{N}$ and $n, m > n(k)$ are arbitrary, $\{f_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C(X)$. □

Although Lemmas 2.7 and 2.8 are well-known (cf. [11, Chap.VIII §57 Section I, Theorem 8, p.435] and [11, Chap.VIII §46 Section XI, Theorem 2, p.165], respectively), for the sake of completeness we give a proof.

LEMMA 2.7. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

(a) $\check{H}^1(X;\mathbf{Z})$ *is trivial.*

(b) *For each connected component X_λ of X, $\check{H}^1(X_\lambda;\mathbf{Z})$ is trivial.*

For a compact Hausdorff space X, it is well-known that

(♯) Every connected component X_λ of X is the intersection of all clopen sets G_μ of X such that $X_\lambda \subset G_\mu$.

We can prove the following as an application of (♯).

(♮) If O is open with $X_\lambda \subset O$ for some connected component X_λ of X, then there is clopen G such that $X_\lambda \subset G \subset O$.

In fact, if $G_\mu \supset X_\lambda$ is clopen with $\bigcap_{\mu\in I} G_\mu = X_\lambda$, then $\{X \setminus G_\mu\}_{\mu\in I}$ becomes an open covering of the closed subset $X\setminus O$, and so $X\setminus O \subset \bigcup_{i=1}^n (X\setminus G_{\mu_i})$ for some $\mu_1, \mu_2, \ldots, \mu_n \in I$. Then the clopen $\bigcap_{i=1}^n G_{\mu_i}$ satisfies $X_\lambda \subset \bigcap_{i=1}^n G_{\mu_i} \subset O$.

PROOF OF LEMMA 2.7. First we show that (a) implies (b). Suppose that (a) is true. Let X_λ be an arbitrary connected component of X. It is enough to show that $C(X_\lambda)^{-1} = \exp C(X_\lambda)$ by a theorem of Arens-Royden. Since $\exp C(X_\lambda) \subset C(X_\lambda)^{-1}$, we show that $C(X_\lambda)^{-1} \subset \exp C(X_\lambda)$. Pick an $f \in C(X_\lambda)^{-1}$ arbitrary. By the Tietze extension theorem, there exists a continuous extension $\tilde{f}$ of f to all of X. Continuity of $\tilde{f}$ implies that $\tilde{f}$ does not vanish on a certain open set O that contains X_λ. Therefore, combining with the condition (♮), we obtain a clopen set G which satisfies that $X_\lambda \subset G \subset O$. Now we define a mapping $\mathfrak{F}$ of X into $\mathbf{C}$ as follows: Let $\mathfrak{F}(x) = \tilde{f}(x)$ if $x \in G$, and $\mathfrak{F}(x) = 1$ otherwise. Then we see that $\mathfrak{F} \in C(X)^{-1}$ with $\mathfrak{F} = f$ on X_λ. Because $\check{H}^1(X;\mathbf{Z})$ is assumed to be trivial, there exists a $g \in C(X)$ such that $\mathfrak{F} = \exp g$. It follows that $f = \exp(g|_{X_\lambda})$. Thus we see that $f \in C(X_\lambda)^{-1}$. Since f was arbitrary, we conclude that $C(X_\lambda)^{-1} \subset \exp C(X_\lambda)$.

Next we show that (b) implies (a). Suppose that (b) is true. It is enough to show that $C(X)^{-1} \subset \exp C(X)$. Pick an $\tilde{f} \in C(X)^{-1}$ arbitrarily. Since (b) is true, to every connected component X_λ of X, the equation $C(X_\lambda)^{-1} = \exp C(X_\lambda)$ holds. Thus to each λ, there corresponds a $g_\lambda \in C(X_\lambda)$ such that $\tilde{f}|_{X_\lambda} = \exp g_\lambda$ holds. Let $\tilde{g}_\lambda$ be a continuous extension of g_λ to the whole space X. If we put $\tilde{h}_\lambda = \tilde{f}/\exp\tilde{g}_\lambda$ on X, then $\tilde{h}_\lambda = 1$ on X_λ. Continuity of $\tilde{h}_\lambda$ implies that there exists an open neighborhood $O_\lambda \supset X_\lambda$ such that $\tilde{h}_\lambda(O_\lambda) \subset \{z \in \mathbf{C} : |z-1| < 1/2\}$. Therefore, combining with (♮), we obtain a clopen set G_λ which satisfies $X_\lambda \subset G_\lambda \subset O_\lambda$. Since $\tilde{h}_\lambda(G_\lambda) \subset \{z \in \mathbf{C} : |z-1| < 1/2\}$, a continuous logarithm log of $\{z \in \mathbf{C} : |z-1| < 1/2\}$ into $\mathbf{C}$ is well-defined. So, we get

$$\tilde{f} = \tilde{h}_\lambda \exp\tilde{g}_\lambda = \exp(\tilde{g}_\lambda + \log\tilde{h}_\lambda) \quad \text{on} \quad G_\lambda .$$

Since $\{G_\lambda\}_\lambda$ is an open covering of the compact space X, this covering has a finite open subcovering $\{G_{\lambda_k}\}_{k=1}^n$. The corresponding mappings to G_k are denoted by $\tilde{g}_k, \tilde{h}_k$ $(k = 1, \ldots, n)$. Since every member of this covering is clopen, without loss of generality, we may assume that $G_{\lambda_{k_1}} \cap G_{\lambda_{k_2}} = \emptyset$ $(k_1 \neq k_2)$. Now we define a mapping $\tilde{g}$ of X into $\mathbf{C}$ as follows. If $x \in X$, then there exists a unique k such that $x \in G_k$; Let $\tilde{g}(x) = \tilde{g}_k(x) + \log \tilde{h}_k(x)$. Then we see that $\tilde{g} \in C(X)$ and $\tilde{f} = \exp \tilde{g}$. Thus we conclude that $C(X)^{-1} \subset \exp C(X)$ and this completes the proof. □

LEMMA 2.8. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

(a) $\dim X \leq 1$.

(b) *For each connected component X_λ of X*, $\dim X_\lambda \leq 1$.

PROOF. A proof of (a) ⇒ (b) is elementary and omitted (cf. [14]).

Conversely, suppose that (b) is true. Let F be a closed subset of X and f an S^1-valued continuous mapping of F. We show that there exists an S^1-valued continuous mapping $\tilde{f}$ on X such that $\tilde{f}|_F = f$. Let X_λ be a connected component of X. Since $\dim X_\lambda \leq 1$, there exists an S^1-valued continuous extension g_λ of $f|_{F \cap X_\lambda}$ to X_λ. We define a mapping h_λ of $F \cup X_\lambda$ into $\mathbf{C}$ as follows: Let $h_\lambda(x) = g_\lambda(x)$ if $x \in X_\lambda$, and $h_\lambda(x) = f(x)$ if $x \in F \setminus X_\lambda$. Then we see that h_λ is an S^1-valued continuous mapping on $F \cup X_\lambda$ satisfying $h_\lambda = f$ on F. Let $\tilde{h}_\lambda$ be a continuous extension of h_λ to all of X. By definition, $|\tilde{h}_\lambda| = |h_\lambda| = 1$ on F. Continuity of $\tilde{h}_\lambda$ implies that there exists an open neighborhood O_λ of X_λ such that $\tilde{h}_\lambda$ never vanishes on O_λ. Therefore, combined with (♮), there exists a clopen set G_λ such that $X_\lambda \subset G_\lambda \subset O_\lambda$. Thus $\tilde{h}_\lambda$ never vanishes on G_λ. Since $\{G_\lambda\}_\lambda$ is an open covering of the compact space X, $\{G_\lambda\}_\lambda$ has a finite subcovering $\{G_{\lambda_k}\}_{k=1}^n$ for X. Since every G_{λ_k} is clopen, without loss of generality, we may assume that $G_{\lambda_{k_1}} \cap G_{\lambda_{k_2}} = \emptyset$ $(k_1 \neq k_2)$. Now we define a mapping $\tilde{f}$ on X as follows. If $x \in X$, then there exists a unique k such that $x \in G_{\lambda_k}$: We put $\tilde{f}(x) = \tilde{h}_{\lambda_k}(x)/|\tilde{h}_{\lambda_k}(x)|$. Since $h_{\lambda_k} = f$ on F for every k, we see that $\tilde{f}$ is an S^1-valued continuous mapping of X such that $\tilde{f}|_F = f$ and this completes the proof. □

3. Proof of results

PROOF OF THEOREM 1.1. (a) ⇒ (b) By Lemma 2.1. (b) ⇒ (c) By Lemma 2.3 and 2.4. (c) ⇒ (d) By Lemma 2.5. (e) ⇒ (a) By definition.

(d) ⇒ (e) Suppose that $\{g^p : g \in C(X)\}$ is uniformly dense in $C(X)$ for every $p \in \mathbf{N}$. Pick $f \in C(X)$ and $p \in \mathbf{N}$ arbitrarily. By hypothesis, there exists a sequence $\{g_n{}^p\}_{n \in \mathbf{N}}$ such that $g_n{}^p$ converges to f as $n \to \infty$. By Lemma 2.6, there is a Cauchy subsequence $\{g_{n_j}\}_{j \in \mathbf{N}}$ of $\{g_n\}_{n \in \mathbf{N}}$. Since $C(X)$ is complete, there exists a $g \in C(X)$ such that g_{n_j} converges to g as $j \to \infty$. It follows that $f = g^p$ and the proof is complete. □

REMARK. Let us consider the following two conditions.

(d′) $\{g^p : g \in C(X)\}$ is uniformly dense in C(X) for some $p \in \mathbf{N}$ with $p \geq 2$.

(e′) There exists a $p \in \mathbf{N}$, $p \geq 2$ with the following property: For each $f \in C(X)$ there is a $g \in C(X)$ such that $f = g^p$.

Then the implications (e) of Theorem 1.1 ⇒ (e′) ⇒ (d′) are obviously true. If, in addition, X is locally connected, then (d′) with Lemma 2.5 implies that every $f \in C(X)$ is the p-th power of a $g \in C(X)$. So, we get (d′) ⇒ (e′). Consequently, both (d′) and (e′) are also equivalent to all of the conditions from (a) to (e) of Theorem 1.1 whenever X is locally connected. Note that Kawamura and Miura [10, Theorem 1.3] proved that if X is a compact Hausdorff space with $\dim X \leq 1$, then the condition (d′) above is equivalent to that $\check{H}^1(X; \mathbf{Z})$ is p-divisible.

It is well-known [13, Theorem 3.3] that if X is locally connected, then $C(X)$ is algebraically closed if and only if $C(X)$ is square root closed as is stated in the following theorem.

THEOREM A ([13]). *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

(1) *$C(X)$ is algebraically closed.*

(2) *$C(X)$ is square-root closed.*

(3) $\dim X \leq 1$ *and $\check{H}^1(X; \mathbf{Z})$ is trivial.*

(4) *X is hereditarily unicoherent.*

PROOF OF COROLLARY 1.2. This is just an application of Theorem 1.1 and Theorem A. □

If X is first-countable, then we see that the condition (a) of Theorem 1.1 holds if and only if $C(X)$ is algebraically closed. To prove this, we need the following result, which was essentially proved by Countryman, Jr. [3] (see also [13]).

THEOREM B ([3, 13]). *Let X be a first-countable compact Hausdorff space. Then the following conditions are equivalent.*

(1) *$C(X)$ is algebraically closed.*

(2) *$C(X)$ is square-root closed.*

(3) *X is almost locally connected and hereditarily unicoherent.*

(4) *X is almost locally connected and for every connected component X_λ of X, X_λ is locally connected*, $\dim X_\lambda \leq 1$ *and $\check{H}^1(X_\lambda; \mathbf{Z})$ is trivial.*

PROOF OF COROLLARY 1.3. (b) ⇔ (d) ⇔ (e): By Theorem B, each of the conditions (b), (d) and (e) implies the other.

(a) ⇒ (b): It follows from Lemmas 2.1 and 2.2 that (a) implies (b).

(e) ⇒ (a): It is obvious that (e) implies (a).

Finally, we show that (c) is equivalent to the condition (4) of Theorem B. It follows from Lemmas 2.7 and 2.8 that (4) of Theorem B implies (c). Conversely, we prove that (c) implies (4) of Theorem B. By [3, Proof of Lemma 2.5], we see that each connected component X_λ of

X is locally connected. It follows from Lemmas 2.7 and 2.8 that (c) implies (4) of Theorem B, and the proof is complete. □

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