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The Partial Vanishing of Victoria's Cohomology of Euclidean Superspaces

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1. Introduction

The notion of supermanifolds is introduced by Berezin–Leites([3]), but we should go back to Berezin's work on the mathematical formulation of the second quantization ([1]) when we talk about the origin of supermanifold theory. Berezin constructed Hamiltonians of relativistic (and also nonrelativistic) spin systems. Formulating spin systems is probably the first motivation of supermanifolds, and supermanifold theory is also used in describing supersymmetry with "superspace formulation" in quantum field theory (*cf.* [4]) in the present. We do not give the definition of supermanifolds in this paper. For the definition and elementary properties of supermanifolds, see [2], [3], [5].

All the results of this paper was announced in [6] without proof, and the purpose of this paper is to show the proofs of these results.

This paper and [6] are works on Victoria's cohomology ([7]) of Euclidean superspace (for the definition see [5]). Although we do not give the definition of Victoria's cohomology concretely, we review Victoria's cohomology shortly in the following.

We denote by $\mathbf{R}^{m|n}$ the $m|n$–dimensional Euclidean superspace. For a supermanifold M, we denote by $C^\infty(M)$ the superalgebra of superfunctions on M, and denote by $\mathfrak{X}(M)$ the $C^\infty(M)$–supermodule of vector fields on M (*i.e.*, $\mathfrak{X}(M)$ is the supermodule of superderivations on $C^\infty(M)$).

By definition, $\Omega^{1|0}(M) = \mathfrak{X}(M)^*$ ($\mathfrak{X}(M)^*$ denotes the $C^\infty(M)$–dual supermodule of $\mathfrak{X}(M)$) and $\Omega^{0|1}(M) = \Pi\mathfrak{X}(M)^*$ (Π is the parity changing functor).

Let $(x_1, \ldots, x_{m+n}) = (u_1, \ldots, u_m, \xi_1, \ldots, \xi_n)$ denote a coordinate system of $\mathbf{R}^{m|n}$. $\mathfrak{X}(\mathbf{R}^{m|n})$ is a free $C^\infty(\mathbf{R}^{m|n})$–supermodule and $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{m+n}}$ form a basis of $\mathfrak{X}(\mathbf{R}^{m|n})$. If we denote $\mathrm{d}^0 x_i := \left(\frac{\partial}{\partial x_i}\right)^*$ and $\mathrm{d}^1 x_i := \Pi\left(\frac{\partial}{\partial x_i}\right)^*$, clearly $\mathrm{d}^0 x_i, \ldots, \mathrm{d}^0 x_{m+n}$ is a $C^\infty(\mathbf{R}^{m|n})$–basis of $\Omega^{1|0}(\mathbf{R}^{m|n})$ and $\mathrm{d}^1 x_1, \ldots, \mathrm{d}^1 x_{m+n}$ is a $C^\infty(\mathbf{R}^{m|n})$–basis of $\Omega^{0|1}(\mathbf{R}^{m|n})$.

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Define

$$\mathrm{d}^0 x_{i_1} \cdots \mathrm{d}^p x_{i_p} \mathrm{d}^1 x_{i_{p+1}} \cdots \mathrm{d}^1 x_{i_{p+q}} :$$
$$= \sum_{\sigma \in \mathfrak{S}_{p+q}} \frac{1}{(p+q)!} \mathrm{sgn}(\sigma) \mathrm{Sgn}_{p|q}(\sigma)$$
$$\mathrm{d}^0 x_{i_{\sigma(1)}} \otimes \cdots \mathrm{d}^0 x_{i_{\sigma(p)}} \otimes \mathrm{d}^1 x_{i_{\sigma(p+1)}} \cdots \mathrm{d}^1 x_{i_{\sigma(p+q)}} . \tag{1}$$

Here $\mathrm{Sgn}_{p|q}$ denotes the supersignature defined by Victoria (see [8]) while sgn denotes the ordinary signature, and (1) is tensored as a $C^\infty(\mathbf{R}^{m|n})$–supermodule. The $C^\infty(\mathbf{R}^{m|n})$–supermodule generated by

$$\{\mathrm{d}^0 x_{i_1} \cdots \mathrm{d}^0 x_{i_p} \mathrm{d}^1 x_{i_{p+1}} \cdots \mathrm{d}^1 x_{i_{p+q}} \neq 0\, ;\, 1 \leq i_1 \leq \cdots \leq i_{p+q} \leq m+n\}$$

is denoted by $\Omega^{p|q}(M)$.

Victoria's complex is a double complex and has two types differentials. D_0 is the differential of the complex

$$0 \to \Omega^{0|q}(M) \overset{D_0}{\to} \Omega^{1|q}(M) \overset{D_0}{\to} \Omega^{2|q}(M) \to \cdots \tag{2}$$

and D_1 is the differential of the complex

$$0 \to \Omega^{p|0}(M) \overset{D_1}{\to} \Omega^{p|1}(M) \overset{D_1}{\to} \Omega^{p|2}(M) \to \cdots \tag{3}$$

We denote by $\mathrm{H}^{p|q}_{D_0}(M)$ the $p|q$–cohomology group of the complex (2), and by $\mathrm{H}^{p|q}_{D_1}(M)$ for the $p|q$–cohomology group of the complex (3).

Victoria([7]) claimed that

CLAIM 1.

$$\mathrm{H}^{p|q}_{D_0}(\mathbf{R}^{m|n}) \cong \bigoplus_{q_1+q_2=q} \mathrm{H}^{0|q_1}_{D_0}(\mathbf{R}^{m|0}) \otimes \mathrm{H}^{p|q_2}_{D_0}(\mathbf{R}^{0|n})$$
$$\mathrm{H}^{p|q}_{D_1}(\mathbf{R}^{m|n}) \cong \bigoplus_{p_1+p_2=p} \mathrm{H}^{p_1|q}_{D_1}(\mathbf{R}^{m|0}) \otimes \mathrm{H}^{p_2|0}_{D_1}(\mathbf{R}^{0|n}) ,$$

and this claim can be regarded as a partial result of the super version of Künneth formula (Indeed, it is not so difficult to generalize Claim 1 to Künneth formula in general case). However, he intended to prove Claim 1 with incorrect argument, so we do not have a proof for Claim 1 now.

Instead of Claim 1, we prove

THEOREM 1.

$$\mathrm{H}^{p|q}_{D_0}(\mathbf{R}^{m|n}) = 0 \quad \textit{if} \;\; p \neq 0$$
$$\mathrm{H}^{p|q}_{D_1}(\mathbf{R}^{m|n}) = 0 \quad \textit{if} \;\; q \neq 0 .$$

with another approach. We can check easily Theorem 1 is contained in Claim 1, but we do not know for certain Claim 1 holds.

This paper is separated to two parts. Sec. 2 is assigned to making clear which step is wrong in Victoria's argument for Claim 1 by giving an example, and we prove Theorem 1 in Sec. 3.

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2. Künneth formula for Victoria's cohomology

We first review Victoria's plan to prove Claim 1. He defined new complexes

$$0 \to \bar{\Omega}^{0|q}(\mathbf{R}^{m|n}) \xrightarrow{\overline{D_0}} \bar{\Omega}^{1|q}(\mathbf{R}^{m|n}) \xrightarrow{\overline{D_0}} \bar{\Omega}^{2|q}(\mathbf{R}^{m|n}) \to \cdots$$

and

$$0 \to \bar{\Omega}^{p|0}(\mathbf{R}^{m|n}) \xrightarrow{\overline{D_1}} \bar{\Omega}^{p|1}(\mathbf{R}^{m|n}) \xrightarrow{\overline{D_1}} \bar{\Omega}^{p|2}(\mathbf{R}^{m|n}) \to \dots ,$$

derived from

$$0 \to \Omega^{0|q}(\mathbf{R}^{m|n}) \xrightarrow{D_0} \Omega^{1|q}(\mathbf{R}^{m|n}) \xrightarrow{D_0} \Omega^{2|q}(\mathbf{R}^{m|n}) \to \cdots$$

and

$$0 \to \Omega^{p|0}(\mathbf{R}^{m|n}) \xrightarrow{D_1} \Omega^{p|1}(\mathbf{R}^{m|n}) \xrightarrow{D_1} \Omega^{p|2}(\mathbf{R}^{m|n}) \to \cdots$$

respectively, by

$$\bar{\Omega}^{p|q}(\mathbf{R}^{m|n}) := \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} \Omega^{p_1|q_1}(\mathbf{R}^{m|0}) \otimes \Omega^{p_2|q_2}(\mathbf{R}^{0|n})$$

$$\overline{D_i} := \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} \Big(D_i|_{\Omega^{p_1|q_1}(\mathbf{R}^{m|0})} \otimes \mathrm{id}_{\Omega^{p_2|q_2}(\mathbf{R}^{0|n})} + \mathrm{id}_{\Omega^{p_1|q_1}(\mathbf{R}^{m|0})} \otimes D_i|_{\Omega^{p_2|q_2}(\mathbf{R}^{0|n})} \Big)$$

Furthermore, he defined

$$\sigma_{p|q} : \Omega^{p|q}(\mathbf{R}^{m|n}) \to \bar{\Omega}^{p|q}(\mathbf{R}^{m|n}) ,$$

and claimed that σ is a cochain isomorphism, *i.e.*, the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega^{p|q}(\mathbf{R}^{m|n}) & \xrightarrow{D_0} & \Omega^{p+1|q}(\mathbf{R}^{m|n}) & \longrightarrow & \cdots \\ & & {\scriptstyle\sigma_{p|q}}\downarrow & & \downarrow{\scriptstyle\sigma_{p+1|q}} & & \\ \cdots & \longrightarrow & \bar{\Omega}^{p|q}(\mathbf{R}^{m|n}) & \xrightarrow{\overline{D_0}} & \bar{\Omega}^{p+1|q}(\mathbf{R}^{m|n}) & \longrightarrow & \cdots \end{array} \tag{4}$$

and the diagram

$$
(5)\qquad
\begin{array}{ccccccc}
\cdots & \longrightarrow & \Omega^{p|q}(\mathbf{R}^{m|n}) & \xrightarrow{D_1} & \Omega^{p|q+1}(\mathbf{R}^{m|n}) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \sigma_{p|q}} & & \Big\downarrow{\scriptstyle \sigma_{p|q+1}} & & \\
\cdots & \longrightarrow & \bar{\Omega}^{p|q}(\mathbf{R}^{m|n}) & \xrightarrow{\overline{D_1}} & \bar{\Omega}^{p|q+1}(\mathbf{R}^{m|n}) & \longrightarrow & \cdots
\end{array}
$$

commute and $\sigma_{p|q}$ are invertible for all p, q. Although the claim can be proven by the argument using a double complex if the diagrams (4) and (5) commute, neither commutes. We can check it by direct computations.

EXAMPLE 1. We see Victoria's cohomology of $\mathbf{R}^{1|1}$. Let (u, ξ) be a coordinate system of $\mathbf{R}^{1|1}$ (u is an even coordinate and ξ is an odd coordinate). Any $\alpha \in \Omega^{0|1}(\mathbf{R}^{1|1})$ can be uniquely written in the form

$$\alpha = \mathrm{d}^1 u(f_{11} + f_{12}\xi) + \mathrm{d}^1\xi(f_{21} + f_{22}\xi)$$

with $f_{11}, f_{12}, f_{21}, f_{22} \in C^\infty(\mathbf{R})$.

We check the diagram 5 does not commute by direct computation using the above notation. We have

$$
(6)\qquad
\begin{aligned}
&\overline{D_0} \circ \sigma_{0|1}(\alpha) \\
&= \mathrm{d}^1 u f_{12} \otimes \mathrm{d}^0\xi + \mathrm{d}^0 u \frac{\mathrm{d} f_{21}}{\mathrm{d}u} \otimes \mathrm{d}^1\xi + \mathrm{d}^0 u \frac{\mathrm{d} f_{22}}{\mathrm{d}u} \otimes \mathrm{d}^1\xi \cdot \xi + f_{22} \otimes \mathrm{d}^0\xi \mathrm{d}^1\xi
\end{aligned}
$$

and

$$
(7)\qquad
\begin{aligned}
&\sigma_{1|1} \circ \overline{D_0}(\alpha) \\
&= \mathrm{d}^0 u \left(-\frac{\mathrm{d} f_{21}}{\mathrm{d}u} + f_{12}\right) \otimes \mathrm{d}^1\xi + \mathrm{d}^0 u \frac{\mathrm{d} f_{22}}{\mathrm{d}u} \otimes \mathrm{d}^1\xi \cdot \xi + f_{22} \otimes \mathrm{d}^0\xi \mathrm{d}^1\xi\,.
\end{aligned}
$$

(6), (7) imply (5) does not commute in the $p = 0, q = 1$ case.

3. A homotopy operator and the super version of Poincaré lemma

We have been able to determine Victoria's cohomology groups of Euclidean superspaces in all degrees, while de Rham cohomology groups of ordinary Euclidean space is completely determined. However, we can prove Theorem 1 as a partial result, by constructing a homotopy operator. We show the proof of Theorem 1 in this section.

In order to explain the situation, we review shortly Poincaré lemma in the commutative case. Let $\pi : \mathbf{R}^{m+1} \to \mathbf{R}^m$ be a projection and $s : \mathbf{R}^m \to \mathbf{R}^{m+1}$ be the embedding which satisfy $\pi \circ s = \mathrm{id}_{\mathbf{R}^m}$. Let $(s \circ \pi)^*$ denote the pull-back homomorphism induced from $s \circ \pi$.

The following is Poincaré lemma for ordinary Euclidean spaces:

PROPOSITION 1.

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbf{R}^m) \cong \mathbf{R}$$

$$\mathrm{H}^p_{\mathrm{dR}}(\mathbf{R}^m) = 0 \quad \textit{if } p \neq 0 \, .$$

In order to prove Proposition 1 we construct a homomorphism $K : \Omega(\mathbf{R}^{m+1}) \to \Omega(\mathbf{R}^{m+1})[-1]$ and a chain homotopy between $(s \circ \pi)^* : \Omega^p(\mathbf{R}^{m+1}) \to \Omega^p(\mathbf{R}^{m+1})$ and the identity satisfying the following equation.

$$d \circ K \pm K \circ d = (s \circ \pi)^* - \mathrm{id}_{\Omega^{\cdot}_{\mathrm{dR}}(\mathbf{R}^{m+1})} \, .$$

Here d is the differential of the de Rham complex.

We can also prove the Victoria's cohomology is vanished in some degrees.

1. Defining projections and embeddings which corresponds to π and s respectively.
2. Constructing homotopy operator like K as above.

We define the following projections and injections between Euclidean superspaces.

$$\pi_0 : \mathbf{R}^{m+1|n} \to \mathbf{R}^{m|n} \quad \pi_1 : \mathbf{R}^{m|n+1} \to \mathbf{R}^{m|n}$$
$$s_0 : \mathbf{R}^{m|n} \to \mathbf{R}^{m+1|n} \quad s_1 : \mathbf{R}^{m|n} \to \mathbf{R}^{m|n+1} \, .$$

Furhtermore, since Victoria's complex is bi-graded, we need to define four types of homotopy operators. Here we investigate only in the case of D_0 cohomology because it is quite parallel to this in the case of D_1–cohomology.

Let $(t, x_1, \ldots, x_{m+n})$ be the canonical coordinate system of $\mathbf{R}^{m+1|n}$ and $(y_1, \ldots, y_{m+n}, \eta)$ be the canoinical coordinate system of $\mathbf{R}^{m|n+1}$. The definition of π_i and s_i $(i = 0, 1)$ is given by

$$\pi_0^{\#}(x_i) := x_i \quad \pi_1^{\#}(y_i) := y_i$$
$$s_0^{\#}(t) := 0 \quad s_0^{\#}(x_i) := x_i$$
$$s_1^{\#}(\eta) := 0 \quad s_1^{\#}(y_i) := y_i \, .$$

The following result ensures the above relation defines morphisms π_i and s_i.

PROPOSITION 2 ([5]). *Let $U = (U_{\mathrm{rd}}, \mathcal{O}_U)$, $V = (V_{\mathrm{rd}}, \mathcal{O}_V)$ be superdomains, and $(y_1, \ldots, y_{m+n})$ be a coordinate system of V. Let $h : C^\infty(V) \longrightarrow C^\infty(U)$ be a superalgebra homomorphism satisfying*

$$(h(y_1)_{\mathrm{rd}}, \ldots, h(y_m)_{\mathrm{rd}}) \in U_{\mathrm{rd}} \, .$$

Then, there exists the unique morphism $(\varphi_{\mathrm{rd}}, \varphi^{\#}) : U \to V$ satisfying $\varphi^{\#}(y_i) = h(y_i)$.

A homotopy operator $K_0 : \Omega^{\cdot|q}(\mathbf{R}^{m+1|n}) \to \Omega^{\cdot|q}(\mathbf{R}^{m+1|n})[-1]$ between $(s_0 \circ \pi_0)^*$ and $\mathrm{id}_{\Omega^{\cdot|q}(\mathbf{R}^{m+1|n})}$ is defined by

$$K_0(\mathrm{d}^{a_1}x_{i_1} \cdots \mathrm{d}^{a_{p+q+1}}x_{i_{p+q+1}}) := 0$$
$$K_0(\mathrm{d}^0 t \mathrm{d}^{a_1}x_{i_1} \cdots \mathrm{d}^{a_{p+q}}x_{i_{p+q}} f) := \mathrm{d}^{a_1}x_{i_1} \cdots \mathrm{d}^{a_{p+q}}x_{i_{p+q}} \int_0^t f \, \mathrm{d}t$$

for $f \in C^{\infty}(\mathbf{R}^{m|n})$. It is the crucial point for our proof that we can carry dt to the top by the sign rule.

Similarly, we can check that $(s_1 \circ \pi_1)^*$ and $\mathrm{id}_{\Omega^{\cdot|q}(\mathbf{R}^{m+1|n})}$ are cochain homotopic, by giving $K_1 : \Omega^{\cdot|q}(\mathbf{R}^{m|n+1}) \to \Omega^{\cdot|q}(\mathbf{R}^{m|n+1})[-1]$ as

$$K_1(\mathrm{d}^{a_1}y_{i_1} \cdots \mathrm{d}^{a_{p+q+1}}y_{i_{p+q+1}}) := 0$$

$$K_1(\mathrm{d}^0\eta\mathrm{d}^{a_1}x_{i_1} \cdots \mathrm{d}^{a_{p+q}}x_{i_{p+q}} f) := \mathrm{d}^{a_1}x_{i_1} \cdots \mathrm{d}^{a_{p+q}}x_{i_{p+q}} f\eta\,.$$

This completes the proof of Theorem 1 for D_0–cohomology, and we can prove Theorem 1 for D_1–cohomology in a quite similar way.

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