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Inductive Construction of Nilpotent Modules of Quantum Groups at Roots of Unity

Yuuki ABE

Sophia University

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Abstract. The purpose of this paper is to prove that we can inductively construct all finite-dimensional irreducible nilpotent modules of type 1 by using the Schnizer homomorphisms for quantum algebras at roots of unity of type A_n, B_n, C_n, D_n, or G_2.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantum algebra of a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbf{C}$. The theory of $U_q(\mathfrak{g})$-modules is almost the same as the one of $\mathfrak{g}$ if q is not a root of unity. However, if q is a root of unity, it is quite different from the one of $\mathfrak{g}$. For example, there are differences as follows:

- Finite-dimensional modules are not always semisimple.
- Finite-dimensional irreducible modules are not necessarily the highest- or lowest-weight modules.
- Among finite-dimensional irreducible modules, there exist maximum dimensional modules.

The theory of $U_q(\mathfrak{g})$-modules at roots of unity is described in [5].

Let ε be a primitive l-th root of unity. A complete classification of finite-dimensional irreducible $U_\varepsilon(\mathfrak{g})$-modules has not been given yet. However, the classification of finite-dimensional irreducible *nilpotent* $U_\varepsilon(\mathfrak{g})$-modules of type 1 has been given by Lusztig (see [7] and [3]). In particular, it is known that these modules are classified by the highest weights. The formal characters of those irreducible modules are also known under mild restrictions on l by the celebrated works of Kazhdan and Lusztig and of Kashiwara and Tanisaki, see, e.g., [12], or more recent work by Arkhipov, Bezrukavnikov, and Ginzburg in [1].

In [9], Nakashima showed that these modules can be constructed by using the modules introduced by Date, Jimbo, Miki, and Miwa in [4] if $\mathfrak{g}$ is of type A_n. Moreover, in [2], we showed that we can construct these modules by using the *Schnizer modules* introduced in [10] if $\mathfrak{g}$ is of type B_n, C_n, or D_n.

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In this paper, we construct these modules inductively in the case that $\mathfrak{g}$ is of type A_n, B_n, C_n, D_n, or G_2 by using the Schnizer homomorphisms introduced in [11]. Then we construct finite-dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1 with the highest weight $(0, \ldots, 0, \lambda_k, \ldots, \lambda_n)$ as a submodule of an $l^{(N_n - N_{k-1})}$-dimensional $U_\varepsilon(\mathfrak{g})$-module, where N_n is the number of positive roots of $\mathfrak{g}$ and n is the rank of $\mathfrak{g}$. In particular, these results cover the ones in [2].

The organization of this paper is as follows. In §2, we recall the quantum algebras at roots of unity. In §3, we introduce the nilpotent modules and their classification theorem. In §4, we introduce the Schnizer homomorphisms. Finally, in §5–§7, we give the inductive construction of all finite-dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1 in the case of $\mathfrak{g} = A_n, B_n, C_n, D_n$, or G_2.

2. Quantum algebras at roots of unity

We will use the following notations. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbf{C}$ of type A_n, B_n, C_n, D_n, or G_2. We define $I := \{1, 2, \ldots, n\}$ with $n = 2$ if $\mathfrak{g}$ is of type G_2. Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots, Δ the root system, and Δ_+ the set of positive roots of $\mathfrak{g}$. Let N be the number of positive roots of $\mathfrak{g}$, that is, $N = \frac{1}{2}n(n+1)$ (resp. n^2, n^2, $(n-1)n$, 6) if $\mathfrak{g} = A_n$ (resp. B_n, C_n, D_n, G_2). We define the root lattice $Q := \bigoplus_{i \in I} \mathbf{Z}\alpha_i$ and the positive root lattice $Q_+ := \bigoplus_{i \in I} \mathbf{Z}_+\alpha_i$, where $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$. Let $(\mathfrak{a}_{i,j})_{i,j \in I}$ be the Cartan matrix associated with $\mathfrak{g}$ such that

$$\begin{cases} \mathfrak{a}_{1,2} = -2 & \mathfrak{g} = B_n\,, \\ \mathfrak{a}_{2,1} = -2 & \mathfrak{g} = C_n\,, \\ \mathfrak{a}_{1,2} = 0,\ \mathfrak{a}_{1,3} = \mathfrak{a}_{2,3} = -1 & \mathfrak{g} = D_n\,, \\ \mathfrak{a}_{1,2} = -3 & \mathfrak{g} = G_2\,. \end{cases}$$

We define $(d_1, \ldots, d_n) := (1, \ldots, 1)$ (resp. $(\frac{1}{2}, 1, \ldots, 1)$, $(2, 1, \ldots, 1)$, $(1, \ldots, 1)$, $(1, 3)$) if $\mathfrak{g} = A_n$ (resp. B_n, C_n, D_n, G_2). We denote the Weyl group of $\mathfrak{g}$ by $\mathcal{W}$, which is generated by the simple reflections $\{s_i\}_{i \in I}$.

Let l be an odd integer which is greater than 2. We assume that l is not divisible by 3 if $\mathfrak{g} = G_2$. Let ε (resp. $\varepsilon^{\frac{1}{2}}$) be a primitive l-th root of unity if $\mathfrak{g} \neq B_n$ (resp. $\mathfrak{g} = B_n$). For $r \in \mathbf{Z}$, $m \in \mathbf{N}$, $d \in \mathbf{Q}$ such that $\varepsilon^{2d} \neq 1$, we define

$$[r]_{\varepsilon^d} := \frac{\varepsilon^{dr} - \varepsilon^{-dr}}{\varepsilon^d - \varepsilon^{-d}}\,, \quad [r] := [r]_\varepsilon\,,$$
$$[m]_{\varepsilon^d}! := [m]_{\varepsilon^d}[m-1]_{\varepsilon^d} \cdots [1]_{\varepsilon^d}\,, \quad [0]_{\varepsilon^d}! := 1\,.$$

DEFINITION 2.1. The quantum algebra $U_\varepsilon(\mathfrak{g})$ is an associative $\mathbf{C}$-algebra generated by

$\{e_i, f_i, t_i^{\pm 1}\}_{i\in I}$ with the relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1\,, \quad t_i t_j = t_j t_i\,,$$
$$t_i e_j t_i^{-1} = \varepsilon_i^{\mathfrak{a}_{i,j}} e_j\,, \quad t_i f_j t_i^{-1} = \varepsilon_i^{-\mathfrak{a}_{i,j}} f_j\,,$$
$$e_i f_j - f_j e_i = \delta_{i,j}\{t_i\}_{\varepsilon_i}\,,$$
$$\sum_{k=0}^{1-\mathfrak{a}_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-\mathfrak{a}_{i,j}-k)} = \sum_{k=0}^{1-\mathfrak{a}_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(1-\mathfrak{a}_{i,j}-k)} = 0 \quad i \neq j\,,$$

where

$$e_i^{(k)} := \frac{1}{[k]_{\varepsilon_i}!} e_i^k\,, \quad f_i^{(k)} := \frac{1}{[k]_{\varepsilon_i}!} f_i^k\,, \quad \{t_i\}_{\varepsilon_i} := \frac{t_i - t_i^{-1}}{\varepsilon_i - \varepsilon_i^{-1}}\,, \quad \varepsilon_i := \varepsilon^{d_i}\,.$$

Let $U_\varepsilon^+(\mathfrak{g})$ (resp. $U_\varepsilon^-(\mathfrak{g})$, $U_\varepsilon^0(\mathfrak{g})$) be the $\mathbf{C}$-subalgebra of $U_\varepsilon(\mathfrak{g})$ generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$, $\{t_i^{\pm 1}\}_{i\in I}$). Moreover, we extend the algebra $U_\varepsilon(\mathfrak{g})$ by adding the elements $\{t_i^{\pm\frac{1}{k}} \mid i, k \in I\}$.

Let w_0 be the longest element of $\mathcal{W}$ and $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression of w_0. We define

$$\beta_1 := \alpha_{i_1}\,,\ \beta_2 := s_{i_1}(\alpha_{i_2}), \dots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})\,.$$

Thus, we have $\Delta_+ = \{\beta_1, \dots, \beta_N\}$, and there exist vectors $\{e_{\beta_i}\}_{i=1}^N$, $\{f_{\beta_i}\}_{i=1}^N$ in $U_\varepsilon(\mathfrak{g})$, which are called "root vectors" (cf. [5], [8]), where $e_{\alpha_i} = e_i$, $f_{\alpha_i} = f_i$ for $i \in I$. These vectors have the following properties.

PROPOSITION 2.2 ([5] Proposition 1.7). (i) *$\{e_{\beta_1}^{m_1} \cdots e_{\beta_N}^{m_N} | m_1, \dots, m_N \in \mathbf{Z}_+\}$ is a $\mathbf{C}$-basis of $U_\varepsilon^+(\mathfrak{g})$.*

(ii) *$\{f_{\beta_1}^{m_1} \cdots f_{\beta_N}^{m_N} | m_1, \dots, m_N \in \mathbf{Z}_+\}$ is a $\mathbf{C}$-basis of $U_\varepsilon^-(\mathfrak{g})$.*

(iii) *$\{t_1^{m_1} \cdots t_n^{m_n} | m_1, \dots, m_n \in \mathbf{Z}\}$ is a $\mathbf{C}$-basis of $U_\varepsilon^0(\mathfrak{g})$.*

(iv) *The multiplication map $\phi : U_\varepsilon^-(\mathfrak{g}) \otimes U_\varepsilon^0(\mathfrak{g}) \otimes U_\varepsilon^+(\mathfrak{g}) \to U_\varepsilon(\mathfrak{g})$ ($u_- \otimes u_0 \otimes u_+ \mapsto u_- u_0 u_+$) is an isomorphism of $\mathbf{C}$-vector spaces.*

Let $Z(U_\varepsilon(\mathfrak{g}))$ be the center of $U_\varepsilon(\mathfrak{g})$.

PROPOSITION 2.3 ([5] Corollary 3.1). *$\{e_\alpha^l, f_\alpha^l, t_i^l \mid \alpha \in \Delta_+, i \in I\} \subset Z(U_\varepsilon(\mathfrak{g}))$.*

Now, for $i \in I$, we define

$$\deg(e_i) := \alpha_i\,, \quad \deg(f_i) := -\alpha_i\,, \quad \deg(t_i) := 0\,. \tag{2.1}$$

Obviously, these are compatible with the relations of $U_\varepsilon(\mathfrak{g})$. Therefore, we can regard $U_\varepsilon(\mathfrak{g})$ as a Q-graded algebra, and we have

$$U_\varepsilon(\mathfrak{g}) = \bigoplus_{\alpha\in Q} U_\varepsilon(\mathfrak{g})_\alpha\,, \quad U_\varepsilon(\mathfrak{g})_\alpha U_\varepsilon(\mathfrak{g})_{\alpha'} \subset U_\varepsilon(\mathfrak{g})_{(\alpha+\alpha')}\,,$$

for $\alpha, \alpha' \in Q$, where $U_\varepsilon(\mathfrak{g})_\alpha := \{u \in U_\varepsilon | \deg(u) = \alpha\}$.

PROPOSITION 2.4 ([6] §8). *$e_\alpha \in U_\varepsilon^+(\mathfrak{g}) \cap U_\varepsilon(\mathfrak{g})_\alpha$ and $f_\alpha \in U_\varepsilon^-(\mathfrak{g}) \cap U_\varepsilon(\mathfrak{g})_{-\alpha}$ for all $\alpha \in \Delta_+$.*

3. Nilpotent modules

DEFINITION 3.1. Let L be a $U_\varepsilon(\mathfrak{g})$-module. If $e_\alpha^l = f_\alpha^l = 0$ on L for all $\alpha \in \Delta_+$, we call L a "nilpotent module". In addition, if $t_i^l = 1$ on L for all $i \in I$, we call L a "nilpotent module of type 1".

REMARK 3.2. Nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1 are the same as $U_\varepsilon^{\mathrm{fin}}(\mathfrak{g})$-modules of type 1, where $U_\varepsilon^{\mathrm{fin}}(\mathfrak{g})$ is the finite-dimensional quantum algebra introduced in [7], [8] (see [2]).

In general, finite-dimensional irreducible $U_\varepsilon^{\mathrm{fin}}(\mathfrak{g})$-modules are divided into 2^n types according to $\{\sigma : Q \to \{\pm 1\}$; group homomorphism $\}$. Without loss of generality, we may assume that finite-dimensional irreducible $U_\varepsilon^{\mathrm{fin}}(\mathfrak{g})$-modules are of type 1.

DEFINITION 3.3. Let L be a $U_\varepsilon(\mathfrak{g})$-module.

(i) We define $P(L) := \{v \in L \mid e_i v = 0$ for all $i \in I\}$ and call the vectors in $P(L)$ "primitive vectors".

(ii) Let $\lambda = (\lambda_i)_{i \in I} \in \mathbf{C}^n$. We assume that L is generated by a nonzero vector $v_0 \in P(L)$ such that $t_i v_0 = \varepsilon_i^{\lambda_i} v_0$ for all $i \in I$. We call L the "highest-weight module with highest weight λ" and v_0 the "highest-weight vector".

Now, we introduce the classification theorem of finite-dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1. We define $\mathbf{Z}_l := \{\lambda \in \mathbf{Z} \mid 0 \le \lambda \le l-1\}$.

THEOREM 3.4 ([7], [3] Proposition 11.2.10). *For any $\lambda \in \mathbf{Z}_l^n$, there exists a unique* (*up to isomorphic*) *finite-dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-module $L_\varepsilon^{\mathrm{nil}}(\lambda)$ of type* 1 *with highest weight λ. Conversely, if L is a finite-dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-module of type* 1, *there exists a $\lambda \in \mathbf{Z}_l^n$ such that L is isomorphic to $L_\varepsilon^{\mathrm{nil}}(\lambda)$.*

In a similar way to the proof of Theorem 5.5(ii) in [9] or Theorem 4.10 in [2], we can prove the following proposition.

PROPOSITION 3.5. *For $\lambda \in \mathbf{Z}_l^n$, let L be a nilpotent highest-weight $U_\varepsilon(\mathfrak{g})$-module of type 1 with highest weight λ. We assume* $\dim(P(L)) = 1$. *Then L is an irreducible $U_\varepsilon(\mathfrak{g})$-module. In particular, as a $U_\varepsilon(\mathfrak{g})$-module, L is isomorphic to $L_\varepsilon^{\mathrm{nil}}(\lambda)$.*

4. Schnizer homomorphisms

In the rest of this paper, we denote $\mathfrak{g}$ by $\mathfrak{g}_n$ if the rank of $\mathfrak{g}$ is n and e_i, f_i, t_i in $U_\varepsilon(\mathfrak{g}_n)$ by $e_{i,n}$, $f_{i,n}$, $t_{i,n}$.

We will use the following notations. Let V_n be an l^n-dimensional $\mathbf{C}$-vector space and $\{v_n(m_n)\,|\,m_n = (m_{1,n},\ldots,m_{n,n}) \in \mathbf{Z}_l^n\}$ a basis of V_n, where $\mathbf{Z}_l = \{m \in \mathbf{Z}\,|\,0 \le m \le l-1\}$. We define $v_n(m_n + lm_n') := v_n(m_n)$ for $m_n \in \mathbf{Z}_l^n$, $m_n' \in \mathbf{Z}^n$. For $i \in I$, we define

$$\varepsilon_{i,n} := (\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i,n}) \in \mathbf{Z}_l^n\,, \tag{4.1}$$

where $\delta_{i,j}$ is the Kronecker delta. For $i \in I$, $a_{i,n} \in \mathbf{C}^\times$, $b_{i,n} \in \mathbf{C}$, we define linear maps $x_{i,n}, z_{i,n} \in \mathrm{End}(V_n)$ as

$$x_{i,n}v_n(m_n) := a_{i,n}v_n(m_n - \varepsilon_{i,n})\,, \quad z_{i,n}v_n(m_n) := \varepsilon^{m_{i,n}+b_{i,n}}v_n(m_n) \quad (m_n \in \mathbf{Z}_l^n)\,. \tag{4.2}$$

For any $z \in \mathrm{End}(V_n)$ such that $z^{-1} \in \mathrm{End}(V_n)$ and $d \in \mathbf{Q}$ such that $\varepsilon^{2d} \neq 1$, we define

$$\{z\}_{\varepsilon^d} := \frac{z - z^{-1}}{\varepsilon^d - \varepsilon^{-d}}\,. \tag{4.3}$$

Then we have

$$\{z_{i,n}\}_{\varepsilon^d}v_n(m_n) = [d^{-1}(m_{i,n} + b_{i,n})]_{\varepsilon^d}v_n(m_n)\,. \tag{4.4}$$

For any $\mathbf{C}$-vector space V, we regard $\mathrm{End}(V) \otimes U_\varepsilon(\mathfrak{g}_n)$ as a $\mathbf{C}$-algebra by using

$$(x \otimes u)(x' \otimes u') := (xx') \otimes (uu') \quad (x, x' \in \mathrm{End}(V), u, u' \in U_\varepsilon(\mathfrak{g}_n))\,.$$

THEOREM 4.1 ([11] Theorem 3.2, 4.10). (a) *Let $\lambda_n \in \mathbf{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbf{C}^\times)^n$, and $b_n = (b_{i,n})_{i=1}^n \in \mathbf{C}^n$. We obtain a $\mathbf{C}$-algebra homomorphism $\rho_n^A := \rho_n^A(a_n, b_n, \lambda_n) : U_\varepsilon(A_n) \to \mathrm{End}(V_n) \otimes U_\varepsilon(A_{n-1})$ such that*

$$\rho_n^A(e_{i,n}) = \{z_{i-1,n}z_{i,n}^{-1}\}x_{i,n} + x_{i-1,n}^{-1}x_{i,n}e_{i-1,n-1}\,, \tag{4.5}$$

$$\rho_n^A(t_{i,n}) = z_{i-1,n}z_{i,n}^{-2}z_{i+1,n}t_{i,n-1}\,, \tag{4.6}$$

$$\rho_n^A(f_{i,n}) = \{z_{i,n}z_{i+1,n}^{-1}t_{i,n-1}^{-1}\}x_{i,n}^{-1} + f_{i,n-1}\,, \tag{4.7}$$

where

$$t_{n,n-1} := \varepsilon^{-\lambda_n}\prod_{i=1}^{n-1} t_{i,n-1}^{-\frac{i}{n}}\,. \tag{4.8}$$

(b) *Let $\lambda_n \in \mathbf{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbf{C}^\times)^n$, $\tilde{a}_{n-1} = (\tilde{a}_{i,n-1})_{i=1}^{n-1} \in (\mathbf{C}^\times)^{n-1}$, $b_n = (b_{i,n})_{i=1}^n \in \mathbf{C}^n$, and $\tilde{b}_{n-1} = (\tilde{b}_{i,n-1})_{i=1}^{n-1} \in \mathbf{C}^{n-1}$. We obtain a $\mathbf{C}$-algebra homomorphism $\rho_n^B := \rho_n^B(a_n, \tilde{a}_{n-1}, b_n, \tilde{b}_{n-1}, \lambda_n) : U_\varepsilon(B_n) \to \mathrm{End}(V_n) \otimes \mathrm{End}(\widetilde{V}_{n-1}) \otimes U_\varepsilon(B_{n-1})$ such that*

$$\rho_n^B(e_{i,n}) = \{z_{i+1,n}z_{i,n}^{-1}\}x_{i,n} + \{\tilde{z}_{i-1,n-1}\tilde{z}_{i,n-1}^{-1}\}x_{i+1,n}^{-1}x_{i,n}\tilde{x}_{i,n-1}$$

$$+x_{i+1,n}^{-1}x_{i,n}\tilde{x}_{i,n-1}\tilde{x}_{i-1,n-1}^{-1}e_{i,n-1} \quad (2 \le i \le n)\,,$$

$$\rho_n^B(e_{1,n}) = \{z_{2,n}z_{1,n}^{-\frac{1}{2}}\}_{\varepsilon_1}x_{1,n} + \{\tilde{z}_{1,n-1}^{-1}z_{1,n}^{\frac{1}{2}}\}_{\varepsilon_1}x_{2,n}^{-1}x_{1,n}\tilde{x}_{1,n-1} + x_{2,n}^{-1}\tilde{x}_{1,n-1}e_{1,n-1}\,, \tag{4.9}$$

$$\rho_n^B(t_{i,n}) = z_{i+1,n}z_{i,n}^{-2}z_{i-1,n}\tilde{z}_{i-2,n-1}\tilde{z}_{i-1,n-1}^{-2}\tilde{z}_{i,n-1}t_{i,n-1} \quad (2 \le i \le n)\,,$$

$$\rho_n^B(t_{1,n}) = z_{2,n}z_{1,n}^{-1}\tilde{z}_{1,n-1}t_{1,n-1}\,, \tag{4.10}$$

$$\rho_n^B(f_{i,n}) = \{z_{i,n}z_{i-1,n}^{-1}\tilde{z}_{i-2,n-1}^{-1}\tilde{z}_{i-1,n-1}^{2}\tilde{z}_{i,n-1}^{-1}t_{i,n-1}^{-1}\}x_{i,n}^{-1}$$

$$+\{\tilde{z}_{i-1,n-1}\tilde{z}_{i,n-1}^{-1}t_{i,n-1}^{-1}\}\tilde{x}_{i-1,n-1}^{-1} + f_{i,n-1} \quad (2 \le i \le n)\,,$$

$$\rho_n^B(f_{1,n}) = \{z_{1,n}^{\frac{1}{2}}\tilde{z}_{1,n-1}^{-1}t_{1,n-1}^{-1}\}_{\varepsilon_1}x_{1,n}^{-1} + f_{1,n-1}\,, \tag{4.11}$$

where

$$t_{n,n-1} := \varepsilon^{-\lambda_n}\prod_{i=1}^{n-1}t_{i,n-1}^{-1}\,. \tag{4.12}$$

(c) *Let* $\lambda_n \in \mathbf{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbf{C}^\times)^n$, $\tilde{a}_{n-1} = (\tilde{a}_{i,n-1})_{i=1}^{n-1} \in (\mathbf{C}^\times)^{n-1}$, $b_n = (b_{i,n})_{i=1}^n \in \mathbf{C}^n$, *and* $\tilde{b}_{n-1} = (\tilde{b}_{i,n-1})_{i=1}^{n-1} \in \mathbf{C}^{n-1}$. *We obtain a* $\mathbf{C}$*-algebra homomorphism* $\rho_n^C := \rho_n^C(a_n, \tilde{a}_{n-1}, b_n, \tilde{b}_{n-1}, \lambda_n) : U_\varepsilon(C_n) \to \mathrm{End}(V_n) \otimes \mathrm{End}(\widetilde{V}_{n-1}) \otimes U_\varepsilon(C_{n-1})$ *such that* $\rho_n^C(e_{i,n})$, $\rho_n^C(t_{i,n})$, $\rho_n^C(f_{i,n})$ *as in* (4.9), (4.10), (4.11) *if* $3 \le i \le n$, *and*

$$\rho_n^C(e_{2,n}) = \{z_{3,n}z_{2,n}^{-1}\}x_{2,n} + \{\tilde{z}_{1,n-1}\tilde{z}_{2,n-1}^{-1}\}x_{3,n}^{-1}x_{2,n}\tilde{x}_{2,n-1} + x_{3,n}^{-1}x_{2,n}\tilde{x}_{2,n-1}\tilde{x}_{1,n-1}^{-1}e_{2,n-1}\,,$$

$$\rho_n^C(e_{1,n}) = \{z_{1,n}^2\tilde{z}_{1,n-1}^{-2}\}_{\varepsilon_1}x_{2,n}^{-2}x_{1,n}\tilde{x}_{1,n-1}^2 + \{z_{2,n}\tilde{z}_{1,n-1}^{-1}\}x_{2,n}^{-1}x_{1,n}\tilde{x}_{1,n-1}$$

$$+\{z_{2,n}^2z_{1,n}^{-2}\}_{\varepsilon_1}x_{1,n} + x_{2,n}^{-2}\tilde{x}_{1,n-1}^2e_{1,n-1}\,, \tag{4.13}$$

$$\rho_n^C(t_{2,n}) = z_{3,n}z_{2,n}^{-2}z_{1,n}^2\tilde{z}_{1,n-1}^{-2}\tilde{z}_{2,n-1}t_{i,n-1}\,, \quad \rho_n^C(t_{1,n}) = z_{2,n}^2z_{1,n}^{-4}\tilde{z}_{1,n-1}^2t_{1,n-1}\,, \tag{4.14}$$

$$\rho_n^C(f_{2,n}) = \{z_{2,n}z_{1,n}^{-2}\tilde{z}_{1,n-1}^2\tilde{z}_{2,n-1}^{-1}t_{2,n-1}^{-1}\}x_{2,n}^{-1} + \{\tilde{z}_{1,n-1}\tilde{z}_{2,n-1}^{-1}t_{2,n-1}^{-1}\}\tilde{x}_{1,n-1}^{-1} + f_{2,n-1}\,,$$

$$\rho_n^C(f_{1,n}) = \{z_{1,n}^2\tilde{z}_{1,n-1}^{-2}t_{1,n-1}^{-1}\}_{\varepsilon_1}x_{1,n}^{-1} + f_{1,n-1}\,, \tag{4.15}$$

$$t_{n,n-1} := \varepsilon^{-\lambda_n}t_{1,n-1}^{-\frac{1}{2}}\prod_{i=2}^{n-1}t_{i,n-1}^{-1} \quad (n \ge 2)\,, \quad t_{1,0} := \varepsilon^{-\lambda_1}\,. \tag{4.16}$$

(d) *Let* $\lambda_n \in \mathbf{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbf{C}^\times)^n$, $\tilde{a}_{n-2} = (\tilde{a}_{i,n-2})_{i=1}^{n-2} \in (\mathbf{C}^\times)^{n-2}$, $b_n = (b_{i,n})_{i=1}^n \in \mathbf{C}^n$, *and* $\tilde{b}_{n-2} = (\tilde{b}_{i,n-2})_{i=1}^{n-2} \in \mathbf{C}^{n-2}$. *We obtain a* $\mathbf{C}$*-algebra homomorphism* $\rho_n^D := \rho_n^D(a_n, \tilde{a}_{n-2}, b_n, \tilde{b}_{n-2}, \lambda_n) : U_\varepsilon(D_n) \to \mathrm{End}(V_n) \otimes \mathrm{End}(\widetilde{V}_{n-2}) \otimes U_\varepsilon(D_{n-1})$ *such that* $\rho_n^D(e_{i,n})$, $\rho_n^D(t_{i,n})$, $\rho_n^D(f_{i,n})$ *as in* (4.9), (4.10), (4.11) *if* $4 \le i \le n$ (*replace* $\tilde{x}_{i,n-1}$ *to* $\tilde{x}_{i-1,n-2}$ *and* $\tilde{z}_{i,n-1}$ *to* $\tilde{z}_{i-1,n-2}$). *Moreover,*

$$\rho_n^D(e_{3,n}) = \{z_{4,n}z_{3,n}^{-1}\}x_{3,n} + \{\tilde{z}_{1,n-2}\tilde{z}_{2,n-2}^{-1}\}x_{4,n}^{-1}x_{3,n}\tilde{x}_{2,n-2} + x_{4,n}^{-1}x_{3,n}\tilde{x}_{2,n-2}\tilde{x}_{1,n-2}^{-1}e_{3,n-1}\,,$$

$$\rho_n^D(e_{2,n}) = \{z_{3,n} z_{2,n}^{-1}\} x_{2,n} + \{z_{1,n} \tilde{z}_{1,n-2}^{-1}\} \tilde{x}_{1,n-2} x_{3,n}^{-1} x_{2,n} + x_{3,n}^{-1} x_{2,n} x_{1,n}^{-1} \tilde{x}_{1,n-2} e_{2,n-1} ,$$

$$\rho_n^D(e_{1,n}) = \{z_{3,n} z_{1,n}^{-1}\} x_{1,n} + \{z_{2,n} \tilde{z}_{1,n-2}^{-1}\} \tilde{x}_{1,n-2} x_{3,n}^{-1} x_{1,n} + x_{3,n}^{-1} x_{1,n} x_{2,n}^{-1} \tilde{x}_{1,n-2} e_{2,n-1} , \tag{4.17}$$

$$\rho_n^D(t_{3,n}) = z_{4,n} z_{3,n}^{-2} z_{2,n} z_{1,n} \tilde{z}_{1,n-2}^{-2} \tilde{z}_{2,n-2} t_{i,n-1} ,$$

$$\rho_n^D(t_{2,n}) = z_{3,n} z_{2,n}^{-2} \tilde{z}_{1,n-2} t_{2,n-1}, \quad \rho_n^D(t_{1,n}) = z_{3,n} z_{1,n}^{-2} \tilde{z}_{1,n-2} t_{1,n-1} , \tag{4.18}$$

$$\rho_n^D(f_{3,n}) = \{z_{3,n} z_{2,n}^{-1} z_{1,n}^{-1} \tilde{z}_{1,n-2}^{2} \tilde{z}_{2,n-2}^{-1} t_{3,n-1}^{-1}\} x_{3,n}^{-1} + \{\tilde{z}_{1,n-2} \tilde{z}_{2,n-2}^{-1} t_{3,n-1}^{-1}\} \tilde{x}_{1,n-2}^{-1} + f_{3,n-1} ,$$

$$\rho_n^D(f_{2,n}) = \{z_{2,n} \tilde{z}_{1,n-2}^{-1} t_{2,n-1}^{-1}\} x_{2,n}^{-1} + f_{2,n-1} ,$$

$$\rho_n^D(f_{1,n}) = \{z_{1,n} \tilde{z}_{1,n-2}^{-1} t_{1,n-1}^{-1}\} x_{1,n}^{-1} + f_{1,n-1} , \tag{4.19}$$

$$t_{n,n-1} := \varepsilon^{-\lambda_n} t_{1,n-1}^{-\frac{1}{2}} t_{2,n-1}^{-\frac{1}{2}} \prod_{i=3}^{n-1} t_{i,n-1}^{-1} , \quad (n \geq 3) , \quad t_{2,1} := \varepsilon^{-\lambda_2} t_{1,1}^{-\frac{1}{2}} , \quad t_{1,0} := \varepsilon^{-\lambda_1} . \tag{4.20}$$

(e) *Let* $\lambda_2 \in \mathbf{C}$, $a_2 = (a_{i,2})_{i=1}^5 \in (\mathbf{C}^\times)^5$, *and* $b_2 = (b_{i,2})_{i=1}^5 \in \mathbf{C}^5$. *We obtain a* $\mathbf{C}$*-algebra homomorphism* $\rho^G := \rho^G(a_2, b_2, \lambda_2) : U_\varepsilon(G_2) \to \mathrm{End}(V_5) \otimes U_\varepsilon(A_1)$ *such that*

$$\rho^G(e_{1,2}) = \{z_{3,2}^3 z_{4,2}^{-2}\} x_{1,2}^{-1} x_{2,2}^{-1} x_{3,2} x_{4,2}^2 + \{z_{4,2} z_{5,2}^{-3}\} x_{1,2}^{-1} x_{2,2}^{-1} x_{4,2}^2 x_{5,2} + \{z_{2,2}^2 z_{3,2}^{-3}\} x_{1,2}^{-1} x_{2,2} x_{3,2}$$
$$+ [2]\{z_{2,2} z_{4,2}^{-1}\} x_{1,2}^{-1} x_{3,2} x_{4,2} + \{z_{1,2}^3 z_{2,2}^{-1}\} x_{2,2} + x_{1,2}^{-1} x_{2,2}^{-1} x_{4,2} x_{5,2} e_{1,1} ,$$

$$\rho^G(e_{2,2}) = \{z_{1,2}^{-3}\}_{\varepsilon^3} x_{1,2} , \tag{4.21}$$

$$\rho^G(t_{1,2}) = z_{1,2}^3 z_{3,2}^3 z_{5,2}^3 z_{2,2}^{-2} z_{4,2}^{-2} t_{1,1} , \quad \rho^G(t_{2,2}) = z_{1,2}^{-6} z_{3,2}^{-6} z_{5,2}^{-6} z_{2,2}^3 z_{4,2}^3 \varepsilon^{\lambda_2} t_{1,1}^{-\frac{3}{2}} , \tag{4.22}$$

$$\rho^G(f_{1,2}) = \{z_{2,2} z_{3,2}^{-3} z_{5,2}^{-3} z_{4,2}^{2} t_{1,1}^{-1}\} x_{2,2}^{-1} + \{z_{4,2} z_{5,2}^{-3} t_{1,1}^{-1}\} x_{4,2}^{-1} + f_{1,1} ,$$

$$\rho^G(f_{2,2}) = \{z_{1,2}^3 z_{3,2}^6 z_{5,2}^6 z_{2,2}^{-3} z_{4,2}^{-3} \varepsilon^{-\lambda_2} t_{1,1}^{\frac{3}{2}}\}_{\varepsilon^3} x_{1,2}^{-1} + \{z_{3,2}^3 z_{5,2}^6 z_{4,2}^{-3} \varepsilon^{-\lambda_2} t_{1,1}^{\frac{3}{2}}\}_{\varepsilon^3} x_{3,2}^{-1}$$
$$+ \{z_{5,2}^3 \varepsilon^{-\lambda_2} t_{1,1}^{\frac{3}{2}}\}_{\varepsilon^3} x_{5,2}^{-1} . \tag{4.23}$$

Here, $x_{i,j}^{\pm 1} := 0$, $z_{i,j}^{\pm 1} := 1$ *if the index* (i, j) *is out of the range,* $e_{0,n} := f_{n,n-1} := 0$, $U_\varepsilon(\mathfrak{g}_0) := \mathbf{C}$, $V_0 := \mathbf{C}$. *Moreover,* $\widetilde{V}_j$, $\tilde{x}_{i,j}$, $\tilde{z}_{i,j}$ *are copies of* V_j, $x_{i,j}$, $z_{i,j}$.

We can regard $\bigotimes_{j=1}^n V_j$ as an l^N-dimensional $U_\varepsilon(A_n)$-module having $\dim\mathfrak{g}_n$-parameters by using the $\mathbf{C}$-algebra homomorphism $\rho_1^A \circ \rho_2^A \circ \cdots \circ \rho_n^A : U_\varepsilon(A_n) \to \bigotimes_{j=1}^n V_j$. We call these modules the *Schnizer modules* of $U_\varepsilon(A_n)$ (see [2] Theorem 1.1). Similarly, by using the homomorphisms in Theorem 4.1, we obtain l^N-dimensional $U_\varepsilon(\mathfrak{g}_n)$-modules having $\dim\mathfrak{g}_n$-parameters if $\mathfrak{g} = A$, B, C, D, or G. We call these modules the *Schnizer modules* of $U_\varepsilon(\mathfrak{g}_n)$. In particular, if $\mathfrak{g} = B$, C, or D these modules coincide with those in [10] (see [11] Remark 5.12).

REMARK 4.2. The actions of $e_{i,n}, t_{i,n}, f_{i,n}$ in [11] are slightly different from those of Theorem 4.1 because we use a $U_\varepsilon(\mathfrak{g}_n)$-automorphism ω such that $(\omega(e_{i,n}), \omega(t_{i,n}), \omega(f_{i,n})) = (f_{i,n}, t_{i,n}^{-1}, e_{i,n})$.

We note the following fact that will be needed later. If $\mathfrak{g}_n = A_n$ $(n \geq 2)$, from (4.6) and (4.8), we obtain

$$\begin{aligned}
\rho_{n-1}^A(t_{n,n-1}) &= \varepsilon^{-\lambda_n} \prod_{i=1}^{n-1} \rho_{n-1}^A(t_{i,n-1}^{-\frac{i}{n}}) = \varepsilon^{-\lambda_n} \prod_{i=1}^{n-1} (z_{i-1,n-1} z_{i,n-1}^{-2} z_{i+1,n-1} t_{i,n-2})^{-\frac{i}{n}} \\
&= \varepsilon^{-\lambda_n} z_{n-1,n-1} \prod_{i=1}^{n-1} t_{i,n-2}^{-\frac{i}{n}} = \varepsilon^{-\lambda_n} z_{n-1,n-1} t_{n-1,n-2}^{-\frac{n-1}{n}} \prod_{i=1}^{n-2} t_{i,n-2}^{-\frac{i}{n}} \\
&= \varepsilon^{-\lambda_n} z_{n-1,n-1} \left(\varepsilon^{-\lambda_{n-1}} \prod_{i=1}^{n-2} t_{i,n-2}^{-\frac{i}{n-1}} \right)^{-\frac{n-1}{n}} \prod_{i=1}^{n-2} t_{i,n-2}^{-\frac{i}{n}} \\
&= \varepsilon^{-\lambda_n + \frac{n-1}{n}\lambda_{n-1}} z_{n-1,n-1} \,. \qquad (4.24)
\end{aligned}$$

Similarly, from (4.10), (4.12), (4.14), (4.16), (4.18), and (4.20), we obtain

$$\rho_{n-1}^B(t_{n,n-1}) = \varepsilon^{-\lambda_n + \lambda_{n-1}} z_{n-1,n-1} \tilde{z}_{n-2,n-2} \quad (n \geq 2)\,, \qquad (4.25)$$

$$\rho_{n-1}^C(t_{n,n-1}) = \varepsilon^{-\lambda_n + \lambda_{n-1}} z_{n-1,n-1} \tilde{z}_{n-2,n-2} \quad (n \geq 3)\,,$$

$$\rho_1^C(t_{2,1}) = \varepsilon^{-\lambda_2 + \frac{1}{2}\lambda_1} z_{1,1}^2\,, \qquad (4.26)$$

$$\rho_{n-1}^D(t_{n,n-1}) = \varepsilon^{-\lambda_n + \lambda_{n-1}} z_{n-1,n-1} \tilde{z}_{n-3,n-3} \quad (n \geq 4)\,,$$

$$\rho_2^D(t_{3,2}) = \varepsilon^{-\lambda_3 + \frac{1}{2}\lambda_2 + \frac{1}{4}} z_{1,1}^{\frac{1}{2}} z_{1,2} \tilde{z}_{2,2}\,, \quad \rho_1^D(t_{2,1}) = \varepsilon^{-\lambda_2 + \frac{1}{2}\lambda_1} z_{1,1}\,, \qquad (4.27)$$

$\rho_{n-1}^{\mathfrak{g}}(t_{1,0}) = \varepsilon^{-\lambda_1}$ for $\mathfrak{g} = A, B, C$, or D.

5. Construction of $L_\varepsilon^{\mathrm{nil}}(\lambda)$: the G_2 case

In this section, we construct all finite-dimensional irreducible nilpotent $U_\varepsilon(G_2)$-modules of type 1 by using the Schnizer homomorphisms in Theorem 4.1(e).

We define

$$\begin{aligned}
&a_{1,1}^{(0)} := a_{i,2}^{(0)} := 1 \quad (1 \leq i \leq 5)\,, \\
&b_{1,1}^{(0)} := b_{1,2}^{(0)} := 1\,, \quad b_{2,2}^{(0)} := 4\,, \quad b_{4,2}^{(0)} := 5\,, \quad b_{3,2}^{(0)} := 3\,, \quad b_{5,2}^{(0)} := 2\,, \\
&a_2^{(0)} := (a_{i,2}^{(0)})_{i=1}^5\,, \quad b_2^{(0)} := (b_{i,2}^{(0)})_{i=1}^5\,. \qquad (5.1)
\end{aligned}$$

For $\lambda \in \mathbf{C}$, we set

$$\rho_1^A(\lambda) := \rho_1^A(a_{1,1}^{(0)}, b_{1,1}^{(0)}, \lambda) : U_\varepsilon(A_1) \to \mathrm{End}(\mathbf{C})\,,$$

(5.2) $$\rho^G(\lambda) := \rho^G(a_2^{(0)}, b_2^{(0)}, \lambda) : U_\varepsilon(G_2) \to \mathrm{End}(V_5) \otimes U_\varepsilon(A_1)\,,$$

(see Theorem 4.1(a), (e)). For $\lambda_1, \lambda_2 \in \mathbf{C}$, we define

(5.3) $$\phi_{1,2} := \phi_{1,2}(\lambda_1, \lambda_2) := \rho_1^{A_1}(\nu_1^{(\lambda_1,\lambda_2)}) \circ \rho^G(\nu_2^{(\lambda_1,\lambda_2)}) : U_\varepsilon(G_2) \to \mathrm{End}(V_5 \otimes V_1)\,,$$

where

$$\nu_1^{(\lambda_1,\lambda_2)} := \lambda_1 + 2\,, \quad \nu_2^{(\lambda_1,\lambda_2)} := \frac{3}{2}\lambda_1 + 3\lambda_2 + 9\,.$$

We denote the $U_\varepsilon(G_2)$-modules associated with $(\phi_{1,2}(\lambda_1, \lambda_2), V_5 \otimes V_1)$ by $V_{1,2}(\lambda_1, \lambda_2)$. For $m_{1,1} \in \mathbf{Z}_l$, $m_5 = (m_{i,5})_{i=1}^5 \in \mathbf{Z}_l^5$, we set

$$v_{1,2}(m_5, m_{1,1}) := v_5(m_5) \otimes v_1(m_{1,1})\,, \quad v_{1,2}^{\mathbf{0}} := v_{1,2}(0, \ldots, 0) \in V_{1,2}(\lambda_1, \lambda_2)\,.$$

We define $y_{i,2}$, $y_{1,1} \in \mathrm{End}(V_{1,2}(\lambda_1, \lambda_2))$ $(1 \le i \le 5)$: for $v = v_{1,2}(m_5, m_{1,1})$,

$$\begin{aligned}
y_{1,2}v &:= [m_{1,2} + 2m_{3,2} + 2m_{5,2} - m_{2,2} - m_{4,2} - m_{1,1} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \varepsilon_{1,2}, m_{1,1})\,,\\
y_{2,2}v &:= [m_{2,2} - 3m_{3,2} - 3m_{5,2} + 2m_{4,2} + 2m_{1,1} - \lambda_1] v_{1,2}(m_5 + \varepsilon_{2,2}, m_{1,1})\,,\\
y_{3,2}v &:= [m_{3,2} + 2m_{5,2} - m_{4,2} - m_{1,1} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \varepsilon_{3,2}, m_{1,1})\,,\\
y_{4,2}v &:= [m_{4,2} - 3m_{5,2} + 2m_{1,1} - \lambda_1] v_{1,2}(m_5 + \varepsilon_{4,2}, m_{1,1})\,,\\
y_{5,2}v &:= [m_{5,2} - m_{1,1} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \varepsilon_{5,2}, m_{1,1})\,,
\end{aligned}$$

(5.4) $$y_{1,1}v := [m_{1,1} - \lambda_1] v_{1,2}(m_5, m_{1,1} + \varepsilon_{1,1})\,.$$

Then, according to Theorem 4.1(a), (e), (4.2), (4.4), and (5.1), we have

$$\begin{aligned}
e_{1,2}v = {}& [3m_{3,2} - 2m_{4,2}] v_{1,2}(m_{1,2} + 1, m_{2,2} + 1, m_{3,2} - 1, m_{4,2} - 2, m_{5,2}, m_{1,1})\\
&+ [m_{4,2} - 3m_{5,2}] v_{1,2}(m_{1,2} + 1, m_{2,2} + 1, m_{3,2}, m_{4,2} - 2, m_{5,2} - 1, m_{1,1})\\
&+ [2m_{2,2} - 3m_{3,2}] v_{1,2}(m_{1,2} + 1, m_{2,2} - 1, m_{3,2} - 1, m_{4,2}, m_{5,2}, m_{1,1})\\
&+ [2][m_{2,2} - m_{4,2}] v_{1,2}(m_{1,2} + 1, m_{2,2}, m_{3,2} - 1, m_{4,2} - 1, m_{5,2}, m_{1,1})\\
&+ [3m_{1,2} - m_{2,2}] v_{1,2}(m_{1,2}, m_{2,2} - 1, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1})
\end{aligned}$$

(5.5) $$\quad + [-m_{1,1}] v_{1,2}(m_{1,2} + 1, m_{2,2} + 1, m_{3,2}, m_{4,2} - 1, m_{5,2} - 1, m_{1,1} - 1)\,,$$

(5.6) $$e_{2,2}v = [-m_{1,2}]_{\varepsilon_2} v_{1,2}(m_{1,2} - 1, m_{2,2}, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1})\,,$$

(5.7) $$t_{1,2}v = \varepsilon^{3m_{1,2}+3m_{3,2}+3m_{5,2}-2m_{2,2}-2m_{4,2}-2m_{1,1}+\lambda_1} v_{1,2}(m_5, m_{1,1})\,,$$

(5.8) $$t_{2,2}v = \varepsilon_2^{-2m_{1,2}-2m_{3,2}-2m_{5,2}+m_{2,2}+m_{4,2}+m_{1,1}+\lambda_2} v_{1,2}(m_5, m_{1,1})\,,$$

(5.9) $$f_{1,2}v = (y_{2,2} + y_{4,2} + y_{1,1}) v_{1,2}(m_5, m_{1,1})\,,$$

(5.10) $$f_{2,2}v = (y_{1,2} + y_{3,2} + y_{5,2}) v_{1,2}(m_5, m_{1,1})\,.$$

Let $P(V_{1,2}(\lambda_1, \lambda_2))$ be as in Definition 3.3 (i).

PROPOSITION 5.1. *Let $\lambda_1, \lambda_2 \in \mathbf{C}$. We have $P(V_{1,2}(\lambda_1, \lambda_2)) = \mathbf{C}v_{1,2}^{\mathbf{0}}$.*

PROOF. Since the actions of $e_{1,2}$, $e_{2,2}$ on $V_{1,2}(\lambda_1, \lambda_2)$ do not depend on λ_1, λ_2, we simply denote $V_{1,2}(\lambda_1, \lambda_2)$ by $V_{1,2}$. From (5.5) and (5.6), obviously, $\mathbf{C}v^{\mathbf{0}}_{1,2} \subset P(V_{1,2})$. So we shall prove $P(V_{1,2}) \subset \mathbf{C}v^{\mathbf{0}}_{1,2}$. Let $v \in V_{1,2}$. There exist $c(m_5, m_{1,1}) \in \mathbf{C}$ ($m_5 \in \mathbf{Z}_l^5$, $m_{1,1} \in \mathbf{Z}_l$) such that

$$v = \sum_{m_5 \in \mathbf{Z}_l^5, m_{1,1} \in \mathbf{Z}_l} c(m_5, m_{1,1}) v_{1,2}(m_5, m_{1,1}) .$$

We assume that $e_{1,2}v = e_{2,2}v = 0$. From (5.6), we get

$$0 = e_{2,2}v = \sum_{m_5 \in \mathbf{Z}_l^5, m_{1,1} \in \mathbf{Z}_l} c(m_5, m_{1,1})[-m_{1,2}] v_{1,2}(m_{1,2} - 1, m_{2,2}, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1}) .$$

Hence, we obtain $c(m_5, m_{1,1}) = 0$ if $m_{1,2} \neq 0$. So, from (5.5), we have

$$\begin{aligned} 0 = e_{1,2}v = \sum_{m_{2,2}, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1} \in \mathbf{Z}_l} & c(0, m_{2,2}, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1}) \\ & \{[3m_{3,2} - 2m_{4,2}] v_{1,2}(1, m_{2,2} + 1, m_{3,2} - 1, m_{4,2} - 2, m_{5,2}, m_{1,1}) \\ & + [m_{4,2} - 3m_{5,2}] v_{1,2}(1, m_{2,2} + 1, m_{3,2}, m_{4,2} - 2, m_{5,2} - 1, m_{1,1}) \\ & + [2m_{2,2} - 3m_{3,2}] v_{1,2}(1, m_{2,2} - 1, m_{3,2} - 1, m_{4,2}, m_{5,2}, m_{1,1}) \\ & + [2][m_{2,2} - m_{4,2}] v_{1,2}(1, m_{2,2}, m_{3,2} - 1, m_{4,2} - 1, m_{5,2}, m_{1,1}) \\ & + [-m_{2,2}] v_{1,2}(0, m_{2,2} - 1, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1}) \\ & + [-m_{1,1}] v_{1,2}(1, m_{2,2} + 1, m_{3,2}, m_{4,2} - 1, m_{5,2} - 1, m_{1,1} - 1)\} . \end{aligned}$$

Since the $(1, 2)$-component of $(0, m_{2,2} - 1, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1})$ is 0 and the one of the other vectors is 1, by the linear independence of these vectors, $c(m_5, m_{1,1}) = 0$ if $m_{2,2} \neq 0$. Therefore we obtain

$$\begin{aligned} 0 = e_{1,2}v = \sum_{m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1} \in \mathbf{Z}_l} & c(0, 0, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1}) \\ & \{[3m_{3,2} - 2m_{4,2}] v_{1,2}(1, 1, m_{3,2} - 1, m_{4,2} - 2, m_{5,2}, m_{1,1}) \\ & + [m_{4,2} - 3m_{5,2}] v_{1,2}(1, 1, m_{3,2}, m_{4,2} - 2, m_{5,2} - 1, m_{1,1}) \\ & + [-3m_{3,2}] v_{1,2}(1, -1, m_{3,2} - 1, m_{4,2}, m_{5,2}, m_{1,1}) \\ & + [2][-m_{4,2}] v_{1,2}(1, 0, m_{3,2} - 1, m_{4,2} - 1, m_{5,2}, m_{1,1}) \\ & + [-m_{1,1}] v_{1,2}(1, 1, m_{3,2}, m_{4,2} - 1, m_{5,2} - 1, m_{1,1} - 1)\} . \end{aligned}$$

Since the $(2, 2)$-component of $(1, -1, m_{3,2} - 1, m_{4,2}, m_{5,2}, m_{1,1})$ (resp. $(1, 0, m_{3,2} - 1, m_{4,2} - 1, m_{5,2}, m_{1,1})$) is -1 (resp. 0) and the one of the other vectors is 1, we get $c(m_5, m_{1,1}) = 0$ if $m_{3,2} \neq 0$ or $m_{4,2} \neq 0$. Hence we have

$$0 = e_{1,2}v = \sum_{m_{5,2}, m_{1,1} \in \mathbf{Z}_l} c(0, 0, 0, 0, m_{5,2}, m_{1,1}) \{[-3m_{5,2}] v_{1,2}(1, 1, 0, -2, m_{5,2} - 1, m_{1,1})$$

$$+[-m_{1,1}]v_{1,2}(1,1,0,-1,m_{5,2}-1,m_{1,1}-1)\}\,.$$

Since the (4, 2)-component of $(1,1,0,-2,m_{5,2}-1,m_{1,1})$ is -2 and the one of $(1,1,0,-1,m_{5,2}-1,m_{1,1}-1)$ is -1, we obtain $c(m_5,m_{1,1})=0$ if $m_{5,2}\neq 0$ or $m_{1,1}\neq 0$. Thus we obtain $v=c(0,\ldots,0)v^{\mathbf{0}}_{1,2}\in \mathbf{C}v^{\mathbf{0}}_{1,2}$. □

Let $y_{i,2}$ $(1\le i\le 5)$, $y_{1,1}$ be as in (5.4). We define $Y_{1,2}:=\{y_{i,2},y_{1,1}\,|\,1\le i\le 5\}$. In addition, let $p_{\mathbf{0}}:V_{1,2}(\lambda_1,\lambda_2)\to \mathbf{C}v^{\mathbf{0}}_{1,2}$ be the projection map.

LEMMA 5.2. *Let* $\lambda_1,\lambda_2\in\mathbf{Z}$.
(a) *For all* $r\in\mathbf{N}$ *and* $g_1,\ldots,g_r\in Y_{1,2}$, *we have*

$$p_{\mathbf{0}}(g_1\cdots g_r v^{\mathbf{0}}_{1,2})=0\quad \textit{in } V_{1,2}(\lambda_1,\lambda_2)\,.$$

(b) *For all* $r\in\mathbf{N}$ *and* $i_1,\ldots,i_r\in\{1,2\}$, *we have*

$$p_{\mathbf{0}}(f_{i_1,2}\cdots f_{i_r,2}v^{\mathbf{0}}_{1,2})=0\quad \textit{in } V_{1,2}(\lambda_1,\lambda_2)\,.$$

PROOF. If we can prove (a), then we can obtain (b) from (5.9) and (5.10).
Let $r\in\mathbf{N}$ and $g_1,\ldots,g_r\in Y_{1,2}$. For $y\in Y_{1,2}$, we define

$$s(y):=\#\{1\le i\le r\,|\,g_i=y\}\ge 0\,,\quad m_g:=\sum_{i=1}^{5}s(y_{i,2})\varepsilon_{i,2}+s(y_{1,1})\varepsilon_{1,1}\,,$$

$$W_g:=\bigoplus_{s=1}^{r}\mathbf{C}(g_sg_{s+1}\cdots g_r v^{\mathbf{0}}_{1,2})\subset V_{1,2}(\lambda_1,\lambda_2)\,,$$

where $g:=g_1\cdots g_r$. Then, $gv^{\mathbf{0}}_{1,2}\in \mathbf{C}v_{1,2}(m_g)$ from (5.4), (5.9), and (5.10). Since $\sum_{i=1}^5 s(y_{i,2})+s(y_{1,1})=r>0$, there exists $1\le i\le 5$ such that $s(y_{i,2})>0$ or $s(y_{1,1})>0$.

Case 1) $s(y_{1,1})>0$: For $1\le r'\le r$, let $m^{(r')}$ be the element in $\mathbf{Z}_l^6$ such that $g_{r'}g_{r'+1}\cdots g_r\in \mathbf{C}v_{1,2}(m^{(r')})$. Let r_1 $(1\le r_1\le r)$ be the integer such that $g_{r_1}=y_{1,1}$ and $g_{r_1+1},\ldots,g_r\neq y_{1,1}$. Then, from (5.4), $m^{(r_1+1)}_{1,1}=0$. Hence, according to the definition of $y_{1,1}$ in (5.4), we get

$$g_{r_1}g_{r_1+1}\cdots g_r v^{\mathbf{0}}_{1,2}\in \mathbf{C}[-\lambda_1]v_{1,2}(m^{(r_1+1)}+\varepsilon_{1,1})\,.$$

Similarly, for $1\le r_2<r_1$ such that $g_{r_2}=y_{1,1}$ and $g_{r_2+1},\ldots,g_{r_1-1}\neq y_{1,1}$, we have

$$g_{r_2}g_{r_2+1}\cdots g_r v^{\mathbf{0}}_{1,2}\in \mathbf{C}[-\lambda_1+1][-\lambda_1]v_{1,2}(m^{(r_2+1)}+\varepsilon_{1,1})\,.$$

By repeating this procedure $s(y_{1,1})$-times, we obtain

$$gv^{\mathbf{0}}_{1,2}\in \mathbf{C}[-\lambda_1+s(y_{1,1})-1]\cdots[-\lambda_1+1][-\lambda_1]v_{1,2}(m_g)\,.$$

Since $\lambda_1\in\mathbf{Z}$ and $[l]=0$, if $s(y_{1,1})\ge l$, $[-\lambda_1+s(y_{1,1})-1]\cdots[-\lambda_1+1][-\lambda_1]=0$. On the other hand, if $0<s(y_{1,1})<l$, $p_{\mathbf{0}}(v_{1,2}(m_g))=0$. Therefore, we obtain $p_{\mathbf{0}}(gv^{\mathbf{0}}_{1,2})=0$.

Case 2) $s(y_{1,1}) = 0$ and $s(y_{5,2}) > 0$: Since $s(y_{1,1}) = 0$, for all $1 \leq r' \leq r$, $m_{1,1}^{(r')} = 0$. Hence, we get

$$y_{5,2}v_{1,2}(m_5, m_{1,1}) = [m_{5,2} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \varepsilon_{5,2}, m_{1,1}) \quad \text{in } W_g\,.$$

Thus, in a similar way to the proof of Case 1, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$.

Case 3) There exists $1 \leq i \leq 4$ such that $s(y_{1,1}) = s(y_{5,2}) = \cdots = s(y_{i+1,2}) = 0$ and $s(y_{i,2}) > 0$: In this case , for all $1 \leq r' \leq r$, $m_{1,1}^{(r')} = m_{5,2}^{(r')} = \cdots = m_{i+1,2}^{(r')} = 0$. Hence we have

$$y_{i,2}v_{1,2}(m_5, m_{1,1}) = [m_{i,2} - \lambda_{\tilde{i}}]_{\varepsilon_{\tilde{i}}} v_{1,2}(m_5 + \varepsilon_{i,2}, m_{1,1}) \quad \text{in } W_g\,,$$

where $\tilde{i} := 1$ if $i = 2, 4$ and $\tilde{i} := 2$ if $i = 1, 3$. Therefore, in a similar way to the proof of Case 1, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$.

From the results of Case 1–3, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$. □

LEMMA 5.3. *For all $\lambda_1, \lambda_2 \in \mathbf{Z}$, $\alpha \in \Delta_+$, we have $f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = 0$ in $V_{1,2}(\lambda_1, \lambda_2)$.*

PROOF. From Lemma 5.2(b) and Proposition 2.4, we obtain $p_{\mathbf{0}}(f_{\alpha,2}^l v_{1,2}^{\mathbf{0}}) = 0$. On the other hand, from Proposition 2.3, 5.1,

$$e_{i,2} f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = f_{\alpha,2}^l e_{i,2} v_{1,2}^{\mathbf{0}} = 0 \quad (i = 1, 2)\,.$$

Hence, from Proposition 5.1, we get $f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} \in \mathbf{C}v_{1,2}^{\mathbf{0}}$. Therefore, we obtain

$$f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = p_{\mathbf{0}}(f_{\alpha,2}^l v_{1,2}^{\mathbf{0}}) = 0\,.$$

□

Now, we shall construct nilpotent $U_\varepsilon(G_2)$-modules (see §3). For λ_1, $\lambda_2 \in \mathbf{C}$, let $L_{1,2}(\lambda_1, \lambda_2)$ be the $U_\varepsilon(G_2)$-submodule of $V_{1,2}(\lambda_1, \lambda_2)$ generated by $v_{1,2}^{\mathbf{0}}$.

THEOREM 5.4. *For any $\lambda_1, \lambda_2 \in \mathbf{Z}_l$, as a $U_\varepsilon(G_2)$-module, $L_{1,2}(\lambda_1, \lambda_2)$ is isomorphic to $L_\varepsilon^{nil}(\lambda_1, \lambda_2)$.*

PROOF. From Proposition 5.1, $e_{1,2}v_{1,2}^{\mathbf{0}} = e_{2,2}v_{1,2}^{\mathbf{0}} = 0$. Moreover, from (5.7) and (5.8),

$$t_{i,2}v_{1,2}^{\mathbf{0}} = \varepsilon_i^{\lambda_i} v_{1,2}^{\mathbf{0}} \quad (i = 1, 2)\,.$$

Thus $L_{1,2}(\lambda_1, \lambda_2)$ is a finite-dimensional highest-weight $U_\varepsilon(G_2)$-module with highest weight (λ_1, λ_2). On the other hand, from Lemma 5.3, $f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = 0$ for all $\alpha \in \Delta_+$. Moreover, from Proposition 2.4, 5.1, we have $e_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = 0$ for all $\alpha \in \Delta_+$. Hence, from Proposition 2.3, $e_{\alpha,2}^l = f_{\alpha,2}^l = 0$ on $L_{1,2}(\lambda_1, \lambda_2)$ for all $\alpha \in \Delta_+$. Thus $L_{1,2}(\lambda_1, \lambda_2)$ is a nilpotent $U_\varepsilon(G_2)$-module. Therefore, the theorem follows from Proposition 5.1 and Proposition 3.5. □

If $\lambda_1 = 0$, we can construct $L_\varepsilon^{\rm nil}(\lambda_1, \lambda_2)$ more easily. For $\lambda \in \mathbf{C}$, let $\rho^G(\lambda)$ be as in (5.2). For $m \in \mathbf{Z}_+$, let $(\pi_m, \mathbf{C})$ be the trivial representation of $U_\varepsilon(\mathfrak{g}_m)$, that is,

$$\pi_m(e_{i,m}) = \pi_m(f_{i,m}) = 0\,, \quad \pi_m(t_{i,m}) = 1 \quad (1 \le i \le m)\,, \tag{5.11}$$

where $e_{i,0} := f_{i,0} := 0$, $t_{i,0} := 1$, $U_\varepsilon(\mathfrak{g}_0) := \mathbf{C}$. For $\lambda_2 \in \mathbf{C}$, we define

$$\phi_{2,2} := \phi_{2,2}(\lambda_2) := \pi_1 \circ \rho^G(\nu_2^{\lambda_2}) : U_\varepsilon(G_2) \to \mathrm{End}(V_5)\,,$$

where $\nu_2^{\lambda_2} := 3\lambda_2 + 9$. We denote the $U_\varepsilon(G_2)$-module associated with $(\phi_{2,2}(\lambda_2), V_5)$ by $V_{2,2}(\lambda_2)$. Let $L_{2,2}(\lambda_2)$ be the $U_\varepsilon(G_2)$-submodule of $V_{2,2}(\lambda_2)$ generated by $v_{2,2}^{\mathbf{0}} := v_5(0,\ldots,0)$. Then, in a similar way to the proof of Proposition 5.1, Lemma 5.2, 5.3, and Theorem 5.4, we can prove the following proposition.

PROPOSITION 5.5. *For any $\lambda_2 \in \mathbf{Z}_l$, as a $U_\varepsilon(G_2)$-module, $L_{2,2}(\lambda_2)$ is isomorphic to $L_\varepsilon^{nil}(0, \lambda_2)$. Moreover, for any $\lambda_2 \in \mathbf{C}$, we have $P(V_{2,2}(\lambda_2)) = \mathbf{C}v_{2,2}^{\mathbf{0}}$.*

6. Inductive construction of $L_\varepsilon^{\rm nil}(\lambda)$: the B_n case

In this section, we inductively construct all finite-dimensional irreducible nilpotent $U_\varepsilon(B_n)$-modules of type 1 by using the Schnizer homomorphisms in Theorem 4.1(b).

We define $a_n^{(0)} = (a_{i,n}^{(0)})_{i=1}^n$, $\tilde a_{n-1}^{(0)} = (\tilde a_{i,n-1}^{(0)})_{i=1}^{n-1}$, $b_n^{(0)} = (b_{i,n}^{(0)})_{i=1}^n$, $\tilde b_{n-1}^{(0)} = (\tilde b_{i,n-1}^{(0)})_{i=1}^{n-1} \in \mathbf{C}^n$ whereby

$$a_{i,n}^{(0)} := \tilde a_{i,n}^{(0)} := 1\,,\ \ b_{i,n}^{(0)} := n-i+1\ (i \ne 1)\,,\ \ b_{1,n}^{(0)} := 2n-1\,,\ \ \tilde b_{i,n-1}^{(0)} := i+n-2\,. \tag{6.1}$$

Let $k \in I$. For $\lambda = (\lambda_k, \ldots, \lambda_n) \in \mathbf{C}^{n-k+1}$, we define $\nu^\lambda = (\nu_k^\lambda, \ldots, \nu_n^\lambda) \in \mathbf{C}^{n-k+1}$ as

$$\nu_i^\lambda := -2i + 1 - \sum_{j=k}^{i} \lambda_j \quad (k \ge 2)\,, \qquad \nu_i^\lambda := -2i + 1 - \frac{1}{2}\lambda_1 - \sum_{j=2}^{i} \lambda_j \quad (k = 1)\,,$$

where $k \le i \le n$. For $\lambda \in \mathbf{C}$, we set $\rho_n^B(\lambda) := \rho_n^B(a_n^{(0)}, \tilde a_{n-1}^{(0)}, b_n^{(0)}, \tilde b_{n-1}^{(0)}, \lambda) : U_\varepsilon(B_n) \to \mathrm{End}(V_n \otimes \widetilde V_{n-1}) \otimes U_\varepsilon(B_{n-1})$ (see Theorem 4.1(b)). Let $(\pi_{k-1}, \mathbf{C})$ be as in (5.11). We define

$$V_{k,n} := \bigotimes_{j=k}^{n} (V_j \otimes \widetilde V_{j-1})\,.$$

For $\lambda = (\lambda_k, \ldots, \lambda_n) \in \mathbf{C}^{n-k+1}$, we define a $U_\varepsilon(B_n)$-representation $\phi_{k,n} := \phi_{k,n}(\lambda) : U_\varepsilon(B_n) \to \mathrm{End}(V_{k,n})$ as

$$\phi_{k,n}(\lambda) := \pi_{k-1} \circ \rho_k^B(\nu_k^\lambda) \circ \cdots \circ \rho_n^B(\nu_n^\lambda)\,, \tag{6.2}$$

and denote the $U_\varepsilon(B_n)$-module associated with $(\phi_{k,n}(\lambda), V_{k,n})$ by $V_{k,n}(\lambda)$.

Let $m_n = (m_{1,n}, \dots, m_{n,n}) \in \mathbf{Z}_l^n$, $\tilde{m}_{n-1} = (\tilde{m}_{1,n-1}, \dots, \tilde{m}_{n-1,n-1}) \in \mathbf{Z}_l^{n-1}$, $w \in V_{k,n-1}$, $v = v_n(m_n) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w \in V_{k,n}(\lambda)$. Then, according to (4.2), (4.4), (4.9), and (6.1), for any $1 < i < n$, we have

$$e_{n,n} v = [-m_{n,n}](v_n(m_n - \varepsilon_{n,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w)\,, \tag{6.3}$$

$$\begin{aligned} e_{i,n} v = {} & [m_{i+1,n} - m_{i,n}](v_n(m_n - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w) \\ & + [\tilde{m}_{i-1,n-1} - \tilde{m}_{i,n-1}](v_n(m_n + \varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{i,n-1}) \otimes w) \\ & + v_n(m_n + \varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{i,n-1} + \tilde{\varepsilon}_{i-1,n-1}) \otimes (e_{i,n-1} w)\,, \end{aligned} \tag{6.4}$$

$$\begin{aligned} e_{1,n} v = {} & [2m_{2,n} - m_{1,n}]_{\varepsilon_1}(v_n(m_n - \varepsilon_{1,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w) \\ & + [m_{1,n} - 2\tilde{m}_{1,n-1}]_{\varepsilon_1} v_n(m_n + \varepsilon_{2,n} - \varepsilon_{1,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{1,n-1}) \otimes w \\ & + v_n(m_n + \varepsilon_{2,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{1,n-1}) \otimes (e_{1,n-1} w)\,. \end{aligned} \tag{6.5}$$

For $i \in \mathbf{N}$, we define $N_i := \frac{1}{2} i(i+1)$. Let $m = (m_{i,j})_{1 \le i \le j, k \le j \le n} \in \mathbf{Z}_l^{N_n - N_{k-1}}$, $\tilde{m} = (\tilde{m}_{i,j-1})_{1 \le i \le j-1, k \le j \le n} \in \mathbf{Z}_l^{N_{n-1} - N_{k-2}}$. We set

$$v_{k,n}(m, \tilde{m}) := \left(\bigotimes_{j=k}^{n} v_j(m_{1,j}, \dots m_{j,j}) \right) \otimes \left(\bigotimes_{j=k}^{n} \tilde{v}_{j-1}(\tilde{m}_{1,j-1}, \dots \tilde{m}_{j-1,j-1}) \right),$$

$$v_{k,n}^{\mathbf{0}} := v_{k,n}(\mathbf{0}, \mathbf{0})\,. \tag{6.6}$$

Let $P(V_{k,n}(\lambda))$ be as in Definition 3.3 (i).

PROPOSITION 6.1. *Let $\lambda \in \mathbf{C}^{n-k+1}$. We have $P(V_{k,n}(\lambda)) = \mathbf{C} v_{k,n}^{\mathbf{0}}$.*

PROOF. Since the actions of $e_{i,n}$ on $V_{k,n}(\lambda)$ do not depend on λ, we simply denote $V_{k,n}(\lambda)$ by $V_{k,n}$. From (6.4) and (6.5), obviously, $\mathbf{C} v_{k,n}^{\mathbf{0}} \subset P(V_{k,n})$. So we shall prove $P(V_{k,n}) \subset \mathbf{C} v_{k,n}^{\mathbf{0}}$ by induction on n.

Let $n = 1$. Then we have $k = 1$. Let $v \in V_{1,1}$. There exist $c(m_{1,1}) \in \mathbf{C}$ ($m_{1,1} \in \mathbf{Z}_l$) such that $v = \sum_{m_{1,1} \in \mathbf{Z}_l} c(m_{1,1}) v(m_{1,1})$. We assume $e_{1,1} v = 0$. Then, from (6.5), we get

$$0 = e_{1,1} v = \sum_{m_{1,1} \in \mathbf{Z}_l} c(m_{1,1})[-m_{1,1}]_{\varepsilon_1} v_{1,1}(m_{1,1} - 1)\,.$$

Hence, we obtain $c(m_{1,1}) = 0$ if $m_{1,1} \neq 0$. Therefore we have $v = c(0) v_{1,1}(0) \in \mathbf{C} v_{1,1}(0) = \mathbf{C} v_{1,1}^{\mathbf{0}}$.

Now, we assume that $n > 1$ and the case of $(n-1)$ holds. We define $v_{n,n-1}^{\mathbf{0}} := 1$ and $V_{n,n-1} := \mathbf{C} v_{n,n-1}^{\mathbf{0}}$. Let $v \in V_{k,n}$. There exist $c(m_n, \tilde{m}_{n-1}) \in \mathbf{C}$ and $v_{m_n, \tilde{m}_{n-1}} \in V_{k,n-1}$ ($m_n \in \mathbf{Z}_l^n$, $\tilde{m}_{n-1} \in \mathbf{Z}_l^{n-1}$) such that

$$v = \sum_{m_n \in \mathbf{Z}_l^n, \tilde{m}_{n-1} \in \mathbf{Z}_l^{n-1}} c(m_n, \tilde{m}_{n-1})(v_n(m_n) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{m_n, \tilde{m}_{n-1}})\,.$$

We assume that $e_{i,n}v = 0$ for all $1 \le i \le n$.

First, we shall prove that $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_n \neq \mathbf{0}$. From (6.3), we get

$$0 = e_{n,n}v = \sum_{m_n,\tilde{m}_{n-1}} c(m_n, \tilde{m}_{n-1})[-m_{n,n}](v_n(m_n - \varepsilon_{n,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}) .$$

Hence, we obtain $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{n,n} \neq 0$. Now, we assume that there exists $2 \le i \le n-1$ such that $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{i+1,n} \neq 0, \ldots, m_{n-1,n} \neq 0$, or $m_{n,n} \neq 0$. Then, from (6.4), we have

$$\begin{aligned}0 = e_{i,n}v = \sum_{m_n,\tilde{m}_{n-1}} & c(m_n, \tilde{m}_{n-1})\{[-m_{i,n}](v_n(m_n - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}) \\ & +[\tilde{m}_{i-1,n-1} - \tilde{m}_{i,n-1}](v_n(m_n + \varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{i,n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}) \\ & +v_n(m_n + \varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{i,n-1} + \tilde{\varepsilon}_{i-1,n-1}) \otimes (e_{i,n-1}v_{m_n,\tilde{m}_{n-1}})\} .\end{aligned}$$

If $m_{i+1,n} = 0$, the $(i+1, n)$-component of $(m_n - \varepsilon_{i,n})$ is 0, and the one of $(m_n + \varepsilon_{i+1,n} - \varepsilon_{i,n})$ is 1. Thus, by the linear independence, $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{i,n} \neq 0$. Hence, we inductively obtain $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{2,n} \neq 0, \ldots, m_{n-1,n} \neq 0$, or $m_{n,n} \neq 0$. Similarly, we have $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{1,n} \neq 0$ by using $e_{1,n}v = 0$. Hence, we obtain $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_n \neq \mathbf{0}$. Therefore, we get

$$v = \sum_{\tilde{m}_{n-1} \in \mathbf{Z}_l^{n-1}} c(\mathbf{0}, \tilde{m}_{n-1})(v_n(\mathbf{0}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{\mathbf{0},\tilde{m}_{n-1}}) .$$

Moreover, we have

$$\begin{aligned}0 &= e_{i,n}v \\ &= \sum_{\tilde{m}_{n-1}} c(\mathbf{0}, \tilde{m}_{n-1})\{[\tilde{m}_{i-1,n-1} - \tilde{m}_{i,n-1}](v_n(\varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{i,n-1}) \otimes v_{\mathbf{0},\tilde{m}_{n-1}}) \\ &\qquad +v_n(\varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{i,n-1} + \tilde{\varepsilon}_{i-1,n-1}) \otimes (e_{i,n-1}v_{\mathbf{0},\tilde{m}_{n-1}})\}\ (i \neq 1) , \\ 0 &= e_{1,n}v \\ &= \sum_{\tilde{m}_{n-1}} c(\mathbf{0}, \tilde{m}_{n-1})\{[-2\tilde{m}_{1,n-1}]_{\varepsilon_1}(v_n(\varepsilon_{2,n} - \varepsilon_{1,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{1,n-1}) \otimes v_{\mathbf{0},\tilde{m}_{n-1}}) \\ &\qquad +v_n(\varepsilon_{2,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\varepsilon}_{1,n-1}) \otimes (e_{1,n-1}v_{\mathbf{0},\tilde{m}_{n-1}})\} .\end{aligned}$$

Then, in a similar way to the above proof, we obtain $c(\mathbf{0}, \tilde{m}_{n-1}) = 0$ if $\tilde{m}_{n-1} \neq \mathbf{0}$ by using $e_{i,n}v = 0$ $(1 \le i \le n)$. Finally, we get

$$\begin{aligned}& v = c(\mathbf{0}, \mathbf{0})(v_n(\mathbf{0}) \otimes \tilde{v}_{n-1}(\mathbf{0}) \otimes v_{\mathbf{0},\mathbf{0}}) , \\ & 0 = e_{i,n}v = c(\mathbf{0}, \mathbf{0})v_n(\varepsilon_{i+1,n} - \varepsilon_{i,n}) \otimes \tilde{v}_{n-1}(-\tilde{\varepsilon}_{i,n-1} + \tilde{\varepsilon}_{i-1,n-1}) \otimes (e_{i,n-1}v_{\mathbf{0},\mathbf{0}})\ (i \neq 1, n) , \\ & 0 = e_{1,n}v = c(\mathbf{0}, \mathbf{0})v_n(\varepsilon_{2,n}) \otimes \tilde{v}_{n-1}(-\tilde{\varepsilon}_{1,n-1}) \otimes (e_{1,n-1}v_{\mathbf{0},\mathbf{0}}) .\end{aligned}$$

Hence $e_{i,n-1}v_{\mathbf{0},\mathbf{0}} = 0$ in $V_{k,n-1}$ for all $1 \le i \le n-1$ if $c(\mathbf{0},\mathbf{0}) \neq 0$. Thus, from the assumption of the induction on n, we obtain $v_{\mathbf{0},\mathbf{0}} \in \mathbf{C}v^{\mathbf{0}}_{k,n-1}$ if $c(\mathbf{0},\mathbf{0}) \neq 0$. Therefore

$$v \in \mathbf{C}(v_n(\mathbf{0}) \otimes \tilde{v}_{n-1}(\mathbf{0}) \otimes v^{\mathbf{0}}_{k,n-1}) = \mathbf{C}v^{\mathbf{0}}_{k.n}\,.$$

□

Let $m = (m_{i,j})_{1\le i\le j, k\le j\le n} \in \mathbf{Z}_l^{N_n - N_{k-1}}$, $\tilde{m} = (\tilde{m}_{i,j-1})_{1\le i\le j-1, k\le j\le n} \in \mathbf{Z}_l^{N_{n-1}-N_{k-2}}$. For $1 \le i \le n$, $\max(k,i) \le j \le n$, we define

$$\nu_{i,j}(m,\tilde{m}) := m_{i+1,j} - 2m_{i,j} + m_{i-1,j} + \tilde{m}_{i-2,j-1} - 2\tilde{m}_{i-1,j-1} + \tilde{m}_{i,j-1} \quad (i \neq 1)\,,$$

$$\nu_{1,j}(m,\tilde{m}) := m_{2,j} - m_{1,j} + \tilde{m}_{1,j-1}\,,$$

$$\mu_{i,i-1}(m,\tilde{m}) := \xi(i > k)(m_{i-1,i-1} + \tilde{m}_{i-2,i-2})\,,$$

$$\mu_{i,j}(m,\tilde{m}) := \mu_{i,i-1}(m,\tilde{m}) + \sum_{r=\max(k,i)}^{j} \nu_{i,r}(m,\tilde{m})\,, \tag{6.7}$$

where

$$\xi(i > j) := \begin{cases} 1 & (i > j) \\ 0 & (i \le j)\,, \end{cases} \qquad \xi(i \ge j) := \begin{cases} 1 & (i \ge j) \\ 0 & (i < j)\,. \end{cases} \tag{6.8}$$

Let $v_{k,n}(m,\tilde{m})$ be as in (6.6). Then, from (4.2), (4.10), (4.25), and (6.1), we obtain

$$t_{i,n}v_{k,n}(m,\tilde{m}) = \varepsilon_i^{\mu_{i,n}(m,\tilde{m})+\xi(i\ge k)\lambda_i} v_{k,n}(m,\tilde{m})\,. \tag{6.9}$$

In particular, we obtain the following lemma.

LEMMA 6.2. *For any* $\lambda = (\lambda_k, \dots, \lambda_n) \in \mathbf{C}^{n-k+1}$, $i \in I$, *we obtain*

$$t_{i,n}v^{\mathbf{0}}_{k,n} = \varepsilon_i^{\xi(i\ge k)\lambda_i} v^{\mathbf{0}}_{k,n} \quad \text{in } V_{k,n}(\lambda)\,.$$

Let $m = (m_{i,j})_{1\le i\le j, k\le j\le n} \in \mathbf{Z}_l^{N_n - N_{k-1}}$, $\tilde{m} = (\tilde{m}_{i,j-1})_{1\le i\le j-1, k\le j\le n} \in \mathbf{Z}_l^{N_{n-1}-N_{k-2}}$, $\lambda = (\lambda_k, \dots, \lambda_n) \in \mathbf{C}^{n-k+1}$. For $i \in I$, $\max(k,i) \le j \le n$, we define $y_{i,j}$, $\tilde{y}_{i-1,j-1} \in \mathrm{End}(V_{k,n}(\lambda))$ as follows: for $v = v_{k,n}(m,\tilde{m})$,

$$\tilde{y}_{i-1,j-1}v := [\tilde{m}_{i-1,j-1} - \tilde{m}_{i,j-1} - \mu_{i,j-1}(m,\tilde{m}) - \xi(i \ge k)\lambda_i]v_{k,n}(m, \tilde{m} + \tilde{\varepsilon}_{i-1,j-1})\,,$$

$$y_{i,j}v := [m_{i+1,j} - m_{i,j} - \mu_{i,j}(m,\tilde{m}) - \xi(i \ge k)\lambda_i]v_{k,n}(m + \varepsilon_{i,j}, \tilde{m}) \quad (i \neq 1)\,,$$

$$y_{1,j}v := [m_{1,j} - 2\tilde{m}_{1,j-1} - \mu_{1,j}(m,\tilde{m}) - \xi(1 \ge k)\lambda_i]_{\varepsilon_1} v_{k,n}(m + \varepsilon_{1,j}, \tilde{m})\,, \tag{6.10}$$

where $\tilde{y}_{0,j-1} := 0$. We define

$$Y_{k,n} := \{y_{i,j}, \tilde{y}_{i-1,j-1} \mid i \in I, \max(k,i) \le j \le n\}\,.$$

Then, from (4.2), (4.4), (4.11), and (6.1), we have

$$f_{i,n}v_{k,n}(m,\tilde{m}) = \sum_{j=\max(k,i)}^{n} (y_{i,j} + \tilde{y}_{i-1,j-1})v_{k,n}(m,\tilde{m}) \quad \text{in } V_{k,n}(\lambda)\,. \tag{6.11}$$

Let $p_{\mathbf{0}} : V_{k,n}(\lambda) \to \mathbf{C}v^{\mathbf{0}}_{k,n}$ be the projection map.

LEMMA 6.3. *Let $\lambda = (\lambda_k, \dots, \lambda_n) \in \mathbf{Z}^{n-k+1}$.*
(a) *For all $r \in \mathbf{N}$ and $g_1, \dots, g_r \in Y_{k,n}$, we have*

$$p_{\mathbf{0}}(g_1 \cdots g_r v^{\mathbf{0}}_{k,n}) = 0 \quad in\ V_{k,n}(\lambda)\,.$$

(b) *For all $r \in \mathbf{N}$ and $i_1, \dots, i_r \in I$, we have*

$$p_{\mathbf{0}}(f_{i_1,n} \cdots f_{i_r,n} v^{\mathbf{0}}_{k,n}) = 0 \quad in\ V_{k,n}(\lambda)\,.$$

PROOF. If we can prove (a), we can obtain (b) from (6.11).
Let $r \in \mathbf{N}$ and $g_1, \dots, g_r \in (Y_{k,n} - \{0\})$. For $y \in Y_{k,n}$, we define

$$s(y) := \#\{1 \le r' \le r \mid g_{r'} = y\} \ge 0\,, \quad s_j := \sum_{i=1}^{j}(s(y_{i,j}) + s(\tilde{y}_{i-1,j-1})) \quad (k \le j \le n)\,,$$

$$m_g := \sum_{j=k}^{n}\sum_{i=1}^{j}(s(y_{i,j})\varepsilon_{i,j} + s(\tilde{y}_{i-1,j-1})\tilde{\varepsilon}_{i-1,j-1})\,, \quad W_g := \bigoplus_{r'=1}^{r} \mathbf{C}(g_{r'} g_{r'+1} \cdots g_r v^{\mathbf{0}}_{k,n})\,,$$

where $g := g_1 \cdots g_r$. Then, $g v^{\mathbf{0}}_{k,n} \in \mathbf{C}v_{k,n}(m_g)$ from (6.10). Since $\sum_{j=k}^{n} s_j = r > 0$, there exists $k \le j \le n$ such that $s_k = \cdots = s_{j-1} = 0$ and $s_j > 0$. Then, $s(y_{p,q}) = s(\tilde{y}_{p-1,q-1}) = 0$ for all $k \le q < j$, $1 \le p \le q$. Thus, for any $1 \le r' \le r$, there exist $m^{(r')} \in \mathbf{Z}_l^{N_n - N_{k-1}}$, $\tilde{m}^{(r')} \in \mathbf{Z}_l^{N_{n-1} - N_{k-2}}$ such that $g_{r'} g_{r'+1} \cdots g_r v^{\mathbf{0}}_{k,n} \in \mathbf{C}v_{k,n}(m^{(r')}, \tilde{m}^{(r')})$ and $m^{(r')}_{p,q} = \tilde{m}^{(r')}_{p-1,q-1} = 0$ for all $k \le q < j$, $1 \le p \le q$. Hence, from (6.7) and (6.10), in W_g, we have

$$\begin{aligned} y_{i,j} v_{k,n}(m, \tilde{m}) &= [m_{i+1,j} - m_{i,j} - \nu_{i,j}(m, \tilde{m}) - \xi(i \ge k)\lambda_i] v_{k,n}(m + \varepsilon_{i,j}, \tilde{m})\,, \\ y_{1,j} v_{k,n}(m, \tilde{m}) &= [m_{1,j} - 2\tilde{m}_{1,j-1} - \xi(1 \ge k)\lambda_1]_{\varepsilon_1} v_{k,n}(m + \varepsilon_{1,j}, \tilde{m})\,, \\ \tilde{y}_{i-1,j-1} v_{k,n}(m, \tilde{m}) &= [\tilde{m}_{i-1,j-1} - \tilde{m}_{i,j-1} - \xi(i \ge k)\lambda_i] v_{k,n}(m, \tilde{m} + \tilde{\varepsilon}_{i-1,j-1})\,, \end{aligned} \tag{6.12}$$

for $2 \le i \le j$, where $\xi(i \ge j)$ as in (6.8).

On the other hand, since $s_j > 0$, there exists i $(1 \le i \le j)$ such that $s(y_{i,j}) > 0$ or $s(\tilde{y}_{i-1,j-1}) > 0$. Now, we assume $s(\tilde{y}_{j-1,j-1}) > 0$. Let r_1 $(1 \le r_1 \le r)$ be the integer such that $g_{r_1} = \tilde{y}_{j-1,j-1}$ and $g_{r_1+1}, \dots, g_r \ne \tilde{y}_{j-1,j-1}$. Then, from (6.10), $\tilde{m}^{(r_1+1)}_{j-1,j-1} = 0$, and from (6.12), we get

$$g_{r_1} g_{r_1+1} \cdots g_r v^{\mathbf{0}}_{k,n} \in \mathbf{C}[-\lambda_j] v_{k,n}(m^{(r_1+1)}, \tilde{m}^{(r_1+1)} + \tilde{\varepsilon}_{j-1,j-1})\,.$$

Thus, in a similar way to the proof of Case 1 in the proof of Lemma 5.2, we have

$$g v^{\mathbf{0}}_{k,n} \in \mathbf{C}[-\lambda_j + s(\tilde{y}_{j-1,j-1}) - 1] \cdots [-\lambda_j + 1][-\lambda_j] v_{k,n}(m_g)\,,$$

and $p_{\mathbf{0}}(gv^{\mathbf{0}}_{k,n}) = 0$. Similarly, if there exists i $(2 \le i \le j-1)$ such that $s(\tilde{y}_{j-1,j-1}) = \cdots = s(\tilde{y}_{i,j-1}) = 0$, $s(\tilde{y}_{i-1,j-1}) > 0$, we have

$$\tilde{y}_{i-1,j-1} v_{k,n}(m, \tilde{m}) = [\tilde{m}_{i-1,j-1} - \xi(i \ge k)\lambda_i] v_{k,n}(m, \tilde{m} + \tilde{\varepsilon}_{i-1,j-1}) \quad \text{in } W_g\,,$$

and $p_{\mathbf{0}}(gv^{\mathbf{0}}_{k,n}) = 0$. If $s(\tilde{y}_{1,j-1}) = \cdots = s(\tilde{y}_{j-1,j-1}) = 0$, and there exists i $(1 \le i \le j)$ such that $s(y_{1,j}) = \cdots = s(y_{i-1,j}) = 0$, $s(y_{i,j}) > 0$, we obtain

$$y_{i,j} v_{k,n}(m, \tilde{m}) = [m_{i,j} - \xi(i \ge k)\lambda_i]_{\varepsilon_i} v_{k,n}(m + \varepsilon_{i,j}, \tilde{m}) \quad \text{in } W_g\,,$$

and $p_{\mathbf{0}}(gv^{\mathbf{0}}_{k,n}) = 0$.

Consequently, we obtain that $p_{\mathbf{0}}(gv^{\mathbf{0}}_{k,n}) = 0$ if $s_j > 0$. Thus, we obtain $p_{\mathbf{0}}(gv^{\mathbf{0}}_{k,n}) = 0$. □

LEMMA 6.4. *For all* $\lambda = (\lambda_k, \ldots, \lambda_n) \in \mathbf{Z}^{n-k+1}$, $\alpha \in \Delta_+$, *we have*

$$f^l_{\alpha,n} v^{\mathbf{0}}_{k,n} = 0 \quad \textit{in } V_{k,n}(\lambda)\,.$$

PROOF. By using Lemma 6.3(b) and Proposition 2.4, we obtain $p_{\mathbf{0}}(f^l_{\alpha,n} v^{\mathbf{0}}_{k,n}) = 0$. On the other hand, from Proposition 2.3, 6.1, we have

$$e_{i,n} f^l_{\alpha,n} v^{\mathbf{0}}_{k,n} = f^l_{\alpha,n} e_{i,n} v^{\mathbf{0}}_{k,n} = 0\,,$$

for all $i \in I$. Hence, by using Proposition 6.1, we get $f^l_{\alpha,n} v^{\mathbf{0}}_{k,n} \in \mathbf{C}v^{\mathbf{0}}_{k,n}$. Therefore

$$f^l_{\alpha,n} v^{\mathbf{0}}_{k,n} = p_{\mathbf{0}}(f^l_{\alpha,n} v^{\mathbf{0}}_{k,n}) = 0\,.$$

□

Now, we shall construct nilpotent $U_\varepsilon(B_n)$-modules (see §3). For $\lambda \in \mathbf{C}^{n-k+1}$, let $L_{k,n}(\lambda)$ be the $U_\varepsilon(B_n)$-submodule of $V_{k,n}(\lambda)$ generated by $v^{\mathbf{0}}_{k,n}$.

THEOREM 6.5. *For any* $k \in I$, $\lambda = (\lambda_k, \ldots, \lambda_n) \in \mathbf{Z}_l^{n-k+1}$, *as a* $U_\varepsilon(B_n)$*-module,* $L_{k,n}(\lambda)$ *is isomorphic to* $L^{nil}_\varepsilon(0, \ldots, 0, \lambda_k, \ldots, \lambda_n)$.

PROOF. According to Proposition 6.1 and Lemma 6.2, $L_{k,n}(\lambda)$ is the highest-weight $U_\varepsilon(B_n)$-module with highest weight $(0, \ldots, 0, \lambda_k, \ldots, \lambda_n)$. On the other hand, from Lemma 6.4, $f^l_{\alpha,n} v^{\mathbf{0}}_{k,n} = 0$ for all $\alpha \in \Delta_+$. Moreover, from Proposition 2.4 and 6.1, $e^l_{\alpha,n} v^{\mathbf{0}}_{k,n} = 0$ for all $\alpha \in \Delta_+$. Hence, from Proposition 2.3, $e^l_{\alpha,n} = f^l_{\alpha,n} = 0$ on $L_{k,n}(\lambda)$ for all $\alpha \in \Delta_+$. Thus, $L_{k,n}(\lambda)$ is a nilpotent $U_\varepsilon(B_n)$-module. Therefore, the theorem follows from Proposition 6.1 and Proposition 3.5. □

In particular, if $k = 1$ we obtain all finite-dimensional irreducible nilpotent $U_\varepsilon(B_n)$-modules of type 1. This case has already been proved in [2].

7. Inductive construction of $L_\varepsilon^{\mathrm{nil}}(\lambda)$: the other cases

In this section, we inductively construct $U_\varepsilon(\mathfrak{g}_n)$-modules $L_\varepsilon^{\mathrm{nil}}(\lambda)$ in the case of $\mathfrak{g}_n = A_n$, C_n, or D_n by using the Schnizer homomorphisms in Theorem 4.1(a), (c), (d).

We define $a_n^{(0)} = (a_{i,n}^{(0)})_{i=1}^n$, $\tilde{a}_n^{(0)} = (\tilde{a}_{i,n}^{(0)})_{i=1}^n$, $b_n^{(0)} = (b_{i,n}^{(0)})_{i=1}^n$, $\tilde{b}_n^{(0)} = (\tilde{b}_{i,n}^{(0)})_{i=1}^n \in \mathbf{C}^n$ whereby,

$$
\begin{aligned}
&a_{i,n}^{(0)} := \tilde{a}_{i,n}^{(0)} := 1 \quad (\mathfrak{g}_n = A_n, C_n, D_n)\,,\\
&b_{i,n}^{(0)} := i \quad (\mathfrak{g}_n = A_n)\,, \quad b_{i,n}^{(0)} := n - i + 1 \quad (\mathfrak{g}_n = C_n)\,,\\
&b_{i,n}^{(0)} := n - i + 1 \quad (i \neq 1)\,, \quad b_{1,n}^{(0)} := n - 1 \quad (n \neq 1)\,, \quad b_{1,1}^{(0)} := 1 \quad (\mathfrak{g}_n = D_n)\,,\\
&\tilde{b}_{i,n}^{(0)} := n + i - 1 \quad (\mathfrak{g}_n = C_n, D_n)\,.
\end{aligned}
$$

Let $k \in I$. For $\lambda = (\lambda_k, \ldots, \lambda_n) \in \mathbf{C}^{n-k+1}$, we define $\nu^\lambda = (\nu_k^\lambda, \ldots, \nu_n^\lambda) \in \mathbf{C}^{n-k+1}$ as

$$
\begin{aligned}
&\nu_i^\lambda := -i - 1 - \frac{1}{i}\sum_{j=k}^{i}(j\lambda_j) \quad (\mathfrak{g}_n = A_n)\,, \quad \nu_i^\lambda := -2i - \sum_{j=k}^{i}\lambda_j \quad (\mathfrak{g}_n = C_n)\,,\\
&\nu_i^\lambda := -2i + 2 - \sum_{j=k}^{i}\lambda_j \quad (k \geq 3)\,, \quad \nu_i^\lambda := -2i + 3 - \frac{1}{2}\lambda_2 - \sum_{j=3}^{i}\lambda_j \quad (k = 2)\,,\\
&\nu_i^\lambda := -2i + 1 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 - \sum_{j=3}^{i}\lambda_j \quad (k = 1)\,, \quad (\mathfrak{g}_n = D_n)\,,
\end{aligned}
$$

where $k \leq i \leq n$. For $\lambda \in \mathbf{C}$, we set $\rho_n^A(\lambda) := \rho_n^A(a_n^{(0)}, b_n^{(0)}\lambda)$, $\rho_n^C(\lambda) := \rho_n^C(a_n^{(0)}, \tilde{a}_{n-1}^{(0)}, b_n^{(0)}, \tilde{b}_{n-1}^{(0)}, \lambda)$, and $\rho_n^D(\lambda) := \rho_n^D(a_n^{(0)}, \tilde{a}_{n-2}^{(0)}, b_n^{(0)}, \tilde{b}_{n-2}^{(0)}, \lambda)$ (see Theorem 4.1(a), (c), (d)). We define

$$
\begin{aligned}
&V_{k,n} := \bigotimes_{j=k}^{n} V_j \quad (\mathfrak{g}_n = A_n)\,, \quad V_{k,n} := \bigotimes_{j=k}^{n}(V_j \otimes \widetilde{V}_{j-1}) \quad (\mathfrak{g}_n = C_n)\,,\\
&V_{k,n} := \bigotimes_{j=k}^{n}(V_j \otimes \widetilde{V}_{j-2}) \quad (\mathfrak{g}_n = D_n)\,.
\end{aligned}
$$

For $\lambda = (\lambda_k, \ldots, \lambda_n) \in \mathbf{C}^{n-k+1}$, we define $U_\varepsilon(\mathfrak{g}_n)$-representations $\phi_{k,n} := \phi_{k,n}(\lambda) : U_\varepsilon(\mathfrak{g}_n) \to \mathrm{End}(V_{k,n})$ as

$$
\phi_{k,n}(\lambda) := \pi_{k-1} \circ \rho_k^{\mathfrak{g}}(\nu_k^\lambda) \circ \cdots \circ \rho_n^{\mathfrak{g}}(\nu_n^\lambda)\,.
$$

We denote the $U_\varepsilon(\mathfrak{g}_n)$-module associated with $(\phi_{k,n}(\lambda), V_{k,n})$ by $V_{k,n}(\lambda)$. We set

$$v_{k,n}^{\mathbf{0}} := \bigotimes_{j=k}^{n} v_j(0,\dots,0) \quad (\mathfrak{g}_n = A_n),$$

$$v_{k,n}^{\mathbf{0}} := \bigotimes_{j=k}^{n} (v_j(0,\dots,0) \otimes \tilde{v}_{j-1}(0,\dots,0)) \quad (\mathfrak{g}_n = C_n),$$

$$v_{k,n}^{\mathbf{0}} := \bigotimes_{j=k}^{n} (v_j(0,\dots,0) \otimes \tilde{v}_{j-2}(0,\dots,0)) \quad (\mathfrak{g}_n = D_n).$$

For $\lambda \in \mathbf{C}^{n-k+1}$, let $L_{k,n}(\lambda)$ be the $U_\varepsilon(\mathfrak{g}_n)$-submodule of $V_{k,n}(\lambda)$ generated by $v_{k,n}^{\mathbf{0}}$. Then we can prove the following theorem in a similar way to that of §6.

THEOREM 7.1. *Let $\mathfrak{g}_n = A_n$, C_n, or D_n. Then, for any $k \in I$, $\lambda = (\lambda_k, \dots, \lambda_n) \in \mathbf{Z}_l^{n-k+1}$, $L_{k,n}(\lambda)$ is isomorphic to $L_\varepsilon^{nil}(0,\dots,0,\lambda_k,\dots,\lambda_n)$ as a $U_\varepsilon(\mathfrak{g}_n)$-module. Moreover, for any $\lambda \in \mathbf{C}^{n-k+1}$, we have $P(V_{k,n}(\lambda)) = \mathbf{C}v_{k,n}^{\mathbf{0}}$.*

In particular, if $k = 1$ we obtain all finite-dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g}_n)$-modules of type 1 if $\mathfrak{g} = A_n$, C_n, or D_n. This case has already been proved in [2].

Consequently, we inductively obtain all finite-dimensional irreducible nilpotent modules of type 1 by using the Schnizer homomorphisms for quantum algebras at roots of unity of types A_n, B_n, C_n, D_n, or G_2.

REMARK 7.2. We expect that the Schnizer modules are also nilpotent.

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Present Address:
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
THE UNIVERSITY OF TOKYO,
KOMABA, MEGURO-KU, TOKYO, 153–8914 JAPAN.
e-mail: abeyuuki@ms.u-tokyo.ac.jp