A Note on Finite Simple Groups with Abelian Sylow $p$-subgroups

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Abstract. In this note, we will make a remark on finite simple groups with abelian Sylow $p$-subgroups using the Classification Theorem of the Finite Simple Groups.

1. Introduction

In our paper Sawabe-Watanabe [6], we verified the Alperin’s weight conjecture [1] for the principal block of a finite group $X$ with an abelian Sylow $p$-subgroup $P$ under the hypothesis $(H_1)$ that $|N_X(P)/C_X(P)| = r$ for a prime $r$. Our method in [6] is as follows. We first reduce the conjecture under $(H_1)$ to that of finite simple groups, and next try to obtain the result [6, Proposition 6.4]; which is saying that, under $(H_1)$ and $X$ is simple, $P$ must be cyclic, $P \cong C_2 \times C_2$, or $X \cong PSL(2, p^e)$ for $p = 2, 3$. As the conjecture is known to be true in those three cases, we could conclude that the conjecture under $(H_1)$ is verified. Note that to prove [6, Proposition 6.4], we used the Classification Theorem of the finite simple groups. On the other hand, in August 2002, the author was informed by Watanabe[8] that the conjecture for the principal block of a finite group $X$ with an abelian Sylow $p$-subgroup $P$, under the another hypothesis $(H_2)$ that $|N_X(P)/C_X(P)| = r^2$ for a prime $r$, can be also reduced to that of finite simple groups. So it is a frequent occurrence in modular representation theory that a problem on finite groups having abelian Sylow $p$-subgroups is reduced to that of finite simple groups. So it is quite valuable to investigate, in general, finite simple groups with abelian Sylow $p$-subgroups. From this reason, the purpose of this note is to prove the following:

**Theorem 1.** Let $X$ be a finite simple group with an abelian Sylow $p$-subgroup $P$. Then one of the following holds.

1. $N_X(P)/C_X(P)$ contains an involution.
2. $P$ is cyclic.
3. $P \cong C_2 \times C_2$.
4. $X \cong PSL(2, p^e)$.
5. \( X \cong J_1 \) or \( 2G_2(3^{2m+1}) \) with \( p = 2 \), and \( N_X(P)/C_X(P) \cong (7 : 3) \).

The following are immediate consequences of Theorem 1.

**Corollary 1.** Let \( X \) be a finite simple group with an abelian Sylow \( p \)-subgroup \( P \). Suppose that \( |N_X(P)/C_X(P)| \) is a prime. Then one of the following holds.
1. \( P \) is cyclic.
2. \( P \cong C_2 \times C_2 \).
3. \( X \cong PSL(2, p^e) \) for \( p = 2 \) or 3.

**Proof.** Set \( \mathcal{E}_X(P) := N_X(P)/C_X(P) \). Suppose that \( \mathcal{E}_X(P) \) contains an involution, then \( \mathcal{E}_X(P) \cong C_2 \). It follows that \( P \) is cyclic by Smith-Tyner\([7]\). Suppose next that \( X \cong PSL(2, p^e) \) with \( p \) odd, then \( |\mathcal{E}_X(P)| = \frac{1}{2}(p^e - 1) \). If \( p \geq 5 \) then \( \frac{1}{2}(p - 1) \neq 1 \) and \( p^{e-1} + \cdots + p + 1 = 1 \). This implies that \( e = 1 \), and thus \( P \) is cyclic. \( \square \)

**Corollary 2.** Let \( X \) be a finite simple group with an abelian Sylow \( p \)-subgroup \( P \). Suppose that \( |N_X(P)/C_X(P)| = r^2 \) for a prime \( r \). Then one of the following holds.
1. \( N_X(P)/C_X(P) \cong C_4 \) or \( C_2 \times C_2 \).
2. \( P \) is cyclic.
3. \( X \cong PSL(2, 3^e) \).

**Proof.** Set \( \mathcal{E}_X(P) := N_X(P)/C_X(P) \). Suppose that \( P \cong C_2 \times C_2 \), then \( \mathcal{E}_X(P) \) is a subgroup of \( S_3 \); but this is impossible. Suppose next that \( X \cong PSL(2, 2^e) \), then \( r^2 = |\mathcal{E}_X(P)| = 2^e - 1 \). Note that \( e \geq 2 \) as \( r \neq 1 \). Now let \( r = 2k + 1 \) then we have that \( 2^e = 4k^2 + 4k + 2 \), a contradiction. Finally suppose that \( X \cong PSL(2, p^e) \) with odd, then \( r^2 = |\mathcal{E}_X(P)| = \frac{1}{2}(p^e - 1) \). If \( p \geq 5 \) then \( p - 1 = 2n \) for \( n \geq 2 \), and \( p^e - 1 = 2nm \) where \( m := p^{e-1} + \cdots + p + 1 \). Suppose further that \( e \neq 1 \). Then \( m \neq 1 \) and \( 2r^2 = p^e - 1 = 2nm \). Thus \( m = n = r \). But it follows that \( p - 1 = 2n = 2m \geq 2(p + 1) \), a contradiction. Therefore we have that if \( p \geq 5 \) then \( e = 1 \); namely \( P \) is cyclic. \( \square \)

Note that the result \([6, \text{Proposition 6.4}]\) mentioned above is exactly Corollary 1, so our result Theorem 1 contains one of the main parts of \([6]\). Furthermore as indicated earlier, the Alperin’s weight conjecture for the principal block of a finite group \( X \) with an abelian Sylow \( p \)-subgroup \( P \) under the hypothesis \((H_2)\) is reduced to that of finite simple groups. So Corollary 2 tells us that, to verify the conjecture under \((H_2)\), it is enough to consider the only three cases described in it.

2. Preliminaries

Throughout this note, denote by \( \pi(G) \) the set of primes dividing the order \( |G| \) of a finite group \( G \), and by \( C_n \) the cyclic group of order \( n \). Furthermore, for a subgroup \( H \) of \( G \), we set the factor group \( \mathcal{E}_G(H) := N_G(H)/C_G(H) \) called the automizer of \( H \) in \( G \). First we prepare the following proposition; which will be used later repeatedly. Although this is shown in \([6]\), we will give a sketch of the proof.
PROPOSITION 1. Let $G$ be a finite group with an abelian Sylow $p$-subgroup $P$.

1. If $Q$ is a subgroup of $P$, then $E_G(Q)$ is involved in $E_G(P)$; that is, there exist a subgroup $M$ of $E_G(P)$ and a normal subgroup $N$ of $M$ such that $E_G(Q) \cong M/N$. In particular $|E_G(Q)|$ divides $|E_G(P)|$.

2. If $H$ is an involved group in $G$ with $p \in \pi(H)$, and $R$ is a Sylow $p$-subgroup of $H$, then $E_H(R)$ is involved in $E_G(P)$. In particular $|E_H(R)|$ divides $|E_G(P)|$.

PROOF. (1) As $P$ is abelian, $P \leq C_G(Q)$. For any $n \in N_G(Q)$, we have that $P^n \leq C_G(Q)^n = C_G(Q) \geq P$, and that there exists $c \in C_G(Q)$ such that $P^{nc^{-1}} = P$. It follows that $N_G(Q) \leq N_G(P)C_G(Q)$, and $N_G(Q) = (N_G(Q) \cap N_G(P))C_G(Q)$ by Modular law. Thus

$$E_G(Q) \cong N_G(Q) \cap N_G(P)/C_G(Q) \cap N_G(P),$$

and which shows that $E_G(Q)$ is a homomorphic image of a subgroup $N_G(Q)\cap N_G(P)/C_G(Q)\cap N_G(P)$ of $E_G(P)$. Therefore $E_G(Q)$ is involved in $E_G(P)$.

(2) Let $N \trianglelefteq H_1$ be subgroups of $G$ such that $H = H_1/N = \overline{H_1}$, and let $Q \in Syl_p(H_1)$ such that $R = QN/N = \overline{Q}$. Then there are natural surjective homomorphisms from $E_{H_1}(Q)$ to $N_{H_1}(Q)/C_{H_1}(Q)$, and from $N_{H_1}(Q)/C_{H_1}(Q)$ to $E_{\overline{H_1}}(\overline{Q}) = E_H(R)$. On the other hand, since $E_G(Q)$ is involved in $E_G(P)$ by (1), and since $E_G(Q)$ possesses a subgroup $N_{H_1}(Q)C_G(Q)/C_G(Q) \cong E_{H_1}(Q)$, we have that $E_{H_1}(Q) \cong L/K$ for some $K \trianglelefteq L \leq E_G(P)$. This implies that there exist surjective homomorphisms $L \twoheadrightarrow E_{H_1}(Q) \twoheadrightarrow E_{H_1}(R)$. Therefore $E_{H_1}(R)$ is involved in $E_G(P)$.

LEMMA 1. Let $G$ be a finite group, and $P$ a $p$-subgroup of $G$ with $p \not\in \pi(Z(G))$. Then $E_{\overline{G}}(\overline{P}) \cong E_{\overline{G}}(\overline{P})$ where $\overline{G} = G/Z(G)$.

PROOF. Straightforward.

3. Alternating groups and sporadic groups

PROPOSITION 2. Let $X$ be the alternating group $A_n$ ($n \geq 5$) with an abelian Sylow $p$-subgroup $P$. Then either $E_X(P)$ contains an involution or $P$ is cyclic; except for $X = A_5 \cong PSL(2, 4)$ and $p = 2$, and in which case $E_{A_5}(P) \cong C_3$ and $P \cong C_2 \times C_2$.

PROOF. If $p = 2$ then, since $P$ is abelian, we have that $X = A_5$ and $E_{A_5}(P) \cong C_3$. Now we may assume that $p \geq 3$, and express $n$ as $pk + h$ ($k \in \mathbb{N}, 0 \leq h \leq p - 1$). If $k = 1$ then $P$ is cyclic. Thus we may also assume that $k \geq 2$. Now we can use at least $2p$ letters $i^{(1)}, \ldots, i^{(2)}$. For $d = 1, 2$, let $Q_d := \langle (i^{(d)}) \rangle \leq X$ and $Q := Q_1 \times Q_2$. Up to conjugacy, we may assume that $Q \leq P$. Furthermore let

$$\alpha_d := (i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)})(i^{(d)}),$$

a permutation on $\{i^{(d)}, \ldots, i^{(d)}\}$ where $r := \frac{1}{2}(p - 1) \geq 1$ as $p \geq 3$. Notice that $\alpha_d$ normalizes $Q_d$ but not centralize $Q_d$. Then an even permutation $\alpha_1\alpha_2$ is an involution lying in $E_X(Q)$.
But since $|E_X(Q)|$ divides $|E_X(P)|$ by Proposition 1(1), we have that $|E_X(P)|$ is even. The proof is complete. □

REMARK. Even if $P$ is cyclic, $E_X(P)$ does not necessarily contain an involution. Indeed, if $p = 2r + 1$ is an odd prime then for $C_p \cong P \leq Syl_p(A_p)$, $E_{Ap}(P)$ is of order $r$. Thus if $r$ is odd then so is $|E_{Ap}(P)|$.

PROPOSITION 3. Let $X$ be a sporadic simple group with an abelian Sylow $p$-subgroup $P$. Then either $E_X(P)$ contains an involution or $P$ is cyclic; except for the first Janko group $X = J_1$ and $p = 2$, and in which case $E_{J_1}(P) \cong (7 : 3)$.

PROOF. See for example [3] or [5, Section 5].

For later use, we prepare the following on the symmetric groups.

PROPOSITION 4. Let $X$ be the symmetric group $S_n$ $(n \geq 3)$ with an abelian Sylow $p$-subgroup $P$ with an odd prime $p$. Then $E_X(P)$ contains an involution.

PROOF. As $p$ is odd, we can write $p$ as $2r + 1$ for $r \geq 1$. Let $x = (i_1, i_2, \ldots, i_p)$ in $P$ of order $p$, and let $Q := \langle x \rangle \cong C_p$ be a subgroup of $P$. Then for an involution

$$\alpha := (i_1, i_p)(i_2, i_{p-1})(i_3, i_{p-2}) \cdots (i_r, i_{r+2})$$

in $X$, we have that $x^\alpha = x^{-1} \neq x$ as $p \neq 2$. Thus $\alpha$ lies in $E_X(Q)$. But since $|E_X(Q)|$ divides $|E_X(P)|$ by Proposition 1(1), we have that $|E_X(P)|$ is even. □

4. Some cases of Lie type groups

In this section, we will consider some special cases of Lie type groups. We refer to [2] for their standard property.

PROPOSITION 5 (Defining characteristic). Let $X$ be a simple group of Lie type over $GF(q)$ where $q = p^e$ for some prime $p$. Suppose that $X$ possesses an abelian Sylow $p$-subgroup $P$. Then $X \cong PSL(2, q)$.

PROOF. This follows from the Chevalley’s commutator formula (see also [6, Proposition 5.1]). □

PROPOSITION 6. Let $X$ be a simple group of Lie type, $X^u$ a universal version of $X$, and $P$ an abelian Sylow $p$-subgroup of $X$ with $p \in \pi(Z(X^u))$ and $p \neq 2$. Then $E_X(P)$ contains an involution.

PROOF. As $p \in \pi(Z(X^u))$, it is enough to consider the following (see also in the proof of [6, Proposition 5.2]):

$$A_l(q)(l \geq 1), \quad p|(l + 1, q - 1) ; \quad E_6(q) , \quad p = 3 ;$$

$$2A_l(q^2)(l \geq 2), \quad p|(l + 1, q + 1) ; \quad 2E_6(q^2) , \quad q + 1 \equiv 0(3), \quad p = 3 .$$
CASE. \( E_6(q), p = 3 \):
A Sylow 3-subgroup of \( E_6(q) \) is not abelian, since the Weyl group \( O^*(6, 2) \) of type \( E_6 \) possesses a non-abelian Sylow 3-subgroup.

CASE. \( A_1(q) = \text{PSL}(l + 1, q), p|(l + 1, q - 1) \):
Let \( X = \text{PSL}(l + 1, q) \). Since \( p|(l + 1, q - 1) \), we have that \( l + 1 \geq p \geq 3 \), and that there exists \( t \in GF(q) \setminus \mathbb{C}_q \) such that \( t^p = 1 \) and \( t \neq 1 \). Let \( \mathcal{Z} = \{ \mathcal{M} = \text{diag}(a_1, \ldots, a_{l+1}) \in X \mid (\mathcal{M})^p = 1 \} \), modulo \( Z(X^n) \), be a \( p \)-subgroup of \( X \) where \( \text{diag}(a_1, \ldots, a_{l+1}) \) is a diagonal matrix in \( SL(l+1, q) \).

Let \( w := A \oplus B \) be an involution of \( X \) where

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and \( B = \text{diag}(1, \ldots, 1, -1) \in GL(l - 1, q) \). Evidently \( w \) normalizes \( D \) but does not centralize an element \( z = \text{diag}(t, t^{-1}, 1, \ldots, 1) \) in \( D \). Note that \( t \neq t^{-1} \) as \( p \neq 2 \). This implies that an involution \( w \) is contained in \( \mathcal{E}_X(D) \). But since \( |\mathcal{E}_X(D)| \) divides \(|\mathcal{E}_X(P)|\) by Proposition 1(1), \(|\mathcal{E}_X(P)|\) is even.

CASE. \( 2A_1(q^2) = \text{PSU}(l + 1, q^2), p|(l + 1, q + 1) \):
Let \( X = \text{PSU}(l + 1, q^2) \). Recall that \( SU(l + 1, q^2) = \{ M \in SL(l + 1, q^2) \mid (M)^\theta(M) = I \} \) where \( \theta \) is an associated field automorphism which is defined by \( \theta(a) = a^q \) for \( a \in GF(q^2) \). \( \theta \) is of order 2. Now since \( p|q + 1 \), there exists \( t \in [\alpha \in GF(q^2) \mid \theta(\alpha) = \alpha^{q+1}] \mathbb{C}_{q+1} \) such that \( t^p = 1 \) and \( t \neq 1 \). Then the same argument as above can be applied. Indeed, define \( D, w, z \) as in the case of \( A_1(q) \). Then \( D \) is a \( p \)-subgroup of \( X = \text{PSU}(l + 1, q^2) \) with \( z \in D \). Furthermore an involution \( w \in X \) lies in \( \mathcal{E}_X(D) \). But since \( |\mathcal{E}_X(D)| \) divides \(|\mathcal{E}_X(P)|\) by Proposition 1(1), \(|\mathcal{E}_X(P)|\) is even.

CASE. \( 2E_6(q^2), q + 1 \equiv 0(3), p = 3 \):
Let \( X = \text{PSU}(q^2) \), and then \( H = \text{PSU}(6, q^2) \) is involved in \( X \). Since \( p = 3|(6, q + 1) \), we have that \( p \) divides \(|Z(SU(6, q^2))|\). Then applying the unitary case above, we have that \( |\mathcal{E}_H(Q)| \) is even for \( Q \in \text{Syl}_p(H) \), and thus so is \(|\mathcal{E}_X(P)|\). The proof is complete.

PROPOSITION 7 (Weyl groups).
Let \( X = \text{d}X_l(q^d) \) be a universal group of Lie type, and \( P \) a Sylow \( p \)-subgroup of \( X \) with \( p \neq 2, p \nmid q \) and \( p \not\equiv \pi(W(X_l)) \) where \( W(X_l) \) is the Weyl group of type \( X_l \). Then \( \mathcal{E}_X(P) \) contains an involution.

REMARK. For formality of notation, \( 1X_l(q^1) \) implies the untwisted group \( X_l(q) \). In the case of Suzuki and Ree groups, namely \( 1X_1(q^2) = 2B_2, 2F_4, 2G_2 \), we set \( dX_l(q^d) = 2B_2(q) (q = 2^{2m+1}), 2F_4(q) (q = 2^{2m+1}), 2G_2(q) (q = 2^{2m+1}) \). The twisted group \( dX_l(q^d) \) \((d \geq 2)\) is a set of elements of \( X_l(q^d) \); which is fixed by a graph-field automorphism of order \( d \) in \( \text{Aut}(X_l(q^d)) \). (In the Atlas [3], \( dX_1(q^d) \) is denoted by \( dX_l(q, q^d) \), and an abbreviated notation \( dX_l(q) \) is also used there.)

PROOF. Concerning the Sylow structure of \( P \), we follow the argument in the proof of [5, (10-1)]. See [5] for the details. Let \( m \) be the multiplicative order of \( q \) modulo \( p \), and set
Let $Y := X_I(q^{dm})$ a universal group. Then there exists a group $F = \langle \rho, \beta \rangle \leq \text{Aut}(Y)$ generated by a graph-field automorphism $\rho = \sigma \theta$ of order $d$ and a field automorphism $\beta$ of order $m$ such that $X \cong C_Y(F)$. (As mentioned above, in the case of Suzuki and Ree groups, we have that $Y = B_2(q^{2m}), F_4(q^{2m}), G_2(q^{2m})$, and thus $\beta$ is of order $2m$ in these cases.) We identify $X$ with $C_Y(F)$. Furthermore we may assume that up to conjugacy, $P$ is contained in a Sylow $p$-subgroup $R$ of a Cartan subgroup $H$ of $Y$; in particular $P$ is abelian. Then we have that $P = C_R(F) = C_P(F) \leq C_R(F) \leq Y$ and $P \in \text{Syl}_p(X)$. Recall that up to conjugacy, $H = \langle h_{r}(t) \mid r \in \Pi, t \in GF(q^{md})^\times \rangle$ where $\Pi$ is a set of fundamental roots of $Y$ and $h_r(t)$ is a standard generator of $H$. Thus letting $E$ the unique Sylow $p$-subgroup of the multiplicative group $GF(q^{md})^\times$, we have that $R = \langle h_r(t) \mid r \in \Pi, t \in E \rangle$.

Now let $\{\omega_r \mid r \in \Pi\}$ be a set of standard generators of the Weyl group of $Y$. Then setting $N = \langle \omega_r, H \mid r \in \Pi \rangle \leq Y$, we have that $N/H \cong W(X_I)$. 

**Case**. $X$ is untwisted: Since $F = \langle \beta \rangle$ in this case, $P = C_R(\beta) = \langle h_r(t) \mid r \in \Pi, t \in E, t^\beta = t \rangle$. Take any $r \in \Pi$. Since $[\omega_r, \beta] = 1$ and $h_r(t)^{\omega_r} = h_{\omega_r(r)}(t)$ for any root $s$, we have that $\omega_r$ is in $X$ and also normalizes $C_R(\beta) = P$; namely $\omega_r \in N_X(P)$. On the other hand, for $t \in E$ with $t \neq 1$ and $t^\beta = t$, an element $h_r(t)$ lies in $P \setminus \{1\}$, and we have that $h_r(t)^{\omega_r} = h_{-r}(t) = h_r(t)^{-1} \neq h_r(t)$ as $p \neq 2$. This implies that $\omega_r \not\in C_X(P)$. Furthermore since $\omega_r^2 \in X \cap H \leq X \cap C_Y(P) = C_X(P)$, $E_X(P)$ contains an involution $\omega_r C_X(P)$.

**Case**. $X$ is twisted: First we recall some $\rho$-invariant subgroups of $C_Y(\rho) \cong d X_I((q^m)^d)$. For a $\sigma$-orbit $J$ on $\Pi$, set $W(J) = \langle \omega_r = \omega_r H \mid r \in J \rangle \leq N/H$. Then there exists a unique element $w_0(J)$ of order 2 in $W(J)$ such that $w_0(J)^2 = w_0(J)$. Then $\langle w_0(J) \mid J = \sigma$-orbit on $\Pi\rangle$ is the Weyl group of $C_Y(\rho)$; which is isomorphic to $N^1/H^1 \cong N^1/H$ where $N^1 = C_Y(\rho) \cap N$ and $H^1 = C_Y(\rho) \cap H$. We may assume that $w_0(J) \in N^1$. Recall that $w_0(J)$ is a reflection along the vector $a(r)$ where $r \in J$ and $a(r)$ is the average of the vectors in the $\sigma$-orbit $J$. Next define an element of $H^1$ as follows:

$$h_J(t) := h_r(t) \quad \text{for} \quad t \in GF(q^{dm})^\times \text{ with } \bar{t} = t, \quad \text{if } J = \{r\},$$

$$h_J(t) := h_r(t\bar{t}) \quad \text{for} \quad r \in J \text{ and } t \in GF(q^{dm})^\times, \quad \text{if } |J| = 2,$$

$$h_J(t) := h_r(t) h_t(\bar{t}) \quad \text{for} \quad r \in J \text{ and } t \in GF(q^{dm})^\times, \quad \text{if } |J| = 3,$$

where $\bar{r} = r^\sigma$ for a root $r$, and $\bar{t} = t^\rho$ for $t \in GF(q^{dm})$. (Note that if the characteristic of Suzuki-Ree groups $C_Y(\rho)$ is 2 or 3, then $h_J(t)$ is defined respectively as $h_r(t) h_t(\bar{t}^2)$ or $h_r(t) h_t(\bar{t})$ for a short root $r$ in $J$.) Any element of $H^1$ can be uniquely expressed as a product $\Pi h_J(t)$ where $J$ runs through all $\sigma$-orbits on $\Pi$. Thus a Sylow $p$-subgroup $R^1$ of $H^1$ is as follows:

$$R^1 = \left\{ h_J(t) \mid J = \sigma$-orbit on $\Pi, t \in E$ such that $\bar{t} = t \right\}$$

if $t$ is a coefficient of $h_J(t)$ with $|J| = 1$. 

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where $E \in Syl_p(GF(q^{dm})^*)$. Then $P = C_R(F) = C_R(\beta)$.

Return back to the proof of Proposition 7. Take a $\sigma$-orbit $J$ on $\Pi$. Then we may assume that $[w_0(J), \rho] = [w_0(J), \beta] = 1$, and thus $w_0(J) \in X$. Furthermore since $w_0(J)$ normalizes $R^1$, a unique Sylow $p$-subgroup of $H^1$, we can see that $w_0(J)$ acts on $C_{R^1}(\beta) = P$; namely $w_0(J) \in N_X(P)$. Now we may assume that $|J| \geq 2$. Then, for $t \in E$ with $t \neq 1$ and $t^\beta = t$, an element $h_J(t)$ lies in $P \setminus \{1\}$. But $h_J(t)^{w_0(J)} = h_J(t)$ as $p \neq 2$; which implies that $w_0(J) \notin C_X(P)$. Furthermore since $w_0(J)^2 \in X \cap H \leq X \cap C_Y(P) = C_X(P)$, $\mathcal{E}_X(P)$ contains an involution $w_0(J)C_X(P)$. The proof is complete. 

PROPOSITION 8 (Primes $p$ with $p|q - 1$). Let $X \cong d X_l(q^d)$ be a universal group of Lie type, and $P$ an abelian Sylow $p$-subgroup of $X$ with $p \neq 2$ and $p|q - 1$. Then $\mathcal{E}_X(P)$ contains an involution.

PROOF. As $p|q - 1$, $p$ divides the order of a Cartan subgroup $H$ of $X$. But we have shown in Proposition 7 implicitly that $|\mathcal{E}_X(Q)|$ is even for $Q \in Syl_p(H)$, and so is $|\mathcal{E}_X(P)|$ (see also [6, Propositions 5.3, 5.4]).

Finally, we mention simple groups with abelian Sylow 2-subgroups (See [4, Chapter 16.6]):

PROPOSITION 9 (Abelian Sylow 2-subgroups). Let $X$ be a nonabelian simple group with an abelian Sylow 2-subgroup $P$. Then one of the following holds.

1. $X \cong PSL(2, q)$ with $q > 3$ and $q \equiv 3, 5$ (mod $8$), or $q = 2^e$.
2. $X \cong J_1$; the first Janko group.
3. $X \cong 2G_2(3^{2m+1})$; the Ree group.

Note that if $X \cong PSL(2, q)$ with $q > 3$ and $q \equiv 3, 5$ (mod $8$) then $P \cong C_2 \times C_2$, and that if $X \cong J_1$ or $2G_2(3^{2m+1})$ then $\mathcal{E}_X(P) \cong (7 : 3)$.

5. Classical groups

The aim of this section is to show the following:

PROPOSITION 10. Let $X$ be a classical simple group, and $P$ an abelian Sylow $p$-subgroup of $X$ with $p \neq 2$ and $p \nmid q$. Then either $\mathcal{E}_X(P)$ contains an involution or $P$ is cyclic.

PROPOSITION 11 (Untwisted classical). Let $X = X_l(q)$ be one of universal groups $A_l(q)(l \geq 1)$, $B_l(q)(l \geq 2, q \equiv 1(2))$, $C_l(q)(l \geq 2)$, $D_l(q)(l \geq 4)$, and $P$ an abelian Sylow $p$-subgroup of $X$ with $p \neq 2$ and $p \nmid q$. Then $\mathcal{E}_X(P)$ contains an involution.

PROOF. Let $W(X_l)$ be the Weyl group of type $X_l$. By Proposition 7, we may assume that $p \in \pi(W(X_l))$. Recall $W(A_l) \cong S_{l+1}$, $W(B_l) \cong W(C_l) \cong 2^l S_l$, and $W(D_l) \cong 2^{l-1} S_l$. As $p \neq 2$, $p$ divides the order of the symmetric group $S_n (n = l + 1)$. Then $|\mathcal{E}_{S_n}(Q)|$ is
even for $Q \in Syl_p(S_n)$ by Proposition 4. But since $|\mathcal{E}_{S_n}(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(2), we have that $|\mathcal{E}_X(P)|$ is even. The proof is complete. 

Let $X = ^2X_l(q^2)$ be a universal version of a classical group. Then the order of $X$ is expressed as

$$|X| = q^N \prod_{m \in \mathcal{O}(^2X_l)} \Phi_m(q)^{r_m}$$

where $\Phi_m(q)$ the cyclotomic polynomial for the $m$th roots of unity, $\mathcal{O}(^2X_l)$ a set of positive integers depending on $^2X_l$, $N$ the number of positive roots in the root system corresponding to $X$, and $r_m$ a positive integer (see [5, Section 10] for the details). Note that $r_m$ is known as in Table 1:

<table>
<thead>
<tr>
<th>$^2A_l$</th>
<th>$r_m = \left\lfloor \frac{l + 1}{lcm(2, m)} \right\rfloor$ if $m \not\equiv 2(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_m = \left\lfloor \frac{2(l + 1)}{m} \right\rfloor$ if $m \equiv 2(4), m &gt; 2$</td>
<td></td>
</tr>
<tr>
<td>$r_2 = l$</td>
<td></td>
</tr>
<tr>
<td>$^2D_l$</td>
<td>$r_m = \left\lfloor \frac{2l}{lcm(2, m)} \right\rfloor$ if $m \not</td>
</tr>
<tr>
<td>$r_m = \left\lfloor \frac{2l}{lcm(2, m)} \right\rfloor - 1$ if $m</td>
<td>l$</td>
</tr>
</tbody>
</table>

Let $e$ be the smallest positive integer such that $p|\Phi_e(q)$, and $m_p(X)$ the maximal $p$-rank of a Sylow $p$-subgroup of $X$. Set

$$\pi := \{ p \in \pi(X) \mid p \not\equiv 2, \text{ } p \nmid q, \text{ } p \not\in \pi(Z(X)) \} .$$

**LEMMA 2** ((10-2) in [5]). For $p \in \pi$, we have that $m_p(X) = m_p(X/Z(X)) = re$.

We will keep the above notation throughout this section.

**PROPOSITION 12** (Unitary groups). Let $X = ^2A_l(q^2) \cong SU(l + 1, q^2)(l \geq 2)$ a universal group with an abelian Sylow $p$-subgroup $P$ for $p \in \pi$. Then either $\mathcal{E}_X(P)$ contains an involution or $P$ is cyclic.

**PROOF.** Set $l = 2k$ or $2k - 1$ for $k \geq 1$.

**STEP 1.** We may assume that $p \not\in \pi(S_k)$:

Suppose that $p \in \pi(S_k)$, and let $Q \in Syl_p(S_k)$. Then $\mathcal{E}_{S_k}(Q)$ contains an involution by Proposition 4. But since $S_k$ is involved in $X$ as the (twisted) Weyl group, we have that $|\mathcal{E}_X(P)|$ is even by Proposition 1(2). Thus we may assume that $p \not\in \pi(S_k)$.

**STEP 2.** We may assume that $e > 1$ and $r_e > 1$:

If $e = 1$ then $p|\Phi_1(q) = q - 1$ and thus $|\mathcal{E}_X(P)|$ is even by Proposition 8. On the other hand if $r_e = 1$ then $m_p(X) = r_e = 1$ by Lemma 2 and thus an abelian Sylow $p$-subgroup $P$ is cyclic.
STEP 3. If \( e = 2i \) and \( i \geq 2 \) is even then \( \mathcal{E}_X(P) \) contains an involution:

Since \( e \neq 2(4) \) and \( 2 \leq r_e = \left\lfloor \frac{i+1}{4} \right\rfloor \), we have that \( 2e \leq l + 1 \) and \( e \leq \frac{i+1}{4} = k + \frac{1}{2} \); which follows that \( e \leq k \) and \( \pi(S_e) \subseteq \pi(S_k) \). Let \( H = \bar{2}A_{l-1}(q^2) \cong SU(e, q^2) \) \((e \geq 4)\) be a subgroup of \( X \). As \( r_e = \left\lfloor \frac{i}{2} \right\rfloor = 1 \) for \( H, p \in \pi(H) \). But since \( \pi(W(A_{l-1})) = \pi(S_e) \subseteq \pi(S_k) \), \( p \notin \pi(W(A_{l-1})) \) by Step 1. Thus \( |\mathcal{E}_H(Q)| \) is even for \( Q \in \text{Syl}_p(H) \) by Proposition 7. Now \( |\mathcal{E}_H(Q)| \) divides \( |\mathcal{E}_X(P)| \) by Proposition 1(2), and hence \( |\mathcal{E}_X(P)| \) is even.

STEP 4. If \( e = 2i \) and \( i \geq 1 \) is odd then \( \mathcal{E}_X(P) \) contains an involution:

Suppose \( e = 2; \) that is, \( p|\Phi_2(q) = q + 1 \). Let \( H = \bar{2}A_1(q^2) \cong SU(2, q^2) \cong SL(2, q) \) be a subgroup of \( X \). As \( |H| = q(q-1)(q+1) \), \( p \in \pi(H) \). Then \( |\mathcal{E}_H(Q)| \) is even for \( Q \in \text{Syl}_p(H) \) by Proposition 11. Thus we may assume that \( i \geq 3 \).

Since \( e \equiv 2(4) \) and \( e > 2 \), we have that \( 2 \leq r_e = \left\lfloor \frac{2l+1}{e} \right\rfloor \) and \( i = \frac{e}{2} \leq \frac{l+1}{2} = k \) or \( k + \frac{1}{2} \); which follows that \( i \leq k \) and \( \pi(S_e) \subseteq \pi(S_k) \). Let \( H = \bar{2}A_{i-1}(q^2) \cong SU(i, q^2) \) \((i \geq 3)\) be a subgroup of \( X \). As \( r_e = \left\lfloor \frac{i}{2} \right\rfloor = 1 \) for \( H, p \in \pi(H) \). But since \( \pi(W(A_{i-1})) = \pi(S_e) \subseteq \pi(S_k) \), \( p \notin \pi(W(A_{i-1})) \) by Step 1. Thus \( |\mathcal{E}_H(Q)| \) is even for \( Q \in \text{Syl}_p(H) \) by Proposition 7.

STEP 5. If \( e \geq 3 \) is odd then \( \mathcal{E}_X(P) \) contains an involution:

Since \( e \neq 2(4) \), we have that \( 2 \leq r_e = \left\lfloor \frac{i+1}{4} \right\rfloor \) and \( 2e \leq \frac{i+1}{4} = k \) or \( k + \frac{1}{2} \); which follows that \( 2e \leq k \) and \( \pi(S_{2e}) \subseteq \pi(S_k) \). Let \( H = \bar{2}A_{2e-1}(q^2) \cong SU(2e, q^2) \) \((2e \geq 6)\) be a subgroup of \( X \). As \( r_e = \left\lfloor \frac{i}{2} \right\rfloor = 1 \) for \( H, p \in \pi(H) \). But since \( \pi(W(A_{2e-1})) = \pi(S_{2e}) \subseteq \pi(S_k) \), \( p \notin \pi(W(A_{2e-1})) \) by Step 1. Thus \( |\mathcal{E}_H(Q)| \) is even for \( Q \in \text{Syl}_p(H) \) by Proposition 7. The proof is complete.

PROPOSITION 13 (Orthogonal groups of type \(-\)). Let \( X = \bar{2}D_{l}(q^2) \cong \Omega^-(2l, q) \) \((l \geq 4)\) a universal group with an abelian Sylow \( p \)-subgroup \( P \) for \( p \in \pi \). Then either \( \mathcal{E}_X(P) \) contains an involution or \( P \) is cyclic.

PROOF. STEP 1. We may assume that \( p \notin \pi(S_{l-1}) \):

Suppose that \( p \in \pi(S_{l-1}) \), and let \( Q \in \text{Syl}_p(S_{l-1}) \). Then \( \mathcal{E}_{S_{l-1}}(Q) \) contains an involution by Proposition 4. But since \( 2^{l-1}S_{l-1} \) is involved in \( X \) as the \((2^{l-1} \times \text{Weyl}) \) group we have that \( |\mathcal{E}_X(P)| \) is even by Proposition 1(2). Thus we may assume that \( p \notin \pi(S_{l-1}) \).

STEP 2. We may assume that \( e > 1 \) and \( r_e > 1 \):

By the same reason as in the proof of Step 2 in Proposition 12.

STEP 3. If \( e = 2i \) is even then \( \mathcal{E}_X(P) \) contains an involution:

Suppose \( e = 2 \) or \( 4; \) that is \( p|\Phi_2(q) = q + 1 \) or \( p|\Phi_4(q) = q^2 + 1 \). Let \( H = \bar{2}D_2(q^2) \cong A_1(q^2) \) be a subgroup of \( X \). As \( |H| = q^2(q^2 - 1)(q^2 + 1) \), \( p \in \pi(H) \). Then \( |\mathcal{E}_H(Q)| \) is even for \( Q \in \text{Syl}_p(H) \) by Proposition 11. But since \( |\mathcal{E}_H(Q)| \) divides \( |\mathcal{E}_X(P)| \) by Proposition 1(2), we have that \( |\mathcal{E}_X(P)| \) is even. Thus we may assume that \( i \geq 3 \).

Since \( 1 \leq r_e = \left\lfloor \frac{l}{2} \right\rfloor \), we have that \( e < 2l \) and \( i = \frac{e}{2} < l \); which follows that \( i \leq l - 1 \) and \( \pi(S_i) \subseteq \pi(S_{l-1}) \). Let \( H = \bar{2}D_{l}(q^2) \cong \Omega^-(2l, q) \) \((i \geq 3)\) be a subgroup of \( X \). (Note that \( \bar{2}D_3(q^2) \cong 2A_3(q^2) \).) As, for \( H, r_e = \left\lfloor \frac{l}{2} \right\rfloor = 1 \) if \( i \geq 4 \) and \( r_e = \left\lfloor \frac{2l+1}{e} \right\rfloor = 1 \) if \( i = 3 \),
we have that \( p \in \pi(H) \). Furthermore if \( i \geq 4 \) then since \( \pi(W(D_{l})) = \pi(2^{i-1}S_{l}) \subseteq \pi(S_{l-1}) \) we have that \( p \notin \pi(W(D_{l})) \) by Step 1, and if \( i = 3 \) then since \( \pi(W(A_{3})) = \pi(S_{3}) \) and \( p > l - 1 \geq i = 3 \) we have that \( p \notin \pi(W(A_{3})) \). In either case, \( p \) does not divide the order of the Weyl group \( W(D_{l}) \) or \( W(A_{3}) \) of \( H \). Thus \( E_{H}(Q) \) is even for \( Q \in Syl_{p}(H) \) by Proposition 7.

STEP 4. If \( e \) is odd then \( E_{X}(P) \) contains an involution:

Since \( 2 \leq r_{e} \leq \frac{|Z(X)|}{2} \), we have that \( e \leq \frac{l}{2} < l - 1 \) and \( e + 1 \leq l - 1 \); which follows that \( \pi(S_{e+1}) \subseteq \pi(S_{l-1}) \). Let \( H = 2^{e}D_{e+1}(q^{2}) \) \((e + 1 \geq 4)\) be a subgroup of \( X \). As \( r_{e} = \frac{|Z(X)|}{2e} \) for \( H \), \( p \in \pi(H) \). But since \( \pi(W(D_{e+1})) = \pi(2^{e}S_{e+1}) \subseteq \pi(S_{l-1}) \), \( p \notin \pi(W(D_{e+1})) \) by Step 1. Thus \( |E_{H}(Q)| \) is even for \( Q \in Syl_{p}(H) \) by Proposition 7. The proof is complete.

PROOF OF PROPOSITION 10. Let \( X^{u} \) be a universal version of \( X \). By Proposition 6, we may assume that \( p \notin \pi(Z(X^{u})) \). Then we have, by Propositions 11, 12, 13, that either \( |E_{X^{u}}(R)| \) is even or \( R \) is cyclic for \( R \in Syl_{p}(X^{u}) \). But this implies that, for \( P = R \in Syl_{p}(X^{u}) \) modulo \( Z(X^{u}) \), either \( |E_{X}(P)| = |E_{X^{u}}(R)| \) is even by Lemma 1, or \( P \equiv R \) is cyclic, as desired.

6. Exceptional groups

The aim of this section is to show the following:

PROPOSITION 14. Let \( X \) be an exceptional simple group, and \( P \) an abelian Sylow \( p \)-subgroup of \( X \) with \( p \neq 2 \) and \( p \nmid q \). Then either \( E_{X}(P) \) contains an involution or \( P \) is cyclic.

PROPOSITION 15 (Untwisted exceptional). Let \( X = X_{i}(q) \) be one of universal groups \( E_{6}(q), E_{7}(q), E_{8}(q), F_{4}(q), G_{2}(q) \), and \( P \) an abelian Sylow \( p \)-subgroup of \( X \) with \( p \neq 2 \) and \( p \nmid q \). Then \( E_{X}(P) \) contains an involution.

PROOF. Let \( W(X_{i}) \) be the Weyl group of type \( X_{i} \). By Proposition 7, we may assume that \( p \notin \pi(W(X_{i})) \). Recall \( W(E_{6}) \cong PSp(4, 3)2 \), \( W(E_{7}) \cong 2 \times Sp(6, 2) \), \( W(E_{8}) \cong 2^{\Omega^{+}(8, 2)2} \), \( W(F_{4}) \cong (2^{3}S_{5})S_{3} \), and \( W(G_{2}) \cong D_{12} \). As \( p \neq 2 \), \( p \) divides the order of a group \( H \); which is a classical group, the symmetric group, or the dihedral group \( D_{12} \). Thus \( |E_{H}(Q)| \) is even for \( Q \in Syl_{p}(H) \) by Propositions 4 or 11. But since \( |E_{H}(Q)| \) divides \( |E_{X}(P)| \) by Proposition 1(2), we have that \( |E_{X}(P)| \) is even. The proof is complete.

PROPOSITION 16 (Twisted exceptional). Let \( X = ^{d}X_{i}(q^{d}) \) be one of universal groups \( ^{d}D_{3}(q^{3}), ^{d}E_{6}(q^{2}), ^{d}F_{4}(2^{2m+1}), ^{d}G_{2}(3^{2m+1}), ^{d}B_{2}(2^{2m+1}) \), and \( P \) an abelian Sylow \( p \)-subgroup of \( X \) with \( p \neq 2 \), \( p \nmid q \), \( p \notin \pi(Z(X)) \). Then either \( E_{X}(P) \) contains an involution or \( P \) is cyclic.

PROOF. If \( X = ^{d}G_{2}(3^{2m+1}) \) or \( ^{d}B_{2}(2^{2m+1}) \) then an abelian Sylow \( p \)-subgroup \( P \) of \( X \) is always cyclic (see [5, (10-2)] or Lemma 2). Thus we may assume that \( X \) is otherwise.
Now let \( W(X_l) \) be the Weyl group of type \( X_l \). By Proposition 7, we may assume that \( p \in \pi(W(X_l)) \).

**CASE.** \( X = 3D_4(q^3) \): Since \( p \in \pi(W(D_4)) = \pi(2^3 S_4) = \{2, 3\} \), we have that \( p = 3 \). Note that \( X \) possesses \( W(G_2) \cong D_{12} \) as the (twisted) Weyl group, and \( |E_{D_{12}}(Q)| \) is even for \( Q \in Syl_3(D_{12}) \). But since \( |E_{D_{12}}(Q)| \) divides \( |\mathcal{E}_X(P)| \) by Proposition 1(2), we have that \( |\mathcal{E}_X(P)| \) is even.

**CASE.** \( X = 2F_4(q) (q = 2^{2m+1}, m \geq 1) \): Since \( p \in \pi(W(F_4)) = \pi(W(D_4)S_3) = \{2, 3\} \), we have that \( p = 3 \). Let \( H = SL(2, q) \) be a subgroup of \( X \). As \( |H| = q(q-1)(q+1) \), \( p \in \pi(H) \). (Note that if \( p = 3 \) does not divide \( q - 1 \) then \( q + 1 \) is divisible by \( p \).) Thus \( |\mathcal{E}_H(Q)| \) is even for \( Q \in Syl_p(H) \) by Proposition 11.

**CASE.** \( X = 2E_6(q^2) \): Since \( p \in \pi(W(E_6)) = \pi(PSp(4, 3)2) = \{2, 3, 5\} \), we have that \( p = 3 \) or \( 5 \). Note that \( X \) possesses \( W(F_4) \cong (2^3 S_4)S_3 \) as the (twisted) Weyl group. So if \( p = 3 \) then, for an involved group \( S_3 \), we have that \( E_{S_3}(R) \cong C_2 \) for \( R \in Syl_3(S_3) \). Thus \( |\mathcal{E}_X(P)| \) is even, and we may assume that \( p = 5 \).

Let \( H = F_4(q) \) be a subgroup of \( X \) of order \( |H| = q^{24} \Phi_1(q)^4 \Phi_2(q)^4 \Phi_3(q)^2 \Phi_4(q)^2 \Phi_5(q)^2 \Phi_6(q)^2 \Phi_1(q_2(q)^2) \), where \( \Phi_m(q) \) is the cyclotomic polynomial for the \( m \)th roots of unity (see [5, Table 4-1] for the existence of \( F_4(q) \) in \( X \)). Now it is easy to see that if \( p = 5 \) does not divide both \( \Phi_1(q) = q - 1 \) and \( \Phi_2(q) = q + 1 \) then \( \Phi_4(q) = q^2 + 1 \) is divisible by \( p \). Thus \( p \) always divides \( |H| \). But since \( \pi(W(F_4)) = \pi((2^3 S_4)S_3) \), \( p = 5 \notin \pi(W(F_4)) \). Thus \( |\mathcal{E}_H(Q)| \) is even for \( Q \in Syl_p(H) \) by Proposition 7. The proof is complete.

**Proof of Proposition 14.** The same as in that of Proposition 10.

**7. Proof of Theorem 1**

Suppose that \( X \) is the alternating group or a sporadic group. Then by Propositions 2 and 3, \( |\mathcal{E}_X(P)| \) is even; (1), \( P \) is cyclic; (2), \( P \cong C_2 \times C_2 \); (3), or \( X = J_1 \); (5).

Suppose next that \( X \) is a Lie type group \( \Gamma X_l(q^d) \). If \( p = 2 \) then by Proposition 9, \( P \cong C_2 \times C_2 \); (3), \( X \cong PSL(2, p^e) \); (4), or \( X \cong 2^2 D_3(q^{2m+1}) \); (5). If \( p \mid q \) then by Proposition 5, \( X \cong PSL(2, p^e) \); (4). Thus we may assume that \( p \neq 2 \) and \( p \nmid q \). Then by Propositions 10 and 14, \( |\mathcal{E}_X(P)| \) is even; (1), or \( P \) is cyclic; (2).

Finally we consider the Tits simple group \( X = 2F_4(2)' \) of order \( 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13 \). Then it easy to see that \( |\mathcal{E}_G(P)| \) is even; (1), or \( P \) is cyclic; (2), (see [3]). The proof is complete.

**References**


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