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Generalized Burgers Equation with Measure Data

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Abstract. A generalized Burgers equation with measure data is studied.The existence of a weak solution of an initial boundary-value problem in a bounded cylindrical domain, is established. Time-periodic solutions are shown to exist and an optimization problem related to an inverse problem is considered.

1. Introduction

Let Ω be a bounded open subset of R^3 with a smooth boundary and consider the initial boundary-value problem

$$u' - \Delta u + \sum_{j=1}^{3} u \frac{\partial u}{\partial x_j} = g(t)\mu(x) \ \text{ in } \Omega \times (0, T)\,, \tag{1.1}$$

$$u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T)\,, \quad u(x,0) = u_0(x) \ \text{ in } \ \Omega$$

with $\{g,\ \mu,\ u_0\} \in H^1(0,T) \times M_b(\Omega) \times L^1(\Omega)$. The set of all Radon measures of bounded variation in Ω, is denoted by $M_b(\Omega)$.

The purpose of this paper is

• to establish the existence of a solution u of (1.1) with

$$\{u,\ u'\} \in \{L^\infty(0,T;\, L^1(\Omega)) \cap L^p(0,T;\, W_0^{1,p}(\Omega))\} \times L^\infty(0,T;\, L^1(\Omega))$$

with $1 < p < 5/4$,

• to prove the existence of a time-periodic solution of the problem

$$u' - \Delta u + \sum_{j=1}^{3} u \frac{\partial u}{\partial x_j} = g(t)\mu(x) \ \text{ in } \Omega \times (0, T)\,, \tag{1.2}$$

$$u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T)\,, \quad u(x,0) = u(x,T) \ \text{ in } \ \Omega\,.$$

• to determine the source and its intensity from the partial measurements of the solution of (1.1) in an interior subdomain.

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Parabolic initial boundary-value problems with Radon measure data were studied by L. Boccardo and T. Gallouet [2], by H.Brezis and A. Friedman [4] and others. Nonlinear elliptic boundary-value problems with Radon measure data have been the subject of extensive investigations by M. F. Betta, A. Mercaldo, F. Murat and M. Porzio [1], L. Boccardo and T. Gallouet [2], L. Boccardo, T, Gallouet and L. Orsina [3].

The strong monotonicity of the elliptic operator plays a crucial role in Boccardo and Gallouet treatment of elliptic and parabolic problems with measure data. In contrast with the case of L^p-data, $1 < p < \infty$,the lower order terms give rise to several technical difficulties. The Burgers equation which exhibits the nonlinear feature of the Navier-Stokes equations, falls outside of their general framework as the elliptic part is not strongly monotone. It is known that for the heat equation with measure data, the solution is in $L^\infty(0,T;L^1(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ for $1 \le p < 5/4$ and thus the expression $\sum_{j=1}^3 uD_ju$ may not belong to some Banach spaces. In this paper, we shall circumvent the difficulty by assuming that g is in $H^1(0,T)$ and establish an $L^\infty(0,T;L^1(\Omega))$ of the time-derivative of the approximate solutions. The estimates allow us to obtain an $L^q(0,T;W^{-1,p}(\Omega))$ estimate of the expression $\sum_{j=1}^3 uD_ju$. The existence of a solution of (1.1) is established in Section 2.

Time-periodic solutions of parabolic equations with measure data have not been treated in the literature.The Poincare method, the abstract operator approach where the periodicity of the problem is incorporated in the definition of the operator, used for $L^p(Q)$ data with $1 < p < \infty$ do not seem applicable in the case of measure data. Appropriate estimates for the time-periodic approximate solutions are obtained by using an associated cut-off function and *not a generic one.* The existence of a solution is shown in Section 3 of the paper.

Let $\{g,\ \mu\}$ be in $\mathcal{G} \times \mathcal{U}$ be some compact convex subsets of $H^1(0,T) \times M_b(\Omega)$. We associate with (1.1) the cost function

$$J(g;\ \mu;\ u_0;\ u;\ t) = \int_t^T \int_G |\ u(x,s) - \chi(x,s)\ |\ dxds \tag{1.3}$$

where u is a solution of (1.1) and $\chi \in L^1(0,T;L^1(G))$ is the observed values of u in an interior subdomain G of Ω. Let

$$\begin{aligned} V(u_0;t) = \inf\{J(g;\ \mu;\ u_0;u\,;t):\ u \text{ is a solution of } (1.1), \\ \forall\{g,\ \mu\} \in \mathcal{G} \times \mathcal{U}\}\,. \end{aligned} \tag{1.4}$$

In Section 4, we shall show the existence of $\{\tilde{g},\ \tilde{\mu}\} \in \mathcal{G} \times \mathcal{U}$ such that

$$V(u_0;t) = J(\tilde{g};\ \tilde{\mu};\ u_0;\ \tilde{u};\ t) \tag{1.5}$$

where $\tilde{u}$ is a solution of (1.1) with source $\{\tilde{g},\ \tilde{\mu}\}$. The equation (1.5) allows us to determine the source from the observed values of the solution in a fixed interior subdomain.

2. Initial boundary-value problem

In this section, we shall establish the existence of a weak solution of (1.1). With the Laplace operator as the main part and a quadratic nonlinearity in u and its derivative, the equation falls outside of the framework of Boccardo and Gallouet's treatment.

Let $\{u_0,\ \mu\} \in L^1(\Omega) \times M_b(\Omega)$, then there exists $\{u_0^n,\ f_n\} \in C_0^\infty(\Omega)$ with

$$\|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{M_b(\Omega)}\ , \quad \{u_0^n,\ f_n\} \to \{u_0,\ \mu\} \text{ in } \mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)\ .$$

Let

$$\mathcal{E}(u_0;\ g;\ \mu) = \|u_0\|_{L^1(\Omega)} + \|g\|_{H^1(0,T)}\|\mu\|_{M_b(\Omega)}\ , \tag{2.1}$$

Consider the initial boundary-value problem

$$u_n' - \Delta u_n + \sum_{j=1}^{3} u_n D_j u_n = g(t) f_n \text{ in } \Omega \times (0, T)\ , \tag{2.2}$$

$$u_n(x,t) = 0 \text{ on } \partial\Omega \times (0,T)\ , \quad u_n(x,0) = u_0^n(x) \text{ in } \Omega\ .$$

LEMMA 2.1. *Let $\{u_0,\ g,\ f_n\}$ be in $L^1(\Omega) \times H^1(0,T) \times C_0^\infty(\Omega)$ with $\|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{M_b(\Omega)}$. Then there exists*

$$u_n \in L^2(0,T;\, H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0,T;\, L^2(\Omega))\ ,$$

solution of (2.1) *with*

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \leq C\{1 + \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\}\ ,$$

where C is independent of n and $\mathcal{E}$ is defined by (2.1).

PROOF. With f_n in $C_0^\infty(\Omega)$, the existence of a weak solution in $L^2(0,T;\, H_0^1(\Omega)) \cap L^\infty(0,T;\, L^2(\Omega))$ of (2.2) may be obtained by using the standard Galerkin approximation method. Since Ω is a bounded open subset of R^3

$$\|u_n(\,.,t)\|^2_{L^4(\Omega)} \leq C\|u_n(\,.,t)\|_{L^2(\Omega)}\|u_n(\,.,t)\|_{H_0^1(\Omega)}\ ,$$

hence u_n is in $L^4(0,T;\, L^4(\Omega))$ and thus $u_n' \in L^2(0,T;\, H^{-1}(\Omega))$. A standard regularity proof shows that u_n is in $L^2(0,T;\, H^2(\Omega))$ and now the usual argument shows that the solution is unique. We shall establish the estimate of the lemma. Let

$$\psi(s) = \begin{cases} 1 & \text{if } 1 < s\ , \\ s & \text{if } -1 \leq s \leq 1\ , \\ -1 & \text{if } s < -1 \end{cases}$$

and set

$$\phi(s) = \int_0^s \psi(\sigma) d\sigma\ .$$

Multiplying (2.2) by $\psi(u_n)$ and we obtain

$$\frac{d}{dt}\int_\Omega \phi(u_n(x,t))dx + \int_\Omega \psi'(u_n) \mid \nabla u_n(x,t) \mid^2 dx$$
$$+\sum_{j=1}^{3}\int_\Omega u_n(x,t)D_j u_n(x,t)\psi(u_n(x,t))dx$$
$$=\int_\Omega g(t) f_n \psi(u_n(x,t))dx \,.$$

We note that

$$\int_\Omega u_n D_j u_n \psi(u_n(x,t))dx = \int_\Omega D_j\left\{\int_0^{u_n(x,t)} s\psi(s)ds\right\}dx$$
$$= \int_{\partial\Omega} e_j . \int_0^{u_n(x,t)} s\psi(s)dsd\sigma(x) = 0 \tag{2.3}$$

as $u_n = 0$ on $\partial\Omega \times (0,T)$. Taking (2.3) into account, we obtain

$$\frac{d}{dt}\int_\Omega \phi(u_n(x,t))dx \le \|g\|_{H^1(0,T)}\|\mu\|_{M_b(\Omega)} \,. \tag{2.4}$$

Integrating between 0 and t and we get

$$\int_\Omega \mid u_n(x,t) \mid dx \le C\left\{ \mid \Omega \mid + \int_0^t \int_\Omega \phi(u_n(x,s))dxds\right\}$$
$$\le C\left\{ \mid \Omega \mid + \int_\Omega \phi(u_0)dx + \|g\|_{H^1(0,T)}\|\mu\|_{M_b(\Omega)}\right\} \tag{2.5}$$
$$\le C\{\mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\} \,,$$

where C is a constant independent of n.The lemma is proved.

LEMMA 2.2. *Suppose all the hypotheses of Lemma* 2.1 *are satisfied. Then*

$$\|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le C\{1+ \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\}$$

for $1 \le p < 5/4$ *with a constant* C *independent of* n.

PROOF. 1) Let m be a positive integer and let $\psi_m(s)$ be the truncated function

$$\psi_m(s) = \begin{cases} 1 & \text{if } s > m+1 \,, \\ s-m & \text{if } m \le s \le m+1 \,, \\ 0 & \text{if } -m \le s \le m \,, \\ s+m & \text{if } -m-1 \le s \le -m \,, \\ -1 & \text{if } -m-1 < s . \end{cases}$$

Taking the pairing of (2.2) with $\psi_m(u_n(x,t))$ and we obtain by taking into account (2.3)

$$\int_\Omega \phi_m(u_n(x,t))dx + \int_\Omega \psi_m'(u_n(x,t)) \mid \nabla u_n(x,t) \mid^2 dx$$
$$\leq \int_\Omega \phi_m(u_0^n(x))dx + \|g\|_{H^1(0,T)}\|\mu\|_{M_b(\Omega)}$$
$$\leq C\{1+ \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\}\,.$$

It follows that

$$\int_{B_m} \mid \nabla u_n(x,t) \mid^2 dxdt \leq C\{1+ \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\} \tag{2.6}$$

with

$$B_m = \{(y,t) :\ (y,t) \text{ in } \ \Omega \times (0,T),\ \ m \leq u_n(y,t) \leq m+1\}\,.$$

2) Let $1 \leq p < 5/4$, then an application of the Hölder inequality gives

$$m \mid B_m \mid \leq \int_{B_m} \mid u_n(x,t) \mid dxdt$$
$$\leq \|u_n\|_{L^{4p/3}(B_m)} \mid B_m \mid^{(4p-3)/4p}\ . \tag{2.7}$$

Therefore

$$\mid B_m \mid \leq m^{-4p/3}\|u_n\|^{4p/3}_{L^{4p/3}(B_m)}\,. \tag{2.8}$$

Again, an application of the Hölder inequality yields

$$\|\nabla u_n\|^p_{L^p(B_m)} \leq \|\nabla u_n\|^p_{L^2(B_m)} \mid B_m \mid^{(2-p)/2}$$
$$\leq Cm^{-2p(2-p)/3}\|u_n\|^{2p(2-p)/3}_{L^{4p/3}(B_m)}\{1+ \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\}\,. \tag{2.9}$$

We have applied the estimates (2.7)–(2.8) in the above inequality.

3) Let m_0 be a fixed positive number and let ψ be the truncated function

$$\psi(s) = \begin{cases} m_0 & \text{if } s > m_0\,, \\ s & \text{if } -m_0 \leq s \leq m_0\,, \\ -m_0 & \text{if } s < -m_0\,. \end{cases}$$

Then a proof exactly as in that of Lemma 2.1 gives

$$\int_{D_{m_0}} \mid \nabla u_n \mid^2 dxdt \leq C\{m_0 + \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\} \tag{2.10}$$

with

$$D_{m_0} = \{(x,t) : (x,t) \text{ in } \Omega \times (0,T);\ \mid u_n(x,t) \mid \leq m_0\}\,.$$

An application of the Hölder inequality yields

$$\int_{D_{m_0}} |\nabla u_n|^p \,dxdt \le \|\nabla u_n\|^{p/2}_{L^2(D_{m_0})} \, |\,\Omega\,|^{(2-p)/2}$$

$$\le C\{m_0 + \mathcal{E}(u_0;\ g;\ \mu)\}^{p/4} \, |\,\Omega\,|^{(4-p)/4}\ . \tag{2.11}$$

It follows from (2.9) and (2.11) that

$$\begin{aligned}\|\nabla u_n\|^p_{L^p(0,T;L^p(\Omega))} &\le C(m_0)\{1+ |\,\Omega\,| + \mathcal{E}(u_0;\ g;\ \mu)\} \\ &\quad + \sum_{m=m_0}^{\infty} \|u_n\|^{2p(2-p)/3}_{L^{4p/3}(B_m)} m^{-(2-p)2p/3} \\ &\le C(m_0)\{1+ |\,\Omega\,| + \mathcal{E}(u_0;\ g;\ \mu)\} \\ &\quad + \|u_n\|^{4p/3}_{L^{4p/3}(0,T;L^{4p/3}(\Omega))} \left\{ \sum_{m=m_0}^{\infty} m^{-4(2-p)/3} \right\}^{p/2}.\end{aligned} \tag{2.12}$$

We have applied the Hölder inequality in (2.12).

4) Again, the Hölder inequality gives

$$\|u_n\|_{L^{4p/3}(\Omega)} \le \|u_n\|^{1/4}_{L^1(\Omega)} \|u_n\|^{3/4}_{L^{3p/(3-p)}(\Omega)}\ . \tag{2.13}$$

With the estimate of Lemma 2.1, we obtain

$$\begin{aligned}\|u_n\|^{2p(2-p)/3}_{L^{4p/3}(0,T;L^{4p/3}(\Omega))} &\le C(1+ |\,\Omega\,| + \mathcal{E}(u_0;\ g;\ \mu)) \\ &\quad \times \|u_n\|^{p(2-p)/2}_{L^p(0,T;L^{3p/(3-p)}(\Omega))}\ .\end{aligned} \tag{2.14}$$

The Sobolev imbedding theorem gives

$$\|u_n\|^p_{L^p(0,T;L^{3p/(3-p)}(\Omega))} \le C\|u_n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))}\ .$$

It follows from (2.12)–(2.14) that

$$\begin{aligned}\|u_n\|^p_{L^p(0,T;L^{3p/(3-p)}(\Omega))} &\le C\{1+ |\,\Omega\,| + \mathcal{E}(u_0;\ g;\ \mu)\}\|u_n\|^p_{L^p(0,T;L^{3p/(3-p)}(\Omega))} \\ &\quad \times \left\{ \sum_{m=m_0}^{\infty} m^{-4(2-p)/3} \right\}^{p/(2p-p)}.\end{aligned} \tag{2.15}$$

Since $1 \le p < 5/4$ the series converges and there exists m_0 such that

$$\|u_n\|^p_{L^p(0,T;L^{3p/(3-p)}(\Omega)))} \le C\{1+ |\,\Omega\,| + \mathcal{E}(u_0;\ g;\ \mu)\}\ . \tag{2.16}$$

Hence (2.14) yields, by taking (2.16) into account

$$\|u_n\|^{2p(2-p)/3}_{L^{4p/3}(0,T;L^{4p/3}(\Omega))} \le C\{1+ |\,\Omega\,| + \mathcal{E}(u_0;\ g;\ \mu)\}^{p(4-p)/2}\ . \tag{2.17}$$

With (2.17), the inequality (2.12) becomes

$$
\begin{aligned}
\|u_n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} &\le C\{1+ \mid \Omega \mid +\mathcal{E}(u_0;\ g;\ \mu)\}^{p(4-p)/2} \\
&\times\left\{\sum_{m=m_0}^{\infty} m^{-4(2-p)/3}\right\}^{p/2} \\
&\le C\{1+ \mid \Omega \mid +\mathcal{E}(u_0;\ g;\ \mu)\}^{p(4-p)/2} .
\end{aligned} \tag{2.18}
$$

The lemma is proved.

REMARK. The restriction on p, namely that $1 < p < 5/4$ is needed so that $4(2-p)/3 > 1$ for the convergence of the series in (2.15). With m_0 large enough, it allows us to take the term $\|u\|^p_{L^p(0,T;L^{3p/(3-p)}(\Omega))}$ to the left hand side.

LEMMA 2.3. *Suppose all the hypotheses of Lemma* 2.1 *are satisfied. Suppose further that* $\mathcal{L}(u_0)$ *is in* $M_b(\Omega)$ *with*

$$
\mathcal{L}(u_0) = \Delta u_0 - \sum_{j=1}^{3} u_0 D_j u_0 .
$$

Then

$$
\|u_n'\|_{L^\infty(0,T;L^1(\Omega))} \le C\{1+ \mid \Omega \mid +\mathcal{E}(u_0;\ g;\ \mu) + \|\mathcal{L}(u_0)\|_{M_b(\Omega)}\} ,
$$

where C *is a constant independent of* $n,\ u_0,\ g,\ \mu$.

PROOF. 1) Differentiating (2.2) with respect to t, we get

$$
\begin{aligned}
u_n'' - \Delta u_n' + \sum_{j=1}^{3} u_n' D_j u_n &= -\sum_{j=1}^{3} u_n D_j u_n' + g' f_n \ \text{ in } \Omega \times (0,T) , \\
u_n'(x,t) &= 0 \text{ on } \partial\Omega \times (0,T) , \\
u'(x,0) &= \mathcal{L}(u_0^n) + g(0) f_n \ \text{ in } \Omega .
\end{aligned} \tag{2.19}
$$

We have

$$
\|\mathcal{L}(u_0^n)\|_{L^1(\Omega)} + \|g(0) f_n\|_{L^1(\Omega)} \le C\{\|\mathcal{L}(u_0)\|_{M_b(\Omega)} + \|g\|_{H^1(0,T)} \|\mu\|_{M_b(\Omega)}\} ,
$$

where C is a constant independent of $n,\ \mu,\ g$ and of u_0. Moreover

$$
\mathcal{L}(u_0^n) \to \mathcal{L}(u_0) \ \text{ in } \mathcal{D}'(\Omega) .
$$

2) Let $\psi(s)$ be the function of Lemma 2.1. Taking the pairing of (2.19) with $\psi(u_n'(x,t))$, we obtain

$$
\frac{d}{dt}\int_\Omega \phi(u_n'(x,t)) + \int_\Omega \psi'(u_n') \mid \nabla u_n'(x,t) \mid^2 dx
$$

$$+\sum_{j=1}^{3}\int_\Omega u_n' D_j u_n \psi(u_n'(x,t))dx \tag{2.20}$$

$$+\int_\Omega u_n(x,t)\sum_{j=1}^{3} D_j u_n'(x,t)\psi(u_n'(x,t))dx$$

$$\leq \|g\|_{H^1(0,T)}\|\mu\|_{M_b(\Omega)}\,.$$

3) We have

$$\int_\Omega u_n'(x,t)D_j u_n(x,t)\psi(u_n'(x,t))dx = -\int_\Omega D_j u_n'(x,t)u_n(x,t)\psi(u_n'(x,t))dx$$
$$-\int_\Omega u_n'(x,t)u_n(x,t)D_j\psi(u_n'(x,t))dx\,.$$

Therefore

$$\int_\Omega \{u_n D_j u_n\}'\psi(u_n'(x,t))dx = -\int_\Omega u_n(x,t)u_n'(x,t)D_j u_n'(x,t)\psi'(u_n'(x,t))dx$$
$$= -\int_\Omega u_n(x,t)D_j\left\{\int_0^{u_n'(x,t)} s\psi'(s)ds\right\}dx$$
$$= \int_\Omega D_j u_n(x,t)\left\{\int_0^{u_n'(x,t)} s\psi'(s)ds\right\}dx\,.$$

A simple calculation yields

$$\left|\int_0^{u_n'(x,t)} s\psi' ds\right| \leq 1\,.$$

It follows that

$$\sum_{j=1}^{3}\left|\int_\Omega \{u_n' D_j u_n + u_n D_j u_n'\}\psi(u_n'(x,t))dx\right| \leq C\|u_n(\,.\,,t)\|_{W_0^{1,p}(\Omega)} \mid \Omega \mid^{1/q}\,.$$

The inequality (2.20) becomes

$$\int_\Omega \phi(u_n'(x,t))dx + \int_0^t\int_\Omega \psi'(u_n'(x,s)) \mid \nabla u_n'(x,s) \mid^2 dxds$$
$$\leq \int_\Omega \phi(u_n'(x,0))dx + C\|u_n\|_{L^p(0,T;W^{1,p}(\Omega))} + \|g\|_{H^1(0,T)}\|\mu\|_{M_b(\Omega)}\,.$$

Therefore

$$\|u_n'(\,.\,,t)\|_{L^1(\Omega)} \leq C\left\{1+\int_\Omega \phi(u_n'(x,t))dx\right\}$$

$$\begin{aligned} &\le C\{1 + \|\mathcal{L}(u_0)\|_{M_b(\Omega)} + \mathcal{E}(u_0;\ g\ ;\ \mu) + \|u_n\|_{L^p(0,T;W^{1,p}(\Omega))}\} \\ &\le C\{1 + \|\mathcal{L}(u_0)\|_{M_b(\Omega)} + \mid \Omega \mid + \mathcal{E}(u_0;\ g;\ \mu)\} \end{aligned} \tag{2.21}$$

by taking into account the estimate of Lemma 2.2 □

LEMMA 2.4. *Suppose all the hypotheses of Lemma* 2.3 *are satisfied. Then*

$$\| \sum_{j=1}^{3} u_n D_j u_n \|_{L^p(0,T;W^{-1,p}(\Omega))} \le C\{1 + \mid \Omega \mid + \|\mathcal{L}(u_0)\|_{M_b(\Omega)} + \mathcal{E}(u_0;\ g;\ \mu)\}$$

with $1/p + 1/q = 1$ *and* $1 \le p < 5/4$ *where* C *is a constant independent of* $n,\ u_0,\ g,\ \mu$.

PROOF. With $1 < p < 5/4$, the Sobolev imbedding theorem implies that

$$W_0^{1,q}(\Omega) \subset L^\infty(\Omega)\ ,\ 1/p + 1/q = 1\,.$$

The assertion is an immediate consequence of the equation and of the estimates of Lemmas 2.2, 2.3. □

The main result of the section is the following theorem.

THEOREM 2.1. *Let* $\{u_0,\ g,\ \mu\}$ *be in* $L^1(\Omega) \times H^1(0,T) \times M_b(\Omega)$ *with* $\mathcal{L}(u_0) \in M_b(\Omega)$. *Then there exists a solution* u *of* (1.1) *with*

$$\begin{aligned} &\|u\|_{L^\infty(0,T;L^1(\Omega))} + \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u'\|_{L^\infty(0,T;L^1(\Omega))} \\ &\qquad \le C\{1 + \mid \Omega \mid + \|\mathcal{L}(u_0)\|_{M_b(\Omega)} + \mathcal{E}(u_0;\ g;\ \mu)\} \end{aligned}$$

for $1 < p < 5/4$, *where* $\mathcal{E}$ *is defined by* (2.1). *Moreover*

$$\| \sum_{j=1}^{3} u D_j u \|_{L^p(0,T;W^{-1,p}(\Omega))} \le C\{1 + \mid \Omega \mid + \|\mathcal{L}(u_0)\|_{M_b(\Omega)} + \mathcal{E}(u_0;\ g;\ \mu)\}$$

with $\mathcal{L}$ *as in Lemma* 2.3.

PROOF. Let u_n be as in Lemmas 2.1–2.4. Then from the estimates of Lemmas 2.1–2.4 and from Aubin's theorem (see, e.g.,[6, Chapt.1,5]), we get by taking subsequences

$$\{u_n,\ u_n'\} \to \{u,\ u'\}$$

in

$$\begin{aligned} &\{(L^\infty(0,T;\,L^1(\Omega))_{weak^*} \cap L^p(0,T;\,L^p(\Omega)) \cap (L^p(0,T;\,W_0^{1,p}(\Omega))_{weak}\} \\ &\quad \times (L^\infty(0,T;\,L^1(\Omega))_{weak^*}\,. \end{aligned}$$

Furthermore $u_n(x,t) \to u(x,t)$ a.e in $\Omega \times (0,T)$.

From Lemma 3.4 we have by taking subsequences

$$\sum_{j=1}^{3} u_n D_j u_n \to F \ \text{ weakly in } L^p(0,T;\,(W^{-1,p}(\Omega))\,.$$

We shall follow the proof of Aubin's theorem and show that

$$F = \sum_{j=1}^{3} u D_j u \, .$$

Since $u_n \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, we have

$$\sum_{j=1}^{3} \int_0^T \int_\Omega u_n D_j u_n v(x,t) dx dt = -\frac{1}{2} \sum_{j=1}^{3} \int_0^T \int_\Omega u_n^2(x,t) D_j v(x,t) dx dt$$
$$\to \langle F, \, v \rangle \;\; \forall v \in L^q(0, T; W_0^{1,q}(\Omega) \cap H^2(\Omega)) \, .$$

The pairing between $L^p(0, T; W_0^{1,p}(\Omega) \cap H^2(\Omega))$ and its dual, is denoted by $\langle \, . \, , \, . \rangle$. On the other hand we have

$$u_n^2(x,t) \to u^2(x,t) \;\text{ a.e. in } \Omega \times (0, T) \, .$$

and

$$\lim_{n\to\infty} \int_0^T \int_\Omega \{\sum_{j=1}^{3} u_n D_j u_n\} v dx dt = \lim_{n\to\infty} \int_0^T \int_\Omega -\frac{1}{2} u_n^2 \sum_{j=1}^{3} D_j v dx dt$$
$$\to \langle F, v \rangle \, .$$

Let

$$Q^N = \{(x,t) : \; (x,t) \in \Omega \times (0,T), \; | \, (u_n^2(x,t) - u^2(x,t)) \, | \leq 1, \; \text{for } n \geq N\} \, .$$

Then Q^N is an increasing sequence of measurable sets and $\text{meas}(Q^N) \to \text{meas}(Q)$ as $N \to \infty$ with $Q = \Omega \times (0, T)$. Let v be in $C_0^\infty(Q)$ with support in Q^{N_0}, then by the Lebesgue convergence theorem

$$\int_0^T \int_\Omega (u_n^2 - u^2) \sum_{j=1}^{3} D_j v dx dt \to 0 \, .$$

Since

$$\lim_{n\to\infty} \int_0^T \int_\Omega u_n v \sum_{j=1}^{3} D_j u_n dx dt = \lim_{n\to\infty} \int_0^T \int_\Omega -\frac{1}{2} u_n^2 \sum_{j=1}^{3} D_j v dx dt$$
$$= \langle F, v \rangle, \quad v \in C_0^\infty(Q), \; \text{supp}(v) \subset Q^{N_0},$$

the expression $u^2 \sum_{j=1}^{3} D_j v$ is integrable on $\Omega \times (0, T)$. Thus,

$$\int_0^T \int_\Omega -\frac{1}{2} u^2(x,t) \sum_{j=1}^{3} D_j v(x,t) dx dt = \langle F, v \rangle$$

for $v \in C_0^\infty(Q)$ with support in Q^{N_0}. Since Q^N are increasing and $\text{meas}(Q^N) \to T \mid \Omega \mid$, we get

$$\langle F\ ,\ v\rangle = -\frac{1}{2}\int_0^T \int_\Omega u^2(x,t)\sum_{j=1}^3 D_j v(x,t)dxdt\ \ \forall v \in C_0^\infty(Q)\,.$$

On the other hand we have

$$\begin{aligned}\langle F, v\rangle &= \lim_{n\to\infty}\int_0^T \int_\Omega \{u_n v' - \nabla u_n.\nabla v + g f_n v\}dxdt \\ &= \int_0^T \int_\Omega \{uv' - \nabla u.\nabla v\}dxdt + \int_0^T g\,\mu(v(\,.\,,t))dt \\ &= -\frac{1}{2}\int_0^T \int_\Omega u^2(x,t)\sum_{j=1}^3 D_j v(x,t)dxdt\ \ \forall v \in C_0^\infty(Q)\,.\end{aligned}$$

It follows that

$$F = \sum_{j=1}^3 u D_j u\ \text{ in } \mathcal{D}'(Q)\,.$$

Now it is straightforward to check that u satisfies the equation (1.1) in $L^p(0,T;(W^{-1,p}(\Omega))$. The stated estimates are immediate consequences of those of Lemmas 2.1–2.4. □

3. Time periodic solution

In this section, we shall establish the existence of a time-periodic weak solution of (1.2). Time-periodic solutions of nonlinear parabolic equations with measure data do not seem to have been treated in the literature.

Let f_n be as in Section 2 and consider the time periodic problem

$$u_n' - \Delta u_n + \sum_{j=1}^3 u_n D_j u_n = g(t) f_n(x)\ \text{ in } \Omega \times (0,T) \tag{3.1}$$

$$u_n(x,0) = 0\ \text{ on } \partial\Omega \times (0,T)\,,\quad u_n(x,0) = u_n(x,T)\ \text{ in } \Omega\,.$$

With $f_n \in H^1(\Omega)$, the existence of a time-periodic solution u_n in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$ is known and can be established by several methods (e.g. by the Poincare's method as in J. Lions' book [6], chapter 4, p. 482–489). We shall now establish some crucial estimates of u_n in terms of $\|g\|_{H^1(0,T)}$ and of $\|\mu\|_{M_b(\Omega)}$.

LEMMA 3.1. *Let* $\{g, \mu, f_n\}$ *be as in Lemma* 2.1. *Then there exists a solution* u_n *in* $L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ *of* (3.1) *with* $u_n' \in L^2(0, T; L^2(\Omega))$. *Moreover*

$$\|u_n(\,.\,, T)\|_{L^1(\Omega)} \leq C\{1 + \mid \Omega \mid + \mathcal{E}(g;\ \mu)\}$$

with

$$\mathcal{E}(g;\ \mu) = \|g\|_{H^1(0,T)} \|\mu\|_{M_b(\Omega)}\,,$$

where C *is a constant independent of* $n,\ g,\ \mu,\ f_n$ *and*

$$\mathcal{E}(g;\ \mu) = \|g\|_{H^1(0,T)} \|\mu\|_{M_b(\Omega)}\,.$$

PROOF. 1) Since u_n is in $C(0, T; L^2(\Omega))$, there exists γ_n such that

$$\sup_{r \in [T/3, T]} \|u_n(\,.\,, r)\|_{L^1(\Omega)} = \|u_n(\,.\,, \gamma_n)\|_{L^1(\Omega)}$$

for some $\gamma_n \in [T/3,\ T]$. Let $\zeta(t)$ be a $C^1(R)$-function with $\zeta(t) = 0$ for $\mid t \mid \leq 1 - \varepsilon$, $\zeta(t) = 1$ for $\mid t \mid \geq 1$ with $0 \leq \zeta(t) \leq 1$. Set

$$\zeta_n(t) = \zeta(t\gamma_n^{-1})\,.$$

Then we have

$$\zeta_n(t) = 0 \text{ for } 0 \leq t \leq \gamma_n(1 - \varepsilon)\ ;\ \ \zeta_n = 1 \text{ for } \gamma_n \leq t$$

with

$$\begin{aligned} \|\zeta_n\|_{C^1(R)} &\leq C\|\zeta\|_{C^1(R)}\{1 + \gamma_n^{-1}\} \\ &\leq C\|\zeta\|_{C^1(R)}(1 + 3T^{-1})\,, \end{aligned}$$

where C is a constant independent of n.

2) Let $\psi(s)$ be the truncated function of Lemma 2.1 and similarly let ϕ be as in that lemma. Multiplying the equation (3.1) by $\zeta_n(t)$ and taking the pairing with $\psi(u_n)$ we obtain

$$\begin{aligned} \zeta_n \frac{d}{dt} \int_\Omega \phi(u_n(x,t))\,dx + \int_\Omega \zeta_n \psi'(u_n) \mid \nabla(u_n) \mid^2 dx \\ \leq -\sum_{j=1}^{3} \int_\Omega \zeta_n(t) u_n D_j(u_n) \psi(u_n) dx \\ + C\{1 + \mid \Omega \mid + \mathcal{E}(g;\ \mu)\}\,. \end{aligned}$$

We have

$$\sum_{j=1}^{3} \int_\Omega \zeta_n(t) u_n D_j(u_n) \psi(u_n) dx = \sum_{j=1}^{3} \int_\Omega \zeta_n u_n D_j(u_n) \psi(u_n) dx$$

$$= \zeta_n(t) \int_\Omega \sum_{j=1}^{3} D_j \left\{ \int_0^{u_n} s\psi(s)ds \right\} dx$$

$$= \zeta_n(t) \sum_{j=1}^{3} \int_{\partial\Omega} e_j . \left\{ \int_0^{u_n(x,t)} s\psi(s)ds \right\} d\sigma = 0\,.$$

Thus we get, by integrating from $T_\varepsilon = T(1-\varepsilon)/3$ to r

$$\zeta_n(r) \int_\Omega \psi(u_n(x,r))dx \le \int_{T_\varepsilon}^{r} \int_\Omega \zeta_n'(s)\phi(u_n(x,s))dxds$$
$$+C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,.$$

It follows that

$$\zeta_n(r)\|u_n(\,.\,,r)\|_{L^1(\Omega)} \le C(\{1+\mid \Omega \mid +\mathcal{E}(g;\mu)\} + \int_\Omega \phi(u_n(x,r))dx)$$
$$\le C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\} + \int_{T_\varepsilon}^{r} \int_\Omega \|\zeta\|_{C^1(R)} \mid u_n(x,s) \mid dxds$$

Hence

$$\sup_{r\in[T(1-\varepsilon)/3,\,t]} \|\zeta_n(r)u_n(\,.\,,r)\|_{L^1(\Omega)} \le C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\}$$
$$+C \int_{T_\varepsilon}^{t} \|u_n(\,.\,,s)\|_{L^1(\Omega)} dxds\,.$$

Thus,

$$\zeta_n(\gamma_n)\|u_n(\,.\,,\gamma_n)\|_{L^1(\Omega)} \le C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\}$$
$$+C \int_{T_\varepsilon}^{t} \|u_n(\,.\,,s)\|_{L^1(\Omega)} dxds\,.$$

Since $\zeta_n(\gamma_n) = 1$, we deduce that

$$\|u_n(\,.\,,t)\|_{L^1(\Omega)} \le \|u_n(\,.\,,\gamma_n)\|_{L^1(\Omega)}$$
$$\le C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\} + C \int_{T_\varepsilon}^{t} \|u_n(\,.\,,s)\|_{L^1(\Omega)} ds \tag{3.2}$$

for $T_\varepsilon = T(1-\varepsilon)/3 \le t$. It follows from the Gronwall lemma that

$$\|u_n(\,.\,,t)\|_{L^1(\Omega)} \le C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\}$$

for all t with $T_\varepsilon \le t \le T$. Therefore

$$\|u_n(\,.\,,T)\|_{L^1(\Omega)} \le C\{1+\mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,. \tag{3.3}$$

Since the solution is periodic in time, we obtain

$$\|u_n(\,.,0)\|_{L^1(\Omega)} = \|u_n(\,.,T)\|_{L^1(\Omega)} \le C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,, \tag{3.4}$$

where C is a constant independent of $n,\ g,\ \mu$. Repeating the proof of Lemma 2.1 and using (3.4) for the $L^1(\Omega)$-estimate of $u_n(x,0)$ we get the assertion.

LEMMA 3.2. *Suppose all the hypotheses of Lemma* 3.1 *are satisfied and let* u_n *be a solution of* (3.1). *Then*

$$\|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}$$

for $1 \le p < 5/4$, *where C is a constant independent of* $n,\ g;\ \mu$.

PROOF. The proof is exactly the same as that of Lemma 2.2 with the $L^1(\Omega)$-estimate of $u_n(x,0)$ replaced by (3.4).

LEMMA 3.3. *Suppose all the hypotheses of Lemma* 3.1 *are satisfied and suppose further that* $g(0) = g(T)$. *Let* u_n *be a solution of* (3.1), *then*

$$\|u_n'\|_{L^\infty(0,T;L^1(\Omega))} \le C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,,$$

where C is a constant independent of $n,\ g,\ \mu$.

PROOF. 1) Let ζ_n be as in Lemma 3.1 with u_n replaced by u_n', then we obtain by differentiating (3.1) with respect to t

$$\begin{aligned}\zeta_n(t)u_n'' - \Delta(\zeta_n u_n') - \zeta_n' u_n' &= -\sum_{j=1}^{3}\{\zeta_n u_n' D_j u_n + u_n D_j(\zeta_n u_n')\}\\ &\quad +\zeta_n g' f_n \ \text{ in } \Omega\times(0,T)\,,\\ u_n'(x,t) = 0 \ \text{ on } \partial\Omega\times(0,T)\,, &\quad u'(x,0) = u'(x,T) \ \text{ in } \Omega\,.\end{aligned} \tag{3.5}$$

Let ψ be the truncated function of Lemma 2.1 and let ϕ be as in that lemma. Then we have

$$\begin{aligned}\zeta_n(t)\frac{d}{dt}\int_\Omega \phi(u_n'(x,t))\,dx &+ \int_\Omega \zeta_n(t)\psi'(u_n'(x,t)) \mid \nabla u_n'(x,t)\mid^2 dx\\ &\le -\sum_{j=1}^{3}\zeta_n(t)\int_\Omega\{u_n' D_j u_n \psi(u_n') + u_n D_j(u_n')\psi(u_n')\}dx\\ &\quad +C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,.\end{aligned}$$

Consider the expression

$$\int_\Omega\{u_n' D_j u_n + u_n D_j(u_n')\}\psi(u_n')dx\,.$$

A simple integration by parts yields

$$\begin{aligned}\int_\Omega \{u_n' D_j u_n + u_n D_j(u_n')\}\psi(u_n')dx &= \int_\Omega u_n(x,t)u_n'(x,t)D_j(\psi(u_n'(x,t))dx \\ &= \int_\Omega u_n(x,t)u_n'(x,t)\psi'(u_n'(x,t))D_j u_n'(x,t)dx \\ &= \int_\Omega u_n(x,t)D_j\left\{\int_0^{u_n'(x,t)} s\psi'(s)ds\right\}dx \\ &= -\int_\Omega D_j u_n(x,t)\left\{\int_0^{u_n'} s\psi'(s)ds\right\}dx\,.\end{aligned}$$

Therefore

$$\left|\int_\Omega \{u_n' D_j u_n + u_n D_j u_n'\}\psi(u_n')dx\right| \le C\|D_j u_n\|_{L^p(\Omega)} \mid \Omega \mid^{1/q}$$

as

$$\left|\int_0^{u_n'(x,t)} s\psi'(s)ds\right| \le 1\,.$$

We have used the property that $\psi'(s) = 1$ for $\mid s \mid \le 1$ and $\psi'(s) = 0$ for $\mid s \mid \rangle 1$. Thus,

$$\begin{aligned}\zeta_n(t)\frac{d}{dt}\int_\Omega \phi(u_n'(x,t))dx &\le C\{1+ \mid \Omega \mid + \mathcal{E}(g;\ \mu)\} \\ &\quad + C\|u_n(\,.\,,t)\|_{W^{1,p}(\Omega)} \mid \Omega \mid^{1/q}\ .\end{aligned} \tag{3.6}$$

Integrating between $T_\varepsilon = T(1-\varepsilon)/3$ and r and we get

$$\begin{aligned}\zeta_n(r)\|u_n'(\,.\,,r)\|_{L^1(\Omega)} &\le C\left\{1+ \mid \Omega \mid + \int_{T_\varepsilon}^r \zeta_n(s)\phi(u_n'(x,s))dxds\right\} \\ &\le \int_{T_\varepsilon}^r \int_\Omega \zeta_n'(s)\phi(u_n'(x,s))dxds + C\{1+ \mid \Omega \mid + \mathcal{E}(g;\ \mu)\} \\ &\quad + C\|u_n\|_{L^p(0,T;\,W^{1,p}(\Omega))}\,.\end{aligned} \tag{3.7}$$

Taking into account the estimate of Lemma 3.2, we obtain

$$\begin{aligned}\sup_{r\in[T_\varepsilon,\,t]} \{\zeta_n(r)\|u_n'(\,.\,,r)\|_{L^1(\Omega)}\} &\le C\{1+ \mid \Omega \mid + \mathcal{E}(g;\ \mu)\} \\ &\quad + C\|\zeta\|_{C^1(R)}\int_{T_\varepsilon}^r \|u_n'(\,,s)\|_{L^1(\Omega)}ds\,.\end{aligned} \tag{3.8}$$

Thus,

$$\|u_n'(\,.\,,t)\|_{L^1(\Omega)} \le \zeta_n(\gamma_n)\|u_n'(\,.\,,\gamma_n)\|_{L^1(\Omega)}$$

$$= \|u_n'(.,\gamma_n)\|_{L^1(\Omega)} = \sup_{r\in[T_\varepsilon,t]} \|u_n'(.,r)\|_{L^1(\Omega)} \tag{3.9}$$
$$\leq C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\} + C\int_{T_\varepsilon}^{t} \|u_n'(.,s)\|_{L^1(\Omega)} ds\,.$$

An application of the Gronwall lemma yields

$$\|u_n'(.,t)\|_{L^1(\Omega)} \leq C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}, \quad t \in [T_\varepsilon, T]\,.$$

Since u_n' is periodic in time, we get

$$\|u_n'(.,0)\|_{L^1(\Omega)} = \|u_n'(.,T)\|_{L^1(\Omega)}$$
$$\leq C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,.$$

Now a proof as of that of Lemma 3.3 gives

$$\|u_n'\|_{L^\infty(0,T;L^1(\Omega))} \leq C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,.$$

LEMMA 3.4. *Suppose all the hypotheses of Lemmas* 3.1–3.3 *are satisfied. Then*

$$\|\sum_{j=1}^{3} u_n D_j u_n\|_{L^q(0,T;W^{-1,q}(\Omega))} \leq C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,,$$

with $1 < p < 5/4$ *and* $1/p + 1/q = 1$, *where C is a constant independent of* $g,\ \mu, n$.

PROOF. It is an immediate consequence of the estimates of Lemmas 3.2, 3.3.

THEOREM 3.1. *Let* $\{g,\ \mu\}$ *be in* $H^1(0,T) \times M_b(\Omega)$ *with* $g(0) = g(T)$. *Then there exists a time-periodic solution u of* (1.1) *with*

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} + \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u'\|_{L^\infty(0,T;L^1(\Omega))} \leq C\{1+ \mid \Omega \mid +\mathcal{E}(g;\ \mu)\}\,,$$

where C is a constant independent of $g,\ \mu$ *and with* $1 < p < 5/4$.

PROOF. Let u_n be a time-periodic solution of (3.1). With the estimates of Lemmas 3.1–3.4, a proof identical to that of Theorem 2.1 gives the stated result.

4. An optimization problem: inverse problem

Let $\mathcal{G}$ be the compact convex subset of $L^2(0,T)$ given by

$$\mathcal{G} = \{g :\ \|g\|_{H^1(0,T)} \leq 1\}$$

and let $\mathcal{U}$ be the closed convex subset of $\mathcal{M}_b(\Omega)$ defined by

$$\mathcal{U} = \{\mu :\ \|\mu\|_{M_b(\Omega)} \leq 1\}\,.$$

We denote by χ, an $L^1(0,T;L^1(G))$-function, representing the observed values of a solution u of the initial boundary-value problem (1.1) in an interior subregion G of Ω. With the control $\{g;\ \mu\} \in \mathcal{G} \times \mathcal{U}$, we associate with (1.1) the cost function

$$J(g;\ \mu;\ u_0; u;\ \tau) = \int_\tau^T \int_G |\ u(x,t) - \chi(x,t)\ |\ dxdt \tag{4.1}$$

where u is a solution of (1.1). The value function $V(u_0;\ \tau)$ of (1.1)–(4.1) is defined by

$$\begin{aligned} V(u_0;\ \tau) = \inf\{J(g;\ \mu;\ u_0; u;\ \tau) :\ \forall u \text{ solution of } (1.1), \\ \forall g \in \mathcal{G},\ \forall \mu \in \mathcal{U}\}\,. \end{aligned} \tag{4.2}$$

For the initial boundary-value problem (1.1), we have the following result.

THEOREM 4.1. *Let $u_0,\ \mathcal{L}_0$ be in $M_b(\Omega))$ and let χ be a $L^1(0,T;\ L^1(G))$-function. Then there exists $\{\tilde g,\ \tilde\mu\}$ in $\mathcal{G} \times \mathcal{U}$ and a solution $\tilde u$ of* (1.1) *with*

$$V(u_0;\ \tau) = J(\tilde g;\ \tilde\mu; u_0;\ \tilde u;\ \tau)\,.$$

Moreover

$$\{\tilde u,\ \tilde u'\} \in\ L^\infty(0,T;\ L^1(\Omega)) \cap L^p(0,T;\ W_0^{1,p}(\Omega)) \times L^\infty(0,T;\ L^1(\Omega))\,.$$

It is an inverse problem as we try to find the source $\tilde\mu$, the intensity of the source $\tilde g$ from the observed values of the solution in an interior subdomain.

PROOF. With u_0 as in the theorem, we know from Theorem 2.1 that there exists a solution u of (1.1) for any given $\{g,\ \mu\}$ in $\mathcal{G} \times \mathcal{U}$. Let $\{g_n;\ \mu_n\}$ be a minimizing sequence of the optimization problem (4.2) with

$$V(u_0;\ \tau) \le J(\,g_n;\ \mu_n; u_0;\ u_n;\ \tau) \le V(u_0;\ \tau) + 1/n\,.$$

From the estimates of Theorem 2.1, we have

$$\begin{aligned} &\|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_n\|_{L^\infty(0,T;L^1(\Omega))} + \|u_n'\|_{L^\infty(0,T;L^1(\Omega))} \\ &\qquad\qquad \le C\{1 +\ |\ \Omega\ |\ + \mathcal{E}(u_0;\ g_n;\ \mu_n) + \|\mathcal{L}u_0\|_{M_b(\Omega)}\}\,. \end{aligned}$$

Furthermore

$$\|\sum_{j=1}^3 u_n D_j u_n\|_{L^q(0,T;W^{-1,q}(\Omega))} \le C\{1 +\ |\ \Omega\ \ | + \mathcal{E}(u_0;\ g;\ \mu) + \|\mathcal{L}(u_0)\|_{M_b(\Omega)}\}\,.$$

2) Thus there exists a subsequence such that

$$\{u_n,\ u_n';\ g_n,\ \mu_n\} \to \{\tilde u,\ \tilde u',\ \tilde g,\ \tilde\mu\}$$

in

$$\begin{aligned} &(L^p(0,T;\ W_0^{1,p}(\Omega)))_{weak} \cap (L^\infty(0,T : L^1(\Omega)))_{weak*} \cap L^p(0,T;\ L^p(\Omega)) \\ &\qquad\qquad \times (L^\infty(0,T;\ L^1(\Omega)))_{weak*} \times (H^1(0,T))_{weak} \cap L^2(0,T) \end{aligned}$$

$$\times \mathcal{D}'(\Omega)\,.$$

Furthermore

$$\sum_{j=1}^{3} u_n D_j u_n \to F \ \text{ weakly in } \ L^q(0,T;\, W^{-1,q}(\Omega))\,.$$

A proof as in that of Theorem 2.1 shows that

$$F = \sum_{j=1}^{3} \tilde{u} D_j \tilde{u} \ \text{ in } \ \mathcal{D}'(Q) \ \text{with} \ Q = \Omega \times (0,T)\,.$$

It is clear that $\tilde{u}$ is a solution of (1.1) with the controls $\{\tilde{g}, \tilde{\mu}\}$ and we have

$$V(u_0;\ \tau) = J(\tilde{g};\ \tilde{\mu}; u_0;\ \tilde{u};\ \tau), \quad \tau \in [0,T].$$

The theorem is proved.

For the time periodic problem of Section 3, we have a similar result.

THEOREM 4.2. *Let χ be a function in $L^1(0,T;L^1(G))$, where G is an interior open subset of Ω. There exists $\{\tilde{g},\ \tilde{\mu}\} \in \mathcal{G} \times \mathcal{U}$ and $\tilde{u}$ such that*

$$\tilde{u}' - \Delta\tilde{u} + \sum_{j=1}^{3} \tilde{u} D_j \tilde{u} = \tilde{g}\tilde{\mu} \ \textit{ in } \ \Omega \times (0,T)\,, \tag{4.3}$$

$$\tilde{u}(x,t) = 0 \ \textit{ on } \ \partial\Omega \times (0,T)\,, \quad \tilde{u}(x,0) = \tilde{u}(x,T) \ \textit{ in } \ \Omega$$

with $\{\tilde{u},\ \tilde{u}'\}$ *in*

$$L^\infty(0,T;L^1(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)) \times L^\infty(0,T;L^1(\Omega))$$

and

$$J(\tilde{u};\ \tilde{g};\ \tilde{\mu};\tau) = \inf\{J(u;\ g;\ \mu):\ \ u \ \textit{is a solution of}\,(4.3),\ \forall g \in \mathcal{G},\ \forall\mu \in \mathcal{U}\}\,.$$

where the cost function J is defined by (4.1).

PROOF. We use the estimates of Theorem 3.1 instead of those of Theorem 2.1 and the proof is the same as that of Theorem 4.1.

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