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New Techniques for Classifying Williams Solenoids

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Abstract. In this paper we compare topological methods and dynamical methods corresponding to recent developments in the classification of inverse limit spaces of one dimensional maps on graphs. We elucidate the role of shifting the periodic base point in applying Williams' theory. We exploit the Fox calculus to define and apply the Bowen-Franks trace—a shift equivalence invariant of free group homomorphisms. We show that augmented cohomology of certain suspensions associated with wrapping rules in a substitution yields augmented dimension groups that have a relatively simple product structure. We complete the classification of a family of examples of generalized solenoids initiated by R. F. Williams.

1. Introduction

The aim of this paper is to contrast older and more recent methods for classifying one dimensional hyperbolic attractors or generalized solenoids. This is a problem with two fronts: dynamical classification (by conjugacy) and the topological classification of the solenoid. We present some new results in both directions, but we want to provide some background first.

In the late 1960's, R. F. Williams (e.g. [16]) discovered a new way to think about the conjugacy class of a one dimensional hyperbolic attractor $h : \mathcal{S} \to \mathcal{S}$. Williams' classification of such attractors $(\mathcal{S}, h)$ (or generalized 1-solenoids) relied on the shift equivalence of various possible "presentations" of $\mathcal{S}$. A presentation is a mapping pair (K, f) such that $f : K \to K$ is a continuous endomorphism of a graph K, with inverse limit space $\varprojlim(f, K)$ whose shift map is conjugate to the given homeomorphism on $\mathcal{S}$. Williams showed (in [16, Theorem 3.3] and Theorem 2.2 below) that two shift maps $\bar{f}_1$ and $\bar{f}_2$, on presentations (K_1, f_1) and (K_2, f_2), are topologically conjugate if and only if the maps f_1 and f_2 are shift equivalent. He was able to show further that shift equivalence is equivalent to "strong shift equivalence" in the category of maps on branched 1-manifolds (connected graphs). This reduces checking shift equivalence to seeking a sequence of "elementary" (or lag 1) shift equivalences.

Williams went on to link the shift equivalence of pointed presentations (corresponding to pointed conjugacy classes of shifts $\bar{f} : (\varprojlim(K, f), \bar{x}) \to (\varprojlim(K, f), \bar{x}), \bar{x} = (x, x, \dots))$ to the shift equivalence of π_1 representations.

Williams defines the *shift class* $S(\bar{f})$ of $\bar{f}$ to be the shift equivalence class of $\pi_1(f, x) : \pi_1(K, x) \to \pi_1(K, x)$. But as Williams observes, "More accurately, $S(\bar{f}, x)$, as all of this

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depends on the choice of base point." This is, indeed, a key distinction: shift equivalence of graph maps does not imply shift equivalence of group endomorphisms unless results are carefully framed in the category of pointed topological spaces. Writing the π_1 representation as f_* rather than, say, $f_{*,x}$ is common enough but perhaps a bit reckless.

In fact, *shifting the basepoint can alter the shift equivalence class of the group endomorphism* (Example 4.5). There is no easy fix for this, as group endomorphisms simply do not have base points. Only graph maps and solenoidal shifts have base points (in a pointed category).

Here is the Williams classification theorem ([16]): See definitions below.

THEOREM. *Suppose the elementary presentations* (K_i, f_i), $i = 1, 2$, *satisfy Axioms* 2.1. *Suppose* $f_i(y_i) = y_i$, $i = 1, 2$. *There is a pointed conjugacy* $\bar{r}$: $(\varprojlim(K_1, f_1), \bar{y}_1) \to (\varprojlim(K_2, f_2), \bar{y}_2)$ *of* $\bar{f}_1$ *with* $\bar{f}_2$ *if and only if the fundamental group homomorphisms* $\pi_1(f_1, y_1)$ *and* $\pi_1(f_2, y_2)$ *are shift equivalent.*

This theorem is most useful when combined with strong shift equivalence invariants for free group homomorphisms. In Section 4, we describe the "bf-trace", due to the authors of [10], which combines the Fox (free) calculus and Bowen-Franks theory. We make use of this to determine (in Proposition 4.7) the pointed and unpointed conjugacy classes of the set of 1-dimensional hyperbolic attractors having the particular characteristic polynomial $X^2 - 3X - 2$, a lingering problem posed by Williams in [16].

The purely topological classification of generalized solenoids has its own history (see references in [1]), which has taken on new life with the discovery that tiling spaces can be viewed as generalized solenoids.

In Section 5.1, we study a new invariant—augmented ordered dimension groups—for topologically distinguishing generalized solenoids. These first appear in the paper [7]. Most recently, these "Matsumoto" groups (e.g. [14]) were constucted more topologically in [4] as augmented Čech ordered first cohomology groups. We prove that augmented dimension groups typically have a simple computable product structure (Theorem 5.4). We show that the order part of the cohomology invariant is preserved, under additional assumptions, only if a new invariant, called the **crossing group**, is preserved by an order isomorphism (Theorem 5.10). We exploit the crossing group to topologically distinguish some specific unpointed solenoids and to complete the classification of a family of generalized solenoids, originally suggested by Williams.

2. Conjugacy and shift equivalence in pointed topological spaces

We will formulate a pointed version of R. F. Williams' classification of 1-solenoids ([16]).

Let K denote a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and suppose $f : K \to K$ is a continuous map.

Consider the following version of Williams' axioms taken from [17]:

AXIOMS 2.1.

1: *K is dynamically indecomposable* (connected, *in this context*).
2: *All points of K are nonwandering under f.*
3: (*Flattening*) *There is a $k \geq 1$ such that for all $x \in K$, there is an open neighborhood U of x such that $f^k(U)$ is an arc.*
4: *f is uniformly expanding on sufficiently small arcs.*
5: *$f^n|_{K \setminus \mathcal{V}}$ is locally 1-1 for $n > 0$.*
6: *$f(\mathcal{V}) \subset \mathcal{V}$.*

A **Williams solenoid** is defined to be the inverse limit space of a pair (K, f) satisfying each of Axioms 2.1. Such a pair will be called a **presentation**. The notation (K, f, x) will mean that $f(x) = x$. Just to fix notation, put

$$\varprojlim(K, f) = \{(x_1, x_2, \dots) : f(x_{n+1}) = x_n, \text{ for } n \geq 1\}$$

If $f(x) = x$, let $\bar{x}$ denote $(x, x, x, \dots)$. Let $\bar{f}$ denote the shift map on $\varprojlim(K, f)$ given by $\bar{f}(x_1, x_2, \dots) = (f(x_1), x_1, x_2, \dots)$.

Two graph presentations (K, f) and (K', f') are **shift equivalent** of lag k if there exist maps $r : K \to K'$ and $s : K' \to K$, and a positive integer k, such that $f^k = s \circ r$, $(f')^k = r \circ s$, $f \circ s = s \circ f'$, and $f' \circ r = r \circ f$. If (r, s) is such a shift equivalence, then $\bar{r} : \varprojlim(K, f) \to \varprojlim(K', f')$ given by $\bar{r}(x_1, x_2, \dots) = (r(x_1), r(x_2), \dots)$ is a homeomorphism defining a conjugacy between $\bar{f}$ and $\bar{f}'$

The presentation (K, f) is **elementary** provided K is homeomorphic with a wedge of circles and f fixes the branch point of K.

A key result about solenoids is in R. F. Williams [16, Theorem 3.3].

THEOREM 2.2. *Suppose that (K, f) and (K', f') are presentations. There is a homeomorphism $h : \varprojlim(K, f) \to \varprojlim(K', f')$ conjugating $\bar{f}$ with $\bar{f}'$ if and only if there is a shift equivalence (r, s) from f to f' such that $h = \bar{r}$.*

REMARK 2.3. It should be noted that this result does not depend on base points, unlike results about π_1 representations, which we now consider.

Shift equivalence categorically extends to homomorphisms of fundamental groups.

Suppose given homomorphisms $\phi : \pi_1(K, x) \to \pi_1(K, x)$ and $\psi : \pi_1(K', x') \to \pi_1(K', x')$, there exist group homomorphisms $r : \pi_1(K', x') \to \pi_1(K, x)$ and $s : \pi_1(K, x) \to \pi_1(K', x')$ such that

$$\phi^\ell = rs\,, \quad \psi^\ell = sr\,, \quad r\psi = \phi r\,, \quad \text{and} \quad s\psi = \phi r\,.$$

Then we say that ϕ and ψ are **shift equivalent of lag** ℓ.

EXAMPLE 2.4. Let K and K' denote wedges of two circles and let (K, f) and (K', f') be elementary presentations defined by the "wrapping rules":

$$\left\{ f : \begin{array}{c} a \to abba \\ b \to aba \end{array} \right\} \quad \text{and} \quad \left\{ f' : \begin{array}{c} \alpha \to \alpha\beta\alpha\beta\alpha \\ \beta \to \alpha \end{array} \right\},$$

where a, b are the oriented edges of K and α, β those of K'. In Example 4.5 we will show that the free group endomorphisms $\phi = \pi_1(f, p)$ and $\psi = \pi_1(f', p')$ are **not** shift equivalent, with p and p' the branch points of K, K'. We show now, however, that f and f' are shift equivalent as maps.

Let x denote the fixed point of f in the interior of edge b, and let x' denote the fixed point of f' in the interior of edge α. Then x splits b into oriented edges b_1 and b_2 and x' splits α into oriented edges α_1 and α_2. Symbolically, $b = b_1 b_2$ and $\alpha = \alpha_1\alpha_2$. Let $r : K \to K'$ and $s : K' \to K$ be given symbolically by

$$\left\{ r : \begin{array}{c} a \to \alpha_2\beta\alpha_1 \\ b \to \alpha_2\alpha_1 \end{array} \right\} \quad \text{and} \quad \left\{ s : \begin{array}{c} \alpha \to b_2 a a b_1 \\ \beta \to b_2 b_1 \end{array} \right\},$$

Then (r, s) is a shift equivalence of (K, f) with (K', f'). In this example, (K, f) and (K', f') are different presentations of the same Williams solenoid, but with different fixed points of the solenoids corresponding to the branch points p and p'.

3. The Fox calculus

We will state some basic facts from the free differential calculus originated by R. H. Fox ([11]). We have adapted the simple treatment due to J. Birman ([6]) to the needs of this paper, and the reader can consult that resource for proofs in this section.

Let $\mathcal{F}(a_1, \ldots, a_n)$ be the free group with basis $a_1, \ldots, a_n$. Suppose G denotes an arbitrary factor group of $\mathcal{F}(a_1, \ldots, a_n)$.

Let $\mathbf{Z}G$ denote the integer group ring of all formal sums

$$\sum_{g \in G} a_g \cdot g\,,$$

with $a_g \in \mathbf{Z}$ and $a_g = 0$ for all but finitely many terms (with $0 \cdot g \equiv 0$). Define addition and multiplication in $\mathbf{Z}G$ by

$$\sum a_g\, g + \sum b_g\, g = \sum (a_g + b_g) g$$

$$\left(\sum a_g\, g\right)\left(\sum b_g\, g\right) = \sum_g \left(\sum_h a_{gh^{-1}} b_h\right) g\,.$$

An element g of G acts by left multiplication to define a $\mathbf{Z}G$ automorphism by

$$g \cdot \left(\sum_k n_k \; g_k \right) \equiv \sum_k n_k \; g \; g_k \, .$$

Of course, $\mathbf{Z}G$ is a $\mathbf{Z}$-module.

For $j = 1, \ldots, n$, there is a unique $\mathbf{Z}$-module homomorphism

$$\frac{\partial}{\partial a_j} : \mathbf{Z}\mathcal{F}(a_1, \ldots, a_n) \to \mathbf{Z}\mathcal{F}(a_1, \ldots, a_n)$$

such that

(i) $\dfrac{\partial a_i}{\partial a_j} = \delta_{i,j}$

(ii) $\dfrac{\partial a_i^{-1}}{\partial a_j} = -\delta_{i,j} a_i^{-1}$

(iii) $\dfrac{\partial (wv)}{\partial a_j} = \left(\dfrac{\partial w}{\partial a_j} \right) + w \cdot \left(\dfrac{\partial v}{\partial a_j} \right)$.

REMARK 3.1. If a word w in the generators $\{a_i\}$ contains no inverses, which is often the case in our applications, the Fox partial derivatives are very easy to compute: $\frac{\partial w}{\partial a_i} = \sum p_{i,j}$ where $p_{i,j}$ denotes the prefix of the j^{th} occurrence of a_i in w. If w begins with a_i, the prefix is defined to be "1". Inverses only slightly complicate this algorithm.

Suppose $\mathcal{F}(a_1, \ldots, a_n)$, $\mathcal{F}(b_1, \ldots, b_m)$, and $\mathcal{F}(c_1, \ldots, c_p)$ are free groups. A homomorphism $\phi : \mathcal{F}(b_1, \ldots, b_m) \to \mathcal{F}(a_1, \ldots, a_n)$ replaces each occurrence of the letter b_k in a word of $\mathcal{F}(b_1, \ldots, b_m)$ by a specific word in $\mathcal{F}(a_1, \ldots, a_n)$. We extend the group homomorphim to a homomorphism $\phi : \mathbf{Z}\mathcal{F}(b_1, \ldots, b_m) \to \mathbf{Z}\mathcal{F}(a_1, \ldots, a_n)$ by

$$\left(\sum a_g \, g \right)^{\phi} \equiv \sum a_g \, \phi(g) \, .$$

Finally, if M denotes the matrix $\left(m_{i,j} \right)$ over $\mathbf{Z}G$, then M^{ϕ} denotes the matrix $\left(\phi(m_{i,j}) \right)$ over $\mathbf{Z}\phi(G)$.

Given $\phi : \mathcal{F}(b_1, \ldots, b_m) \to \mathcal{F}(a_1, \ldots, a_n)$, define the free jacobian matrix with entries in $\mathbf{Z}\mathcal{F}(a_1, \ldots, a_n)$ as follows:

$$\mathcal{D}\phi = \begin{pmatrix} \dfrac{\partial \phi(b_1)}{\partial a_1} & \dfrac{\partial \phi(b_1)}{\partial a_2} & \cdots & \dfrac{\partial \phi(b_1)}{\partial a_n} \\ \dfrac{\partial \phi(b_2)}{\partial a_1} & \dfrac{\partial \phi(b_2)}{\partial a_2} & \cdots & \dfrac{\partial \phi(b_2)}{\partial a_n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \dfrac{\partial \phi(b_m)}{\partial a_1} & \dfrac{\partial \phi(b_m)}{\partial a_2} & \cdots & \dfrac{\partial \phi(b_m)}{\partial a_n} \end{pmatrix} .$$

Consider the chain of free group homomorphisms

$$\mathcal{F}(c_1, \ldots, c_p) \xrightarrow{\psi} \mathcal{F}(b_1, \ldots, b_m) \xrightarrow{\phi} \mathcal{F}(a_1, \ldots, a_n)\,.$$

Then the chain rule $\mathcal{D}(\phi\,\psi) = (\mathcal{D}\psi)^{\phi} \cdot \mathcal{D}\phi$ holds.

REMARK 3.2. Although we cannot find a particular reference with the chain rule written in this fashion, it is equivalent to the chain rule in J. Birman ([6]), originally due to R. H. Fox.

EXAMPLE 3.3. Define a homomorphism $\rho : \mathcal{F}(a, b) \to \mathcal{F}(\alpha, \beta)$ by $\rho(a) = \alpha\beta\alpha$ and $\rho(b) = \alpha$. Also define $\sigma : \mathcal{F}(\alpha, \beta) \to \mathcal{F}(a, b)$ by $\sigma(\alpha) = ab$ and $\sigma(\beta) = b$.

Then $\rho\sigma(\alpha) = \alpha\beta\alpha\alpha$ and $\rho\sigma(\beta) = \alpha$. Also

$$\mathcal{D}(\rho\sigma) = \begin{pmatrix} 1 + \alpha\beta + \alpha\beta\alpha & \alpha \\ 1 & 0 \end{pmatrix}$$

$$(\mathcal{D}\sigma)^{\rho} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{\rho} = \begin{pmatrix} 1 & \alpha\beta\alpha \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{D}\rho = \begin{pmatrix} 1 + \alpha\beta & \alpha \\ 1 & 0 \end{pmatrix}$$

4. Bowen-Franks factor groups and trace invariants

Given a free group endomorphism $\phi : \mathcal{F}(a_1, \ldots, a_n) \to \mathcal{F}(a_1, \ldots, a_n)$. The factor group G_ϕ generated by the relations $[\{\phi(a_k) \equiv a_k\}_k, gh \equiv hg]$ is called the **Bowen-Franks group** of ϕ ([5]). Let $bf : \mathcal{F}(a_1, \ldots, a_n) \to G_\phi$ denote the natural homomorphism.

It is easily checked that if $\tau : G_\phi \to G_\psi$ is an isomorphism, then the induced homomorphism $\tau : \mathbf{Z}G_\phi \to \mathbf{Z}G_\psi$ is an isomorphism of the associated group rings.

In the absence of strong clues, it can be quite arduous to decide whether two distinct endomorphisms are shift equivalent. As part of his program to classify hyperbolic attractors, and as a kind of test case, R. F. Williams ([16]) sought to determine the shift equivalence classes of all—there are 46—free group endomorphisms ϕ arising as actions of orientation preserving presentations on fundamental groups whose induced abelianizations A share the characteristic polynomial $t^2 - 3t - 2$.

By restricting to shift equivalences that preserve the number of generators, which are more computable, Williams found the following (see Table 1):

FACT 4.1. Under the equivalence relation of shift equivalence of group endomorphisms induced on the fundamental group, based at the branch point, the elementary presentations with characteristic polynomial $t^2 - 3t - 2$ fall into at most the four equivalence classes represented in Table 1.

TABLE 1. Elementary Presentations with Characteristic Polynomial $t^2 - 3t - 2$

I	II	III	IV
$g_1(a) = a^2b^2a$ $g_1(b) = a$	$g_2(a) = ababa$ $g_2(b) = a$	$g_3(a) = a^2b^4$ $g_3(b) = ab$	$g_4(a) = ab^2a$ $g_4(b) = aba$

Previously obtained results on this classification problem are as follows: The fact that classes (I) and (II) are distinct shift equivalence classes was first established in [15], using combinatorial group theory, then in [17], using shift of finite type covers.

The strongest result along these lines (see [1]) is that the pair of 1-solenoids $\varprojlim(K, g_1)$ and $\varprojlim(K, g_2)$ not only fail to have conjugate shift maps but are not even homeomorphic topological spaces. The authors of [10] use the bf-trace invariant (see below) to show the following:

FACT 4.2. Classes (I) and (III) in Table 1 are distinct and are each distinct from (II) and (IV).

This leaves only classes (II) and (IV). We will show (in Example 4.5 below) that $\pi_1(g_2, p)$ and $\pi_1(g_4, p)$ are not shift equivalent in the category of free group endomorphisms, p the branch point, using the same bf-trace invariant applied to the squares $\pi_1(g_2, p)^2$ and $\pi_1(g_4, p)^2$.

We will need the following definition (see [10]).

Let $(\mathcal{D}\phi)^{bf}$ be the Bowen-Franks reduced Fox jacobian matrix as in Section 3. Then the element $\mathcal{T}(\phi) \equiv \mathrm{Trace}((\mathcal{D}\phi)^{bf})$ in $\mathbf{Z}G_\phi$ will be called the **bf-trace of** ϕ.

Provided the Bowen-Franks groups aren't too large, the bf-trace is useful for separating shift equivalence classes. Comparing traces is greatly facilitated by further reducing them modulo some integer. The following result was proved in [10, Theorem 4.3] by a different argument.

PROPOSITION 4.3. *If ϕ and ψ are shift equivalent of lag 1, then there is an isomorphism (of Bowen-Franks groups) $r_* : G_\phi \to G_\psi$ such that $\mathcal{T}(\phi) = \mathcal{T}(\psi)^{r_*}$.*

PROOF. So we have $\phi = rs$ and $\psi = sr$. As observed in [5], there are canonical isomorphisms

$$\phi_* : G_\phi \to G_\phi\,, \quad \psi_* : G_\psi \to G_\psi\,, \quad r_* : G_\psi \to G_\phi \quad \text{and} \quad s_* : G_\phi \to G_\psi\,,$$

that are factor maps, respectively, of the homomorphisms ϕ, ψ, r, and s.

For ease in notation, the choice of Bowen Franks reduction ($bf = bf_\phi$) is to be understood by the context.

In particular, $bf\, r = r_*\, bf$ and $bf\, s = s_*\, bf$. Also $r_*s_* = id$ and $s_*r_* = id$.

Applying the chain rule and Bowen-Franks reductions yields

$$(\mathcal{D}\phi)^{bf} = (\mathcal{D}rs)^{bf} = ((Ds)^r Dr)^{bf} = (Ds)^{r\,bf}(Dr)^{bf}$$

$$((D\psi)^{bf})^{r_*} = ((Dr)^{s\,bf}(Ds)^{bf})^{r_*} = (Dr)^{bf\,s_* r_*}(Ds)^{bf\,r_*} = (Dr)^{bf}(Ds)^{r\,bf}\,.$$

That completes the proof, since, for all compatible matrix pairs A, B, we know that trace$(AB) =$ trace(BA). □

COROLLARY 4.4. *If ϕ and ψ are shift equivalent (of arbitrary lag) then for all $k \geq 1$, $\mathcal{T}(\phi^k) = \mathcal{T}(\psi^k)^{r_{k*}}$, where r_{k_*} denotes an isomorphism of Bowen-Franks groups of the powers ϕ^k and ψ^k.*

PROOF. If ϕ and ψ are shift equivalent, then so are ϕ^k and ψ^k by [16]. Every shift equivalence can be decomposed into a finite chain of lag 1 shift equivalences (Lemma 4.6 in [16]). By Proposition 4.3, each lag 1 shift equivalence preserves the trace up to Bowen-Franks isomorphism. □

EXAMPLE 4.5. Define a pair of free group homomorphisms in terms of generators, as follows:

$$[a, b] \xrightarrow{\phi} [abba, aba] \quad [\alpha, \beta] \xrightarrow{\psi} [\alpha\beta\alpha\beta\alpha, \alpha]\,.$$

These endomorphisms arise from the shift equivalence classes (IV) and (II) in the table above. Our aim is to show that these lie in distinct shift equivalence classes.

We note $G_{\phi^2} \cong \mathbf{Z}_8$, where $b^8 = 1, a = b^6$. Similarly $G_{\psi^2} \cong \mathbf{Z}_8$, where $\beta^8 = 1, \alpha = \beta^5$.

A computation shows $\mathcal{T}(\phi) = [1+ab^2+a]^{bf} = 2+b^2$ and $\mathcal{T}(\psi) = [1+\alpha\beta+(\alpha\beta)^2]^{bf} = 2 + \beta^2$, so not much help here. Passing to the second powers, we obtain the Bowen-Franks groups: $G_{\phi^2} = \langle b \rangle \cong \mathbf{Z}_8,\ a = b^6$ and $G_{\psi^2} = \langle \beta \rangle \cong \mathbf{Z}_8,\ \alpha = \beta^5$. The integer term in the monomial expansion of $\mathcal{T}(\phi^2)$ is 4, whereas the integer part of $\mathcal{T}(\psi^2)$ equals 2. By Corollary 4.4, there is no shift equivalence between ϕ and ψ.

PROPOSITION 4.6. *There exists an elementary presentation, satisfying Axioms 2.1, $f : K \to K$ with branch point p and a fixed point x such that $\pi_1(f, p) = \phi$ and $\pi_1(f, x) = \theta$ with ϕ and θ* **not** *shift equivalent.*

PROOF. Let (K, f) be as in Example 2.4, with branch point $p \in K$ and second fixed point x in edge b splitting b as $b = b_1 b_2$. Then $\pi_1(K, x)$ has generators $\hat{a} = b_2 a b_2^{-1}$ and $\hat{b} = b_2 b b_2^{-1}$. Apply f to these generators to get

$$f(\hat{a}) = f(b_2 a b_2^{-1}) = (b_2 a)(abba)(a^{-1} b_2^{-1}) = \hat{a}\hat{a}\hat{b}\hat{b}$$

$$f(\hat{b}) = f(b_2 b b_2^{-1}) = (b_2 a)(aba)(a^{-1} b_2^{-1}) = \hat{a}\hat{a}\hat{b}\,.$$

Thus, $\theta = \pi_1(f, x)$ has the form $[\hat{a}, \hat{b}] \to [\hat{a}\hat{a}\hat{b}\hat{b}, \hat{a}\hat{a}\hat{b}]$. Let (K', f') be as in Example 2.4 with K' the wedge of circles α and β, p' the branch point of K'. Define the group homomorphisms $s : \pi_1(K, x) \to \pi_1(K', p')$ and $r : \pi_1(K', p') \to \pi_1(K, x)$ by

$$\begin{array}{ll} s(\hat{a}) = \alpha\beta & r(\alpha) = \hat{a}\hat{a}\hat{b} \\ s(\hat{b}) = \alpha & r(\beta) = \hat{b}\,. \end{array}$$

Then (r, s) is a shift equivalence of $\theta = \pi_1(f, x)$ with $\pi_1(f', p') = \psi : [\alpha, \beta] \to [\alpha\beta\alpha\beta\alpha, \alpha]$. In Example 4.5 we saw that ψ and $\phi = \pi_1(f, p)$ are not shift equivalent; therefore, neither are $\phi = \pi_1(f, p)$ and $\theta = \pi_1(f, x)$. □

We return to the four part classification of R. F. Williams' endomorphisms in Table 1.

PROPOSITION 4.7. *Suppose K is a wedge of two circles. There are exactly three conjugacy classes of shift homeomorphisms $\bar{f} : \varprojlim(K, f) \to \varprojlim(K, f)$, for which (K, f) is an elementary presentation such that the abelianizations have the characteristic polynomial $t^2 - 3t - 2$. There are exactly five pointed conjugacy classes of such shifts.*

PROOF. There are 46 positive endomorphisms of the free group on two generators whose abelianizations share the characteristic polynomial $t^2 - 3t - 2$, and each of these is shift equivalent to one induced by an elementary presentation in Table 1 by Fact 4.1.

Each of the presentations (K, g_i), $i = 1, \dots, 4$ fixes the branch point p and an additional point $x_i \neq p$. For each i there is an elementary presentation (K_i, f_i) with (f_i, p) pointedly shift equivalent to (g_i, x_i). We need to make this precise.

Moving the basepoint to x_i (as in Example 4.5), one finds $K_i = K$, $i = 1, 2, 4$, is a wedge of two circles, K_3 is a wedge of three circles and the wrapping rules are as follows: $f_1 : [a, b] \to [baa, ba^4]$; $f_2 : [a, b] \to [ab^2a, aba]$; $f_3 : [a, b, c] \to [ab^4c, abc, ac]$ and $f_4 : [a, b] \to [ababa, a]$. Note that $f_2 = g_4$ and $f_4 = g_2$.

We explicitly show that $\pi_1(f_1, p)$ is shift equivalent to $\pi_1(g_1, p)$ as follows: The conjugacy $[a, b] \to [\alpha, \beta]$, $\alpha = a^{-1}$, $\beta = b^{-1}$, yields the homomorphism $\theta : [\alpha, \beta] \to [\alpha\alpha\beta, \alpha^4\beta]$ shift equivalent with $\pi_1(f_1, p)$. Now the pair of homomorphisms $r(\alpha) = a$, $r(\beta) = bba$ and $s(a) = \alpha\alpha\beta$, $s(b) = \alpha$ defines a shift equivalence between θ and $\pi_1(g_1, p)$.

Since a conjugacy between $\bar{f}$ on $\varprojlim(K, f)$ and $\bar{g}$ on $\varprojlim(K, g)$ must arise from a shift equivalence between f and g (Theorem 2.2), the foregoing calculations show that there are at most three conjugacy classes of shifts $\bar{f} : \varprojlim(K, f) \to \varprojlim(K, f)$ represented by $\bar{g}_1$, $\bar{g}_2$, and $\bar{g}_3$. Now if there is a conjugacy from $\bar{g}_i$ to $\bar{g}_j$,for some $i \neq j$, then the conjugacy either fixes the point $\bar{p} = (p, p, \dots)$ or maps $\bar{p}$ to $\bar{x_j}$. Thus, either the group endomorphisms $\pi_1(g_i, p)$ and $\pi_1(g_j, p)$ are shift equivalent, or $\pi_1(g_i, p)$ and $\pi_1(f_j, p)$ are shift equivalent.

We will know there are exactly three (unpointed) conjugacy classes if we verify that $\pi_1(f_3, p)$ is not shift equivalent to either $\pi_1(g_1, p)$ or $\pi_1(g_2, p)$, as group endomorphisms. The bf-trace $\tau(\pi_1(f_3, p)) = 3$ is distinct from the constant terms of both $\tau(\pi_1(g_1, p))$ and $\tau(\pi_1(g_2, p))$ (see Example 4.5). Thus, (using Fact 4.2 and Example 4.5) the conjugacy classes of $\bar{g}_i$, $i = 1, 2, 3$, are all distinct.

We now want to show that $\pi_1(f_3, p)$ and $\pi_1(g_3, p)$ are not shift equivalent as group endomorphisms[1]. We claim that $\bar{f}_3 : \varprojlim(K_3, f_3) \to \varprojlim(K_3, f_3)$ and $\bar{g}_3 : \varprojlim(K, g_3) \to \varprojlim(K, g_3)$ are not conjugate by a conjugacy taking the fixed point $\bar{p}$ of $\bar{g}_3$ to the fixed point $\bar{p}$ of $\bar{f}_3$. The branched manifolds K and K_3 are naturally oriented and g_3, f_3 preserve the orientation. Hence, the inverse limits are orientable and any homeomorphism from $\varprojlim(K_3, f_3)$ to $\varprojlim(K_3, f_3)$ either preserves or reverses the orientation. Suppose that h is a conjugacy of $\bar{g}_3$ with $\bar{f}_3$ taking $\bar{p}$ to $\bar{p}$ with corresponding shift equivalences (r, s) from g_3 to f_3 (as in Theorem 2.2). Then r and s both preserve or both reverse the orientation. Thus, the free group homomorphism r_* takes letters a, b to words in a, b, c or to words in $\alpha = a^{-1}$, $\beta = b^{-1}$, $\gamma = c^{-1}$, and similarly for s_*. Now $(g_3)_*$ has a unique periodic right infinite word: $w = w_1 w_2 \cdots$ with each $w_i \in \{a, b\}$ such that $(g_3)_*^k(w) := (g_3)_*^k(w_1)(g_3)_*^k(w_2) \cdots = w$. In our case, $k = 1$ and $w = aabbbbaabbbbabababab\cdots$. Similarly, $(f_3)_*$ has unique positive and negative periodic right infinite words $u = abbbbcabcabcabcabcabcac\cdots$ and $v = \gamma\alpha\gamma\beta\beta\beta\beta\alpha\gamma\alpha\cdots$. The homomorphism r_* must either take w to u or w to v (if orientation reversing). A straight forward inductive argument shows that neither of the words u or v has a nonempty prefix that is a square. Since $r_*(w) = r_*(a)r_*(a)r_*(b)\cdots = xxr_*(b)\cdots$, we see that $\bar{g}_3$ and $\bar{f}_3$ cannot be conjugate by a homeomorphism that takes $\bar{p}$ to $\bar{p}$.

We can now infer that there are five distinct pointed conjugacy classes represented by $(\bar{g}_i, \bar{p})$, for $i = 1, \ldots, 4$ and $(\bar{g}_3, \bar{x}_3)$. □

5. Ordered Čech equivalence of augmented solenoids

If the graph K is oriented and the presentation (K, f) is orientation preserving, there is a natural nonsingular flow φ_t on the Williams solenoid $\varprojlim(K, f)$ that satisfies $\bar{f}(\varphi_t(\bar{x})) = \varphi_{\lambda t}(\bar{f}(\bar{x}))$, for all $\bar{x} = (x_1, x_2, \ldots) \in \varprojlim(K, f)$, and λ the Perron-Frobenius eigenvalue of the abelianization of f. In case $\varprojlim(K, f)$ is homeomorphic to the inverse limit of $z \mapsto z^n$ on the circle—a *classical* solenoid—then φ_t is a translation, and isometry, on a compact abelian group. Otherwise, there are a finite number of **forward asymptotic orbits** of the flow; that is, there exist $\bar{x} \neq \bar{y}$ such that $d(\varphi_t(\bar{x}), \varphi_t(\bar{y})) \to 0$ as $t \to \infty$ (similarly for **backward asymptotic orbits**). In case (K, f) and (K', f') are two orientation preserving presentations, an orientation preserving homeomorphism of $\varprojlim(K, f)$ onto $\varprojlim(K', f')$ must take forward (backward) asymptotic orbits to forward (backward) asymptotic orbits ([1], Lemma 3.5). Moreover, each asymptotic orbit contains a unique periodic point of $\bar{f}$ so that a homeomorphism between $\varprojlim(K, f)$ and $\varprojlim(K', g)$ must, up to isotopy, map the collection of asymptotic periodic points, say $\mathcal{P}_f^{\pm}$, to the like collection $\mathcal{P}_g^{\pm}$ of $\varprojlim(K', g)$ ([1], Proof of Thm. 3.10), where the $\pm$ superscript refers to forward $(+)$ or backward $(-)$ asymptotic orbits.

[1] One can prove that it is not possible to distinguish the shift equivalence classes of $\pi_1(f_3, p)$ and $\pi_1(g_3, p)$ based solely on Bowen-Franks traces of powers.

The orientation on $\varprojlim(K, f)$ induces an order structure on the Čech cohomology $\check{\mathrm{H}}^1(\varprojlim(K, f), \mathcal{P}_f)$, with integer coefficients, and $\mathcal{P}_f = \mathcal{P}_f^+,\ \mathcal{P}_f^-,$ or $\mathcal{P}_f^+ \cup \mathcal{P}_f^-$. This ordered group is called the **augmented cohomology group** of $\varprojlim(K, f)$. The augmented cohomology group is closely related to the augmented dimension group employed by Carlsen and Eilers to study substitutive systems ([7], [8]). The latter group is, in turn, a manifestation of the Matsumoto K_0 group ([14]) arising in C^*-algebra theory. In this section we use the following theorem to separate Williams solenoids.

THEOREM 5.1. [4] *If (K, f) and (K', g) are orientation preserving presentations, and there is an orientation preserving homeomorphism between the Williams solenoids $\varprojlim(K, f)$ and $\varprojlim(K', g)$, then the augmented cohomology groups $\check{\mathrm{H}}^1(\varprojlim(K, f), \mathcal{P}_f)$ and $\check{\mathrm{H}}^1(\varprojlim(K', g), \mathcal{P}_g)$ are order isomorphic.*

The group $\check{\mathrm{H}}^1(\varprojlim(K, f))$ is determined entirely by the abelianization of f. The space $\varprojlim(K, f)/\mathcal{P}_f$ contains some additional cocycles that may intertwine in an algebraically nontrivial fashion with the generators of $\check{\mathrm{H}}^1(\varprojlim(K, f))$. In this way, $\check{\mathrm{H}}^1(\varprojlim(K, f), \mathcal{P}_f)$ can capture at least some of the nonabelian character of f, as we will see in the sequel. For any such augmented cohomology group, there is a recipe (see [4]) for constructing an "augmented matrix" whose dimension group is order isomorphic to the augmented cohomology group. In the extended discussion at the end of this section, we will explain in more detail this recipe in the context of the Williams examples.

5.1. Ordered dimension groups for augmented matrices. Suppose A denotes a $d \times d$ nonsingular matrix over the integers called the **base matrix**. The nonsingularity will greatly simplify the exposition (see Remark 5.3). Let I denote the $k \times k$ identity matrix, and E denote a $k \times d$ rational matrix. Then an **augmented matrix** (A, E) is a $d + k$ square matrix of the special form

$$(A, E) = \begin{pmatrix} A & O \\ E & I \end{pmatrix}.$$

We will say A is augmented by E. The reader can verify that $(A, E)^{-1} = (A^{-1}, -EA^{-1})$, and $(A, E) * (B, F) = (AB, EB + F)$.

DEFINITION 5.2. The **dimension group** of a nonsingular matrix M is defined to be the infinite union $\mathcal{D}_M = \bigcup_{m \geq 0} M^{-m}\mathbf{Z}^n$. When $M = (A, E)$ with E a $k \times d$ integral matrix, we will say that $\mathcal{D}_{A,E}$ is a k-augmented dimension group.

REMARK 5.3. While, for simplicity, we assume M is nonsingular in the sequel, suitably adapted results would hold for the singular case. If $\det M = 0$, one replaces M by the restriction of M to the eventual range $\mathcal{R}(M) = \bigcap_m M^m(\mathbf{Q}^n)$.

We now show that $\mathcal{D}_{A,E}$ has a very simple product structure in most cases.

THEOREM 5.4. *Given the k-augmented dimension group $\mathcal{D}_{A,E}$ such that A and $A-I$ are nonsingular, there exist a subgroup $\mathcal{D}(E) \subset \mathcal{D}_A$ and a lattice $\Lambda_k = \sum_{i=1}^{k} \mathbf{Z} \cdot (p_i, q_i)$ isomorphic to $\mathbf{Z}^k$, for some $p_i \in \mathbf{Z}^d$ and $q_i \in \mathbf{Q}^k$, such that $\mathcal{D}_{A,E}$ is isomorphic to $\mathcal{D}(E) \oplus \Lambda_k$. The subgroup $\mathcal{D}(E)$ has the form $\bigcup_{m\geq 0} A^{-m}\mathcal{P}_E$ where $\mathcal{P}_E = \{p \in \mathbf{Z}^d : E(A-I)^{-1}p \in \mathbf{Z}^k\}$.*

REMARK 5.5. Even though (A, E) can be block diagonalized to $(A, 0) = A \oplus I_k$, this similarity transformation does not imply that the dimension group $\mathcal{D}_{A,E}$ is isomorphic to $\mathcal{D}_A \oplus \mathbf{Z}^k$ as might be expected, unless the similarity transformation is invertible over $\mathbf{Z}$. In the sequel we will equate $\mathcal{D}_{A,E}$ with the particular, easy to describe, product structure $\mathcal{D}'_{A,E} = \mathcal{D}(E) \oplus \Lambda_k$ of Theorem 5.4.

Let $\mathcal{M}_{k,d}$ denote the group of $k \times d$ integral matrices. Theorem 5.4 shows that there are at most $k_0 = \left|\mathcal{M}_{k,d}/\mathcal{M}_{k,d}(A-I)\right|$ isomorphism classes of augmented dimension groups. We can relate this to the Bowen-Franks group (Section 4) $bfg(A) = \mathbf{Z}^d/\mathbf{Z}^d(A-I)$.

COROLLARY 5.6. *There are at most*

$$k_0 = |bfg(A)|^k \leq |\det(A-I)|^k$$

isomorphism classes for the family of k-augmented dimension groups $\mathcal{D}_{A,E}$.

Now we prove Theorem 5.4.

PROOF. It will be convenient to replace $\mathcal{D}_{A,E}$ with the isomorphic group

$$\mathcal{D}'_{A,E} = \begin{pmatrix} I & 0 \\ -E(A-I)^{-1} & I \end{pmatrix} \mathcal{D}_{A,E} = \{(A^{-m}p, -E(A-I)^{-1}p + z)\}_{m,p,z}\,,$$

for all nonnegative integers m, and all $(p, z) \in \mathbf{Z}^d \times \mathbf{Z}^k$.

The group $(-E(A-I)^{-1}\mathbf{Z}^d) + \mathbf{Z}^k$ is a submodule of the free $\mathbf{Z}$-module $\det((A-I)^{-1}))\mathbf{Z}^k$ and hence is a free $\mathbf{Z}$-module having a set of k generators (nonunique) $\{q_1, \ldots, q_k\} \subset \mathbf{Q}$.

As a consequence, there exist vectors $(p_i, z_i) \in \mathbf{Z}^d \times \mathbf{Z}^k$ such that $-E\,(A-I)^{-1}\,p_i + z_i = q_i$. Put $v_i = (p_i, q_i)$ for each i.

Define the subgroup $\mathcal{P}_E = \{\tilde{p} \in \mathbf{Z}^d : E\,(A-I)^{-1}\,\tilde{p} \in \mathbf{Z}^k\}$.

Then we want to establish the following splitting:

$$\mathcal{D}'_{A,E} = \bigcup_{m\geq 0} A^{-m}\mathcal{P}_E \oplus \sum \mathbf{Z} \cdot v_i\,.$$

We need to solve the equation $(A^{-m}p, -E(A-I)^{-1}p+z) = (A^{-m}\tilde{p}, 0) + \sum_i \ell_i v_i$ for $\tilde{p} \in \mathcal{P}_E$ and $\ell_i \in \mathbf{Z}$. From the definition of the set $\{q_i\}$, there exist unique integers $\{\ell_i\}$ such that $\sum_i \ell_i q_i = -E\,(A-I)^{-1}\,p + z$.

We are left with $A^{-m}\tilde{p} = A^{-m}p - \sum \ell_i p_i$ to be solved for $\tilde{p} \in \mathcal{P}_E$. But we can rewrite this as

$$\tilde{p} = p - A^m\Big(\sum_i \ell_i p_i\Big) = p - (A^m - I)\Big(\sum_i \ell_i p_i\Big)$$
$$- \sum_i \ell_i p_i = p - \sum_i \ell_i p_i + (A - I)t$$

for some $t \in \mathbf{Z}^d$. So, it is enough to show $p - \sum_i \ell_i p_i \in \mathcal{P}_E$. We have the two equations:

$$-E\,(A-I)^{-1}\,p + z = \sum_i \ell_i q_i \quad \text{and} \quad -\,E\,(A-I)^{-1}\left(\sum_i \ell_i p_i\right) + z_0 = \sum_i \ell_i q_i\ .$$

Subtracting yields the conclusion that $E(A-I)^{-1}(p - \sum_i \ell_i p_i) \in \mathbf{Z}^k$.

For the reverse inclusion, fixing $m = 0$, $p = p_i$ and $z = z_i$, then $(p_i, q_i) \in \mathcal{D}'_{A,E}$, for each $i = 1, \ldots, k$. If each $\ell_i = 0$, we want to show that $A^{-m}\tilde{p} \in \mathcal{D}'_{A,E}$, when there exists $z \in \mathbf{Z}^k$ such that $-E\,(A-I)^{-1}\,\tilde{p} + z = 0$. Hence, $p = \tilde{p}$ and $z = E\,(A-I)^{-1}\,\tilde{p}$ will do. □

EXAMPLE 5.7. Consider the 1-augmented matrices $A_j = \begin{pmatrix} 8 & 0 \\ j & 1 \end{pmatrix}$. For $j = 0, 1$, these matrices appear in [7]. An explicit computation (using Theorem 5.4) shows that if $j \neq 0$ and $a_j = (7-j)^{-1} \bmod 7$, then

$$\mathcal{D}'_{A_j} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \mathbf{Z}[1/2] \oplus \begin{pmatrix} a_j \\ 1/7 \end{pmatrix} \mathbf{Z}\,.$$

If $j = 0$, $\mathcal{D}'_{A_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{Z}[1/2] \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{Z}$. All seven dimension groups are isomorphic.

As described at the beginning of the section, the topological invariant underlying augmented dimension groups is ordered Čech cohomology. In case u is a nonnegative left Perron-Frobenius eigenvector for an augmented matrix (A, E), an order is determined in $\mathcal{D}_{A,E}$ by $x \geq y \Leftrightarrow (x-y)\cdot u \geq 0$. A homeomorphism between oriented Williams solenoids either preserves or reverses orientations and the induced isomorphism on augmented cohomology groups (augmented dimension groups) either preserves or reverses the order. In either event, the induced isomorphism must preserve orthogonality to Perron-Frobenius eigenvectors. With this in mind, for $v \in \mathbf{R}^n$ define $v^\perp = \{u \in \mathbf{R}^n :\ v \cdot u = 0\}$.

In terms of augmented dimension groups, the order structure is reflected in the following definition ((see [4] and [7] for more details).

Suppose we are given two augmented matrices (A, E) and (B, F) having, respectively, the left Perron-Frobenius eigenvectors u and v. Then the isomorphism $C : \mathcal{D}_{A,E} \to \mathcal{D}_{B,F}$ is an **augmented order isomorphism** if $C(u^\perp) = v^\perp$.

Order isomorphisms yield a new topological invariant, as follows:

Suppose $A - I$ is nonsingular. Define the **crossing group** of the augmented matrix (A, E) to be the quotient group

$$\mathcal{CG}(A, E) = \Big(E(A - I)^{-1}\mathbf{Z}^d + \mathbf{Z}^k\Big)/\mathbf{Z}^k.$$

The assignment $v + (A - I)\mathbf{Z}^d \mapsto v^T + \mathbf{Z}^d(A - I)$ yields an isomorphism between the Bowen-Franks groups $bfg(A)$ and $bfg(A^T)$. Some additional dual characterizations of the crossing group are as follows:

PROPOSITION 5.8. (a) *$\mathcal{CG}(A, E)$ is isomorphic to a factor group of the Bowen-Franks group $bfg(A)$.*

(b) *If $k = 1$, $\mathcal{CG}(A, E) \cong \langle E + \mathbf{Z}^d\rangle \subset bfg(A)$. Hence, $\mathcal{CG}(A, E)$ is isomorphic to a cyclic subgroup of $bfg(A)$. If $k > 1$, and E has the row vectors $e_1, e_2, \ldots, e_k$, then $\mathcal{CG}(A, E)$ is isomorphic to the subgroup of $bfg(A)^k$ corresponding to the direct product of the cyclic subgroups $\langle e_i + \mathbf{Z}^d(A - I)\rangle$, for $i = 1, 2, \ldots, k$.*

PROOF. (a): The coset map $\eta : v + (A - I)\mathbf{Z}^d \mapsto E(A - I)^{-1}v + \mathbf{Z}^k$ is well-defined and defines a surjective homomorphism $\eta : bfg(A) \to \mathcal{CG}(A, E)$.

(b): First, suppose $k = 1$. Fix a generator $E(A - I)^{-1}v_1 + \mathbf{Z}$ of $\mathcal{CG}(A, E)$, for some $v_1 \in \mathbf{Z}^d$.

Consider the homomorphism $h : v + \mathbf{Z}^d(A - I) \mapsto v(A - I)^{-1}v_1 + \mathbf{Z}$ for $v \in \mathbf{Z}^d$. Then $h(E + \mathbf{Z}^d(A - I))$ generates $\mathcal{CG}(A, E)$. The homomorphism h maps the cyclic subgroup $\langle E + \mathbf{Z}^d(A - I)\rangle$ onto $\mathcal{CG}(A, E)$. Choose the least m (equal to the order of $\mathcal{CG}(A, E)$) such that $mE(A - I)^{-1}v_1 \in \mathbf{Z}$. If $mE(A - I)^{-1} \notin \mathbf{Z}^d$ then there exists $w \in \mathbf{Z}^d$ with $E(A - I)^{-1}mw \notin \mathbf{Z}$. But this would mean $\mathcal{CG}(A, E)$ contains a cyclic subgroup whose order does not divide m. Thus the restriction of h to $\langle E + \mathbf{Z}^d(A - I)\rangle$ is an isomorphism.

Now suppose $k > 1$ with E a matrix containing the rows e_i, $i = 1, 2, \ldots, k$ of $1 \times d$ vectors. The crossing group $(E(A - I)^{-1}\mathbf{Z}^d + \mathbf{Z}^k)/\mathbf{Z}^k$ is naturally isomorphic to the product $\prod_i^k (e_i(A - I)^{-1}\mathbf{Z}^d + \mathbf{Z}^k)/\mathbf{Z}$. Each group summand $(e_i(A - I)^{-1}\mathbf{Z}^d + \mathbf{Z})/\mathbf{Z} = \langle e_i(A - I)^{-1}v_i + \mathbf{Z}\rangle$ for some choice of $v_i \in \mathbf{Z}^d$. By the above remarks for $k = 1$, each component cyclic group is isomorphic to the cyclic subgroup $\langle e_i + \mathbf{Z}^d(A - I)\rangle$ of the Bowen-Franks group. □

EXAMPLE 5.9. For instance, the crossing group for $(A, E) = \left(\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, (1\ \ 0)\right)$ is

$$\mathcal{CG}(A, E) = \frac{1}{4}\mathbf{Z}/\mathbf{Z} \cong \left\langle \frac{1}{4} + \mathbf{Z} \right\rangle \cong \mathbf{Z}_4\,.$$

THEOREM 5.10. *Suppose* $\det(A), \det(B) \neq 0, \pm 1$, *and the characteristic polynomials of A, B are irreducible over* $\mathbf{Q}$. *Then the condition $\mathcal{CG}(A, E) \cong \mathcal{CG}(B, F)$ is necessary for the existence of an order isomorphism from $\mathcal{D}_{A,E}$ to $\mathcal{D}_{B,F}$.*

EXAMPLE 5.11. If $\det(A) = \pm 1$, then $\mathcal{CG}(A, E)$ need not be an invariant. For instance, let $A = B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, $E = (1\ \ 1)$, and $F = (2\ \ 3)$.

Then $\mathcal{CG}(A, E) \cong \mathbf{Z}_3$ and $\mathcal{CG}(A, F) \cong \mathbf{Z}_2$. The dimension group of both examples is $\mathbf{Z}^3$. The identity map is certainly an isomorphism. Since the base matrices are the same, the identity preserves the orthogonal space to the left Perron eigenvector, and, thus, is an order isomorphism, despite the crossing group difference. In fact, these two augmented matrices arise from specific wrapping rules, given by $[a, b] \mapsto [ab^2ab, a^3b^4]$ and $[a, b] \mapsto [a^2b^3, a^3b^4]$ (see the recipe in 5.1 below for obtaining augmented matrices from wrapping rules).

PROOF. Suppose that there is an order isomorphism of $\mathcal{D}_{A,E}$ with $\mathcal{D}_{B,F}$. Then there is an order isomorphism C given by a rational $(d + k) \times (d + k)$-matrix, acting on the left, mapping $\mathcal{D}'_{A,E}$ onto $\mathcal{D}'_{B,F}$.

Since C is rational, we have $C(u_i^{\perp}) = w_i^{\perp}$ for the left Perron conjugate eigenvectors u_i, w_i for $i = 1, 2, \ldots, d$, of (A, E) and (B, F) respectively. The intersection $\bigcap u_i^{\perp} = 0 \oplus \mathbf{R}^k$ is C-invariant.

By Theorem 5.4, since C fixes $0 \times \mathbf{R}^k$ and preserves dimension groups, and, therefore, their intersection $0 \times \mathbf{Z}^k = \mathcal{D}'_{A,E} \bigcap (0 \oplus \mathbf{R}^k) = \mathcal{D}'_{B,F} \bigcap (0 \oplus \mathbf{R}^k)$, it follows that $0 \oplus \mathbf{Z}^k$ is C-invariant and that C has the (d, k) block decomposition

$$C = \begin{pmatrix} C_d & 0 \\ H & C_k \end{pmatrix}$$

with C_k invertible over $\mathbf{Z}$.

Since $C(\mathcal{D}'_{A,E}) = \mathcal{D}'_{B,F}$, we know that

$$(C_d(A^{-m}p), H(A^{-m}p) + C_k(-E(A - I)^{-1}p + z)) = (B^{-\ell}p', -F(B - I)^{-1}p' + z')$$

for $p, p' \in \mathbf{Z}^d$ and $z, z' \in \mathbf{Z}^k$.

From Minkowski's Theorem, and $\det(A) \neq 0, \pm 1$, the element zero is not isolated in the dimension group $\mathcal{D}_A$. The nonzero rational vector space $W = \bigcap_{\varepsilon > 0} \operatorname{span}_{\mathbf{Q}}(\mathcal{D}_A \cap B_\varepsilon \backslash \{0\})$ is A-invariant, where B_ε denotes the ε ball about 0. It follows that $\operatorname{span}_{\mathbf{Q}}(\mathcal{D}_A \cap B_\varepsilon \backslash \{0\}) = W = \mathbf{Q}^d$, since the nested intersections must stabilize at some fixed $\varepsilon > 0$ and A has no proper nontrivial rational invariant subspaces. Thus, $\operatorname{span}_{\mathbf{Z}}(\mathcal{D}_A \cap B_\varepsilon \backslash \{0\}) \subset \mathcal{D}_A$ is ε-dense in $\mathbf{R}^d$.

It follows that $H(\mathbf{R}^d) = 0$, since H is continuous with discrete values, and that C is block diagonal, relative to $\mathbf{R}^d \times \mathbf{R}^k$.

We now know that $C_k(-E(A-I)^{-1}(\mathbf{Z}^d)+\mathbf{Z}^k) = -F(B-I)^{-1}(\mathbf{Z}^d)+\mathbf{Z}^k$ and $C_k(\mathbf{Z}^k) = \mathbf{Z}^k$. Therefore C_k induces the quotient map $\tilde{C}_k : \mathcal{CG}(A, E) \to \mathcal{CG}(B, F)$. Since C_k is invertible, $\tilde{C}_k$ is a group isomorphism. $\square$

EXAMPLE 5.12. Which matrices in Example 5.7 have order isomorphic dimension groups? The crossing group for $j = 0 \bmod 7$ is trivial while all the others have crossing groups isomorphic to $\mathbf{Z}_7$. The cases $j \neq 0 \bmod 7$ correspond to order isomorphic dimension groups, as the reader can check.

We now can write the "final chapter" on the Williams examples featured in Section 4 (Table 1) on the dynamical classification of solenoids. We want to indicate how the augmented matrices are obtained from those presentations.

Suppose we consider a solenoid with an elementary presentation on a wedge of two circles (K, f), where f denotes one of the four classes of wrapping rules g_i, $i = 1, 2, 3, 4$ described in Table 1. The general construction can be found in [4]. The space $\varprojlim(K, f)$ has exactly one pair of forward (and one pair of backward) orbits asymptotic under the natural flow, each invariant under the square $\bar{f}^2$. Each of the flow orbits contains a unique fixed point of $\bar{f}^2$. If, for some letters $i \neq j$, $f(i)$ is a prefix (suffix) of $f(j)$, we say that f has a **prefix problem** (**suffix** problem). In the recipe for finding augmented matrices given below, it is somewhat simpler to start with wrapping rules with no prefix or suffix problem. In the present case, we can readily find wrapping rules (Williams presentations) that are shift equivalent to g_i in Table 1 and do not have a prefix or suffix problem.

Assume this has been done and select fixed points (underlined below), one on a backward composant, one on a forward composant, that correspond to word factorings of the form:

(backward) $\quad f^2(\underline{a}) = p\,\underline{a}\,x\,, \quad f^2(\underline{b}) = p\,\underline{b}\,y \quad p$ a nonempty prefix, and

(forward) $\quad f^2(\underline{a}) = u\,\underline{a}\,s\,, \quad f^2(\underline{b}) = v\,\underline{b}\,s \quad s$ a nonempty suffix.

Then the 1-augmented matrix for (K, f^2) is (M^2, E) where M is the transpose of the 2×2 transition matrix for f, as cohomology suggests. (The group $\mathcal{D}_{M^2,E}$ is order isomorphic with $\check{H}^1(\varprojlim(K, f), \mathcal{P}_f)$, for the case $\mathcal{P}_f = \mathcal{P}_f^+ \cup \mathcal{P}_f^-$). The (2×1)-matrix $E = (e_1 \;\; e_2)$ has e_1 equal to the number of occurrences of a in the word $p\,s$ and e_2 equal to the number of occurrences of b in $p\,s$.

It should be of some interest that one can obtain the crossing group at this juncture, without passing to the augmented matrix first, as follows:

Suppose we are in the setting with $k = 1$, and there are pairs of forward and backward asymptotic composants. The **asymptotic group** of the wrapping rule f^2 above is given by the

$$\mathcal{AG}(f^2) \equiv \langle (p\,s)^{bf} \rangle$$

viewing $\{a, b\}$ as generators of the Bowen-Franks group, and the word $p\,s$ as in the description of the asymptotic composants. In other words, $\mathcal{AG}$ is the cyclic group generated by the reduction of $p\,s$ to the Bowen-Franks group of f^2.

Since this is just the second representation of the crossing group in Propositon 5.8, expressed in terms of the original symbols, we obtain the following result:

PROPOSITION 5.13. *The crossing group and asymptotic group are isomorphic*; *i.e.*, $\mathcal{CG}(A, E) \cong \mathcal{AG}(f^2)$

We now return to the Williams examples, described in Table 1. For example, given g_3 with wrapping rule, $g_3(a) = a^2b^4$, $g_3(b) = ab$, then the images of g_3^2 factor as follows:

$$\text{(backward)} \quad a \mapsto \overbrace{a^2b^4a}^{p}\ \underline{a}\ \overbrace{b^4(ab)^4}^{x} \qquad \text{and} \quad b \mapsto \overbrace{a^2b^4a}^{p}\ \underline{b}\ \overbrace{\{\,\}}^{y}, \quad \text{and}$$

$$\text{(forward)} \quad b \mapsto \overbrace{a^2b^4a^2b^4abab}^{u}\ \underline{a}\ \overbrace{bab}^{s} \quad \text{and} \quad b \mapsto \overbrace{a^2b^2}^{v}\ \underline{b}\ \overbrace{bab}^{s}.$$

The wrapping rules for g_3 do not have a prefix/suffix problem. Consequently, we can directly apply the foregoing algorithm. It follows that points of K fixed by g_3^2 corresponding to the "a" in $g_3^2(a) = p\,a\,s$ and the "b" in $g_3^2(b) = p\,b$, lifted to fixed (by $\bar{g}_3^2$) points in $\varprojlim(K, g_3)$ that lie on backward asymptotic composants, and there are corresponding fixed points on forward asymptotic composants in $\varprojlim(K, g_3)$. The relevant word is $p\,s = aabbbbabab$ and the augmented matrix is

$$(M_3^2, E_3) = \begin{pmatrix} 8 & 12 & 0 \\ 3 & 5 & 0 \\ 4 & 6 & 1 \end{pmatrix}.$$

The Bowen-Franks group of g_3^2 has presentation $\langle a, b \mid b^8 = 1,\ a = b^4 \rangle$. The asymptotic group for g_3^2 is

$$\mathcal{CG}(A, E) \cong \mathcal{AG}(g_3^2) \equiv \langle (p, s) \rangle^{bf} = \langle (a^2b^4(ab)^2)^{bf} \rangle = \langle b^2 \rangle \cong \mathbf{Z}_4 .$$

To treat the prefix/suffix problem for the rules g_1, g_2, replace g_1 with $\tilde{g}_1 : [a, b] \to [abb, abab]$ and g_2 with $\tilde{g}_2 : [a, b] \to [aba, aabaa]$ (taken from Williams original reduction to Table 1).

In like manner to the construction for g_3, one computes the pair of augmented matrices

$$(M_2^2, E_2) = \begin{pmatrix} 8 & 3 & 0 \\ 12 & 5 & 0 \\ 6 & 2 & 1 \end{pmatrix}, \quad \text{for the wrapping rule } \tilde{g}_2 : [a, b] \to [aba, aabaa],$$

$$(M_1^2, E_1) = \begin{pmatrix} 5 & 6 & 0 \\ 6 & 8 & 0 \\ 6 & 8 & 1 \end{pmatrix}, \quad \text{for the wrapping rule } \tilde{g}_1 : [a, b] \to [abb, abab].$$

Here M_1, M_2, M_3 are the transposes of the transition matrices for the wrapping rules generated by $\tilde{g}_1, \tilde{g}_2, g_3$, respectively.

The crossing groups are $\mathcal{CG}(M_1^2, E_1) \cong \mathbf{Z}/4\mathbf{Z}$ $\mathcal{CG}(M_3, F_3) \cong \mathbf{Z}/4\mathbf{Z}$ and $\mathcal{CG}(M_2^2, E_2) \cong \mathbf{Z}/2\mathbf{Z}$. As a consequence, $\varprojlim(K, g_1)$ and $\varprojlim(K, g_3)$ are not orientation preserving homeomorphic to $\varprojlim(K, g_2)$. The symmetry of g_2 induces an orientation reversing self homeomorphism of $\varprojlim(K, g_2)$. Thus, $\varprojlim(K, g_1)$ and $\varprojlim(K, g_3)$ are not orientation reversing homeomorphic with $\varprojlim(K, g_2)$ either.

Unfortunately, the dimension groups of (M_3^2, E_3) and (M_1^2, E_1) are order isomorphic, with an explicit isomorphism given by the matrix

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Can we topologically distinguish the unpointed solenoids $\varprojlim(K, g_3)$ and $\varprojlim(K, g_1)$ by some other method? The answer is "yes". Because the Perron-Frobenius eigenvalue of the base matrices is a Pisot number, there is an additional topological invariant associated with so-called *proximal composants*. While this invariant does topologically separate these two Williams solenoids, the topic is not in the scope of this paper. We refer the interested reader to the preprint [2].

This observation does allow us to obtain the following topological classification corresponding to the Williams examples.

PROPOSITION 5.14. *Consider the four classes of wrapping rules given in Table* 1 *in Section* 5.1. *The solenoids corresponding to these rules have exactly three homeomorphism classes, the same as the* (*unpointed*) *conjugacy classes.*

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