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Volumes and Degeneration of Cone-structures on the Figure-eight Knot

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1. Introduction

A geometrical construction of manifolds on the figure-eight knot appeared originally in Thurston's lectures [22]. He constructed a hyperbolic three-dimensional manifold by gluing faces of two ideal tetrahedra. The manifold obtained in this way is homeomorphic to the complement of the figure-eight knot in the three-dimensional sphere $\mathbf{S}^3$. In addition, this manifold has a complete hyperbolic structure.

We define the *three-dimensional Euclidean cone-manifold* [11] to be the complete metric space obtained as the quotient of (possibly non-compact) geodesic 3-simplices in the three-dimensional Euclidean space $\mathbf{E}^3$ by an isometric gluing of faces in such a fashion that the underlying topological space is a manifold. In this case, the metric structure around each edge is defined by the cone angle, which is equal to the sum of the dihedral angles corresponding to the identified edges. We define the *singular set* of the cone-manifold to be the closure of the set of edges whose cone angle is not equal to 2π. By definition, on the complement of the singular set, the constructed space has a Euclidean structure. Analogously, we define *spherical* and *hyperbolic* cone-manifolds. A cone-manifold is said to be an *orbifold* if the cone angles are equal to $2\pi/n$ for an integer $n \in \mathbf{N}$.

Let $\mathcal{C}(2\pi/n)$ be an orbifold whose singular set forms a figure-eight knot, and whose cyclic isotropy group has order $n \geq 1$. It is well known [3, 9, 22], that the orbifold $\mathcal{C}(2\pi/n)$ is spherical for $n = 2$, Euclidean for $n = 3$ and hyperbolic for $n \geq 4$. In [8] Hilden, Lozano and Montesinos constructed a family of three dimensional cone-manifolds $\mathcal{C}(\theta)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot. They showed that the cone-manifold obtained is hyperbolic for $\theta \in [0, 2\pi/3)$, Euclidean for $\theta = 2\pi/3$ and spherical for $\theta \in (2\pi/3, \pi]$. They also calculated geometrical parameters for the fundamental polyhedra and volume formulas for complicated cone manifolds. The question of the existence of spherical structure for $\theta > \pi$ was left open.

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In [7] it was shown that the Fibonacci manifolds are n-fold cyclic coverings of the three-dimensional sphere branched over the figure-eight knot. This generated interest in the Fibonacci manifolds. In a paper of Helling, Kim and Mennicke [6], fundamental polyhedra were constructed for the Fibonacci manifolds with even indexes. The authors proved the arithmeticity of the groups for the corresponding Fibonacci manifolds for $n = 4, 5, 6, 8, 12$. In a paper of Vesnin and the second named author [20], the isometry group of the Fibonacci manifolds was calculated. In [23] Mednykh and Vesnin calculated the volumes of the hyperbolic Fibonacci manifolds.

The main goal of this paper is to construct the simplest possible fundamental polyhedron for cone-manifolds $\mathcal{C}(\theta)$, $0 \leq \theta < 4\pi/3$ in the sense that it has just two pairs of piecewise linear faces, identified by the isometries generating the holonomy group. Then the constructed polyhedron is hyperbolic for $0 \leq \theta < 2\pi/3$, Euclidean for $\theta = 2\pi/3$ and spherical for $2\pi/3 \leq \theta < 4\pi/3$. The construction is universal, that is a similar construction can be carried out for any two-bridge link or knot. Its geometrical parameters can be found as the roots of an algebraic equation which is the sequence of the main relation in the group. This construction is found below in the second Section. This construction was first performed in [15].

The third Section of the work is devoted to calculations of the volumes of the cone-manifolds in hyperbolic and spherical spaces. Some more complicated formulae were obtained for special cases by Kojima [13].

In the fourth and final Section we prove that cone-manifolds on the figure-eight knot have spherical structures for $2\pi/3 < \theta < 4\pi/3$ and that these inequalities are the limiting values. That is, for $\theta = 2\pi/3$ the spherical structure degenerates to a Euclidean structure and for $\theta = 4\pi/3$ the cone-manifold $\mathcal{C}(\theta)$ singular set becomes a degenerate singular knot with two vertices of valency four and four arcs joining the vertices. We note that our approach is general and can be applied to other two-bridge links and knots.

The authors' interest in geometrical structures on the figure-eight knot was initiated by J. H. Przytycki during the International Congress of Mathematicians ICM'82 in Warsaw in August, 1983.

2. The construction of the fundamental set of the cone-manifolds $\mathcal{C}(\theta)$

2.1. The fundamental set of the spherical cone-manifold $\mathcal{C}(\pi)$. Denote by $\mathbf{S}^3$ the three-dimensional sphere. Let us consider a lens L bounded by two spherical planes in $\mathbf{S}^3$ with dihedral angle $\frac{\pi}{5}$. Choose 10 points on the line (circle) of intersection of these planes P_i, $i = 0, \ldots, 9$ such that the spherical distance between P_i and P_{i+1} ($i \ mod\ 10$) is equal to $\frac{\pi}{5}$. Draw the line P_0P_p on the bottom of the lens and the line P_2P_7 on the top of the lens (See Figure 1).

LEMMA 1. *Let S and T be spherical isometries which identify faces of the lens L in the following way*:

$$S : \ Q_1P_5P_6 \ \cdots \ P_0 \rightarrow Q_1P_5P_4 \ \cdots \ P_0$$

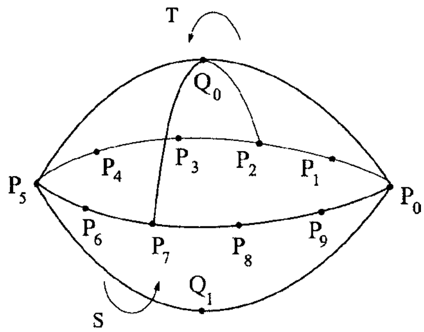

FIGURE 1. The fundamental set L of the spherical cone-manifold $\mathcal{C}(\pi)$.

$$T : Q_0P_2P_1 \cdots P_7 \to Q_0P_2P_3 \cdots P_7$$

then

(i) *the group* $\langle S, T\rangle$ *is a discrete group of isometries of the sphere* $\mathbf{S}^3$ *and has the following presentation*:

$$\langle S, T \mid S^2 = T^2 = (TS)^5 = 1\rangle$$

(ii) *the lens* L *is a fundamental set for the group* $\langle S, T\rangle$

(iii) *the orbifold* $\mathbf{S}^3/\langle S, T\rangle$ *is isomorphic to* $\mathcal{C}(\pi)$.

PROOF. The statements (i) and (ii) are immediate consequence of the Poincare's polyhedron theorem [4]. We verify the statement (iii).

To show that the orbifold $\mathbf{S}^3/\langle S, T\rangle$ coincides with the orbifold $\mathcal{C}(\pi)$ consider the two-fold covering M of $\mathbf{S}^3/\langle S, T\rangle$ branched over the union of lines P_0P_5 and P_2P_7. The fundamental group of this covering is given by the kernel of the epimorphism $\phi : \langle S, T\rangle \to \mathbf{Z}_2 = \{0, 1\}$ defined by $\phi(S) = 1$ and $\phi(T) = 1$. The group $Ker(\phi)$ is the cyclic group of order 5 generated by TS. The fundamental set of this group is the lens space $L \bigcup S(L)$ formed by L and S(L) identified along the bottom hemisphere of L.

Denote the poles of the lens $L \cup S(L)$ as $North = Q_0$, $South = S(Q_0)$, and set $V_i = P_{2i}, i = 0, \ldots, 5$ and $V_5 = V_0$. Then the isometry $TS = TS^{-1}$ acts on the set $L \cup S(L)$ in the following way: the triangle $South V_i V_{i+1}$ is identified with the triangle $North V_{i+2} V_{i+3}$ for every $i = 0, \ldots, 4$. Hence, M is the standard lens space $L(5, 2)$. The orbifold $\mathbf{S}^3/\langle S, T\rangle$ is obtained as a factor space of $M = L(p, q)$ by means of involution S and, hence, coincides with the orbifold $\mathcal{C}(\pi)$ [10]. □

2.2. The fundamental set of the Euclidean cone-manifold $\mathcal{C}(2\pi/3)$. The aim of this Section is to show that the orbifold $\mathcal{C}(2\pi/3)$ on the figure-eight knot has the fundamental set in the Euclidean space $\mathbf{E}^3$ shown in Figure 2 which is topologically similar to the fundamental set of the orbifold $\mathcal{C}(\pi)$ in the sphere shown in Figure 1.

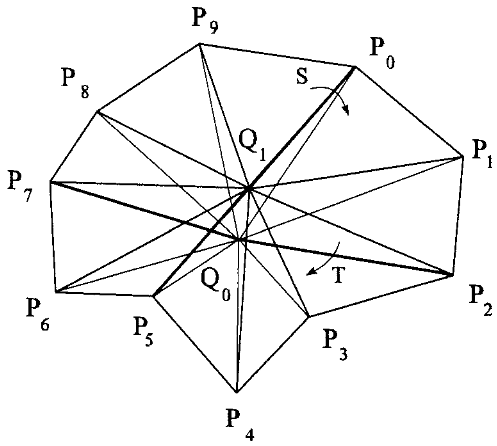

FIGURE 2. The fundamental set of the Euclidean cone-manifold $\mathcal{C}(2\pi/3)$.

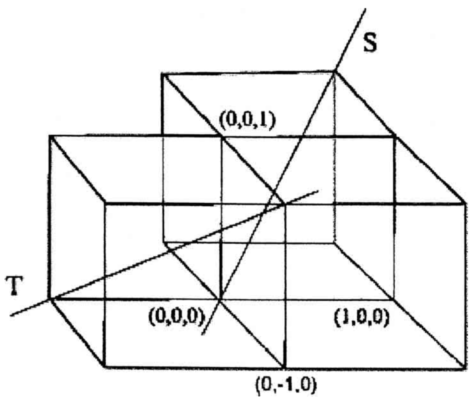

FIGURE 3. The axes of elliptic elements S and T in the Euclidean space.

More precisely, we shall describe an algorithm for the construction of the fundamental set $\mathcal{P}$ for the figure-eight orbifold which effectively works in every space of constant sectional curvature. With slight modifications this algorithm can also be used to construct the fundamental set of any 2-bridge link orbifold.

Since the cone-manifold $\mathcal{C}(2\pi/3)$ is the orbifold, it is possible to consider it as the quotient space of $\mathbf{E}^3$ by the action of an isometry group.

It was shown in [22] that the orbifold $\mathcal{C}(2\pi/3)$ can be obtained as a quotient space $\mathbf{E}^3/\langle S, T\rangle$, where $\langle S, T\rangle$ is the group generated by two rotations S and T of order three whose axes are shown in Figure 3 and are given by lines $[(0, 0, 0), (1, 1, 1)]$ and $[(-1, 0, 0), (0, -1, 1)]$ respectively.

We use the following properties of the isometries S and T:

$$S(-1,0,0) = (0,-1,0)\,, \quad S(0,-1,1) = (1,0,-1)\,,$$
$$TS(-1,0,0) = (0,0,1)\,, \quad TS(0,-1,1) = (-1,1,2)\,.$$

Consider the half-turns b and e about the lines

$$(-1/2,-1/4,0) + k(1,0,1)\,, \quad (-1/4,-1/4,-1/4) + k(1,0,-1)\,.$$

One can easily verify the following properties of the isometry b:

$$bSb^{-1} = T\,, \quad b^2 = 1$$

and e:

$$eS = se\,, \quad eT = te\,, \quad e^2 = 1\,, \quad be = eb\,,$$

where $s = S^{-1}$ and $t = T^{-1}$. Consider the element $TSTst$. It has an axis that is the image of the axis of rotation of T under the transformation TS. So this axis coincides with the line $[TS(-1,0,0), TS(0,-1,1)] = [(0,0,1), (-1,1,2)]$. Thus to find the fundamental polyhedron let us denote by P_1 the point of intersection of the axes b and $TSTst$. By direct calculation one can find that $P_1 = (1/4, -1/4, 3/4)$. All other vertices of the fundamental polyhedra can be obtained in the following way:

$$P_3 = T^{-1}(P_1)\,, \quad P_7 = S^{-1}(P_3)\,, \quad P_9 = S^{-1}(P_1)\,,$$
$$P_5 = T^{-1}(P_9)\,, \quad P_6 = e(P_1)\,, \quad P_2 = e(P_7)\,,$$
$$P_4 = e(P_9)\,, \quad P_8 = e(P_3)\,, \quad P_0 = e(P_5)\,.$$

Define the points Q_0, Q_1 to be the points of intersection of the axis e with the axes S, T respectively.

Hence, we have the following coordinates for the vertices of the fundamental polyhedron:

$Q_1 = (-1/4, -1/4, -1/4)$	$P_5 = (-3/4, -3/4, -3/4)$
$Q_0 = (-3/4, -1/4, 1/4)$	$P_6 = (-5/4, -1/4, -3/4)$
$P_1 = (1/4, -1/4, 3/4)$	$P_7 = (-5/4, 1/4, -1/4)$
$P_2 = (-1/4, -3/4, 3/4)$	$P_8 = (-3/4, 3/4, -1/4)$
$P_3 = (-1/4, -5/4, 1/4)$	$P_9 = (-1/4, 3/4, 1/4)$
$P_4 = (-3/4, -5/4, -1/4)$	$P_0 = (1/4, 1/4, 1/4)$

Consider the polyhedron $\mathcal{P}(2\pi/3)$ formed by the vertices P_i, Q_j, the edges P_iQ_j, P_iP_{i+1} and the faces $P_iP_{i+1}Q_j$, where $i = 0,\ldots,9$, $j = 0,1$ and $P_{10} = P_0$. It is more convenient to consider the non-convex polyhedron $\mathcal{P}(2\pi/3)$ as a curvilinear polyhedron with ten vertices P_i, $i = 0,\ldots,9$, twelve edges P_iP_{i+1}, $i = 0,\ldots,9$, P_0P_5, P_2P_7 and four curvilinear faces:

$$Q_1P_5\cdots P_0\,, \quad Q_1P_0\cdots P_5\,, \quad Q_0P_7\ \cdots P_2\,, \quad Q_0P_2\cdots P_7\,.$$

It is easy to see that each of these curvilinear faces consists of five triangles.

REMARK. The curvilinear polyhedron thus described is combinatorially equivalent to the spherical polyhedron shown in Figure 1. As in the spherical case the lines $[P_0, P_5]$, $[P_2, P_7]$ are fixed by the elements S and T respectively. Using the transformation $x \to 4x + (2, 1, 0)$, $x \in \mathbf{E}^3$ one can easily find more convenient coordinates for vertices of the polyhedron $\mathcal{P}(2\pi/3)$:

$$\begin{array}{lll} Q_1 = (1, 0, -1) & P_3 = (1, -4, 1) & P_7 = (-3, 2, -1) \\ Q_0 = (-1, 0, 1) & P_4 = (-1, -4, -1) & P_8 = (-1, 4, -1) \\ P_1 = (3, 0, 3) & P_5 = (-1, -2, -3) & P_9 = (1, 4, 1) \\ P_2 = (1, -2, 3) & P_6 = (-3, 0, -3) & P_0 = (3, 2, 1) \end{array}$$

Now we will show that the constructed polyhedron $\mathcal{P}(2\pi/3)$ is a fundamental set for the group $\langle S, T\rangle$ in $\mathbf{E}^3$. We first check the conditions of the Poincare theorem and define the presentation of the group $\langle S, T\rangle$. For this define the interior dihedral angles between adjacent faces occurring in the polyhedron. We have:

$$\cos(\angle P_3P_4) = \cos(\angle P_8P_9) = \frac{7}{9}, \quad \angle P_0P_5 = \angle P_2P_7 = \frac{2\pi}{3},$$

$$\cos(\angle P_0P_1) = \cos(\angle P_1P_2) = \cos(\angle P_2P_3) = \cos(\angle P_4P_5) =$$

$$= \cos(\angle P_5P_6) = \cos(\angle P_6P_7) = \cos(\angle P_7P_8) = \cos(\angle P_9P_0) = \sqrt{\frac{2}{3}}.$$

In such a way the conditions of the Poincare theorem are fulfilled and the following statement is true.

TABLE 1

Edge cycle	Sum of angles	Relation in group
P_2P_7	$\frac{2\pi}{3}$	$T^3 = 1$
P_0P_5	$\frac{2\pi}{3}$	$S^3 = 1$
$P_5P_6,\ P_6P_7,\ P_7P_8,\ P_8P_9,\ P_9P_0,\ P_0P_1,\ P_1P_2,\ P_2P_3,\ P_3P_4,\ P_4P_5$	2π	$sTStstSTst = 1$
$Q_1P_6,\ Q_1P_4$	2π	Ss
$Q_1P_7,\ Q_1P_3$	2π	Ss
$Q_1P_8,\ Q_1P_2$	2π	Ss
$Q_1P_9,\ Q_1P_1$	2π	Ss
$Q_0P_6,\ Q_0P_8$	2π	Tt
$Q_0P_5,\ Q_0P_9$	2π	Tt
$Q_0P_3,\ Q_0P_1$	2π	Tt

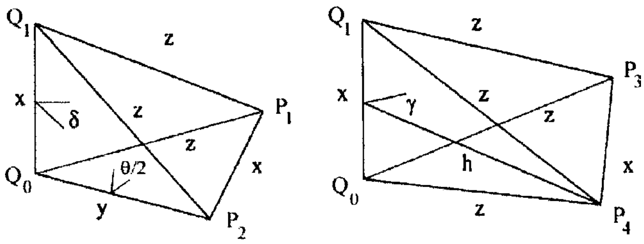

FIGURE 4. Tetrahedra $\mathcal{T}_1$ and $\mathcal{T}_3$ in the Euclidean situation.

THEOREM 2. *Let S and T be Euclidean isometries which identify curvilinear faces of the polyhedron $\mathcal{P}(2\pi/3)$ in the same way as in Lemma 1. Then*

(i) *the group $\langle S, T\rangle$ is a discrete subgroup of isometries of the space $\boldsymbol{E}^3$ and has the following presentation*:

$$\langle S,\ T \mid S^3 = T^3 = sTStstSTst = 1\,\rangle$$

(ii) *the polyhedron $\mathcal{P}$ is a fundamental set for the group $\langle S, T\rangle$*

(iii) *the orbifold $\boldsymbol{E}^3/\langle S, T\rangle$ is isomorphic to $\mathcal{C}(2\pi/3)$.*

Another approach to the proof of Theorem 2 can be obtained by the following scheme. Note that the fundamental polyhedron $\mathcal{P}(2\pi/3)$ can be decomposed into ten tetrahedra $\mathcal{T}_i = Q_0Q_1P_iP_{i+1}$, $i = 0, \ldots, 9$. It is easy to verify directly computing the lengths of the edges of $\mathcal{P}(2\pi/3)$ that the tetrahedra $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_4$, $\mathcal{T}_5$, $\mathcal{T}_6$, $\mathcal{T}_7$, $\mathcal{T}_9$ are congruent to each other and the remaining two tetrahedra $\mathcal{T}_3$ and $\mathcal{T}_8$ are congruent to each other too. Let us denote the angles and lengths of the edges of the tetrahedra $\mathcal{T}_1$, $\mathcal{T}_3$ as shown in Figure 4.

Then we have the following system of equations:

$$\begin{cases} \sin\frac{\gamma}{2} = \frac{x}{2h} \\ \sin\delta = \frac{x\sqrt{3}}{2h} \\ \gamma + 4\delta = \pi \\ h^2 = z^2 - (\frac{x}{2})^2 \\ \frac{x}{2y} = \frac{\mathrm{tg}\delta}{\sqrt{3}} \\ x^2 + y^2 = z^2 \end{cases} \tag{1}$$

Up to similarity the system (1) has a unique solution. The polyhedron mentioned above corresponds to $x = 2\sqrt{2}$ and $\frac{\theta}{2} = \frac{\pi}{3}$.

This scheme can be realized also for construction of the fundamental set of the 2-bridge link orbifolds and cone-manifolds in the spherical and hyperbolic cases.

2.3. The fundamental set of hyperbolic and spherical cone-manifolds $\mathcal{C}(\theta)$. We note that the topological structure of the fundamental set for the orbifold $\mathcal{C}(\pi)$ in the spherical space and of the fundamental set for the orbifold $\mathcal{C}(2\pi/3)$ in the Euclidean space are the same. Now the main idea is to construct a fundamental set for the cone-manifold $\mathcal{C}(\theta)$, $0 \leq \theta < 2\pi/3$ in hyperbolic space and $\mathcal{C}(\theta)$, $2\pi/3 < \theta < 4\pi/3$ in spherical space of the same topological type. Thus, after the identification of the faces of the fundamental polyhedron we shall have a cone-manifold whose singular set is the figure-eight knot and whose underlying space is the three-dimensional sphere $\mathbf{S}^3$.

This Section will be divided into two parts. In the first part we just suppose that the above mentioned polyhedron $\mathcal{P}(\theta)$ exists in the hyperbolic space $\mathbf{H}^3$. In such a way it is possible to obtain a system of equations for the dihedral angles and the lengths of the edges of this polyhedron and show that these solutions are sufficient for constructing the polyhedron. In the second part the fundamental polyhedron for the spherical cone-manifolds will be constructed.

Suppose that the corresponding fundamental set $\mathcal{P}(\theta)$ is already realized in the hyperbolic space. Identifying faces of the $\mathcal{P}(\theta)$ by the isometries S and T we have that polyhedron $\mathcal{P}(\theta)$ consists of ten tetrahedra which are each congruent to $\mathcal{T}_1$ or $\mathcal{T}_3$. Assume that equalities for the angles and the lengths of the edges of the tetrahedra $\mathcal{T}_1$ and $\mathcal{T}_3$ in the hyperbolic case are similar to the equalities in the Euclidean case. Then, by replacing the Euclidean relations between angles and lengths by the hyperbolic ones we have the following system of equations:

$$\begin{cases} \sin\frac{\gamma}{2} = \frac{\operatorname{sh}\frac{x}{2}}{\operatorname{sh} h} \\ \sin\delta = \frac{\operatorname{sh} x \sin\frac{\theta}{2}}{\operatorname{sh} h} \\ \gamma + 4\delta = \pi \\ \operatorname{ch} h = \frac{\operatorname{ch} z}{\operatorname{ch}\frac{x}{2}} \\ \coth\ y \operatorname{th}\frac{x}{2} = \operatorname{ctg}\frac{\theta}{2}\operatorname{tg}\delta \\ \operatorname{ch} z = \operatorname{ch} x \operatorname{ch} y \end{cases} \tag{2}$$

Solving the system of equations (2) directly, we get that it is equivalent to the algebraic equation of order seven with respect to the variable $u = \cos 2\delta$

$$\begin{aligned} &(Y + 4u^4 p^2)((1-u)^2(2-p^2) - (1+u)Yp^2) \\ &= 4u^2(1-u)(2-p^2)(Y + 2u^2p^2)^2 \end{aligned} \tag{3}$$

Where

$$Y = 1 - u - 4u^2 p^2 \tag{4}$$

$$u = \cos 2\delta \tag{5}$$

$$p^2 = 2\sin^2\frac{\theta}{2} \tag{6}$$

Equation (3) is equivalent to the following equation:

$$(2u^2p^2 + 2u - 1)(u + 1/2)(u - 1)(4u^2p^2 + u - 1)(u + p^2 - 1) = 0 \tag{7}$$

We are interested in the following factor of the equation (7):

$$2u^2p^2 + 2u - 1 = 0 \tag{8}$$

In this case

$$\operatorname{ch} x = \frac{1}{2up^2}\,, \tag{9}$$

$$\operatorname{ch} h = \sqrt{2(1 + \operatorname{ch} x)}\cos\frac{\theta}{2}\,, \tag{10}$$

$$\operatorname{ch} z = (1 + \operatorname{ch} x)\cos\frac{\theta}{2}\,, \tag{11}$$

$$\operatorname{ch} y = \frac{1 + \operatorname{ch} x}{\operatorname{ch} x}\cos\frac{\theta}{2}\,, \tag{12}$$

$$\gamma = \pi - 4\delta\,. \tag{13}$$

All other factors of (7) $u + 1/2 = 0$, $u - 1 = 0$, $4u^2p^2 + u - 1 = 0$, $u + p^2 - 1 = 0$, with a few exceptions in the case of multiple roots lead to degenerate geometrical situations. Solving the equation (8) we have

$$u = \cos 2\delta = \frac{-1 + \sqrt{1 + 4\sin^2\frac{\theta}{2}}}{4\sin^2\frac{\theta}{2}}\,, \tag{14}$$

REMARK 1. Formula (14) to determine the dihedral angle δ is also valid in the Euclidean case ($\theta = 2\pi/3$) and in the spherical case ($\theta > 2\pi/3$).

REMARK 2. We note from (14) that $\delta = \frac{1}{2}\arccos u$ and

$$0 < \delta < \frac{\pi}{4}\,. \tag{15}$$

Hence, $\gamma = \pi - 4\delta$ satisfies

$$(0 < \gamma < \pi)\,. \tag{16}$$

Inequalities (15) and (16) are very important in our case. Taken together with the condition $2\gamma + 8\delta = 2\pi$ they give us that the ten tetrahedra $\mathcal{T}_0, \ldots, \mathcal{T}_9$ forming the polyhedron $\mathcal{P}(\theta)$ have mutually disjoint interiors.

Check the Poincare theorem conditions for the fundamental group of orbifold $\mathcal{C}(\theta)$. Let us consider $\mathcal{P}(\theta)$ as curvilinear polyhedron with edges P_0P_5, P_2P_7, P_iP_{i+1}, $i = 0, \ldots, 9$, and with four curvilinear faces

$$Q_1P_5 \cdots P_0\,, \quad Q_1P_0 \cdots P_5\,, \quad Q_0P_7 \cdots P_2\,, \quad Q_0P_2 \cdots P_7\,.$$

Then under the action of the group $\langle S, T\rangle$ the polyhedron $\mathcal{P}(\theta)$ has exactly three edge cycles. Two of them, P_0P_5 and P_2P_7, correspond to the identification of faces $Q_1P_5\cdots P_0$, $Q_1P_0\cdots P_5$ and $Q_0P_7\cdots P_2$, $Q_0P_2\cdots P_7$. The third edge cycle P_iP_{i+1}, $i = 0, \ldots, 9$ has as its sum of angles $2\gamma + 8\delta = 2\pi$ and corresponds to the relation $sTStstSTst = 1$.

And as a result we have the following theorem:

THEOREM 4. *For all $0 \leq \theta < 2\pi/3$ the polyhedron $\mathcal{P}(\theta)$ is fundamental for the hyperbolic cone-manifold $\mathcal{C}(\theta)$. The holonomy group of the cone-manifold $\mathcal{C}(\theta)$ is generated by isometries S and T, which identify curvilinear faces of the polyhedron $\mathcal{P}(\theta)$ in the same way as in Lemma 1.*

In particular, for $\theta = 2\pi/n$, $n \in \mathbf{N}$ the cone-manifolds $\mathcal{C}(\theta)$ are orbifolds and the following theorem holds:

THEOREM 5. *Let S and T be hyperbolic isometries which identify curvilinear faces of the polyhedron $\mathcal{P}(2\pi/n)$, $n \geq 4$ in the same way as in Lemma 1.*
Then

(i) *the group $\langle S, T\rangle$ is a discrete isometry group of the space $\mathbf{H}^3$ and has the following presentation*:

$$\langle S,\ T \mid S^n = T^n = S^{-1}TST^{-1}S^{-1}T^{-1}STS^{-1}T^{-1} = 1\rangle$$

(ii) *the polyhedron $\mathcal{P}$ is a fundamental set for the group $\langle S, T\rangle$*

(iii) *the orbifold $\mathbf{H}^3/\langle S, T\rangle$ is isometric to $\mathcal{C}(\theta)$, $\theta = 2\pi/n$, $n \geq 4$.*

It is interesting to compare the values of angle δ of the polyhedra $\mathcal{P}(2\pi/n)$ for different n (Table 2). We recall that for $n = 2$ the $\mathcal{P}(2\pi/n)$ is realized in the spherical space, for $n = 3$ in the Euclidean space and for $n > 3$ in the hyperbolic space.

Consider the spherical case. For $2\pi/3 < \theta < 4\pi/3$ the following theorem holds.

THEOREM 6. *For all $2\pi/3 < \theta < 4\pi/3$ the polyhedron $\mathcal{P}(\theta)$ is fundamental for the spherical cone-manifold $\mathcal{C}(\theta)$. The holonomy group of the cone-manifold $\mathcal{C}(\theta)$ is generated*

TABLE 2

n	p^2	u	δ
2	2	0.30901	36^0
3	1.5	0.33333	35.2643^0
4	1	0.36602	34.2646^0
5	0.690983	0.39915	33.4246^0
6	0.5	0.41421	32.7652^0
7	0.376510	0.43028	32.2571^0
8	0.29289	0.44262	31.8644^0
9	0.23395	0.45217	31.5583^0
10	0.19098	0.45965	31.3117^0
∞	0	0.5	30^0

by the isometries S and T, which identify curvilinear faces of the polyhedron $\mathcal{P}(\theta)$ in the same way as in Lemma 1.

The proof of this theorem is based on the fact that in the spherical space the geometrical parameters of tetrahedra $\mathcal{T}_0, \ldots, \mathcal{T}_9$ can be found from the spherical analog of system (2):

$$\begin{cases} \sin\frac{\gamma}{2} = \frac{\sin\frac{x}{2}}{\sin h} \\ \sin\delta = \frac{\sin x \sin\frac{\pi}{n}}{\sin h} \\ \gamma + 4\delta = \pi \\ \cos h = \frac{\cos z}{\cos\frac{x}{2}} \\ \mathrm{ctg}\ y\ \mathrm{tg}\ \frac{x}{2} = \mathrm{ctg}\ \frac{\pi}{n} \mathrm{tg}\ \delta \\ \cos z = \cos x \cos y \end{cases} \tag{17}$$

Hence, as above, we have the following equation with respect to the variable $u = \cos 2\delta$:

$$2u^2p^2 + 2u - 1 = 0\,.$$

Thus,

$$u = \cos 2\delta = \frac{-1 + \sqrt{1 + 4\sin^2\frac{\pi}{n}}}{4\sin^2\frac{\pi}{n}} \tag{18}$$

$$\cos x = \frac{1}{2up^2}\,, \tag{19}$$

$$\cos h = \sqrt{2(1 + \cos x)}\cos\frac{\theta}{2}\,, \tag{20}$$

$$\cos z = (1 + \cos x)\cos\frac{\theta}{2}\,, \tag{21}$$

$$\cos y = \frac{1 + \mathrm{ch}\ x}{\mathrm{ch}\ x}\cos\frac{\theta}{2}\,, \tag{22}$$

The remaining part of the theorem can be proved in the same way as in Theorem 4.

3. Schläfli formula and volumes of the cone-manifolds $\mathcal{C}(\theta)$

Essentially, all the known results about volumes in the spherical and hyperbolic space either are contained in papers of Lobachevsky [14] and Schläfli [21] or based on the ideas of these papers. Among numerical papers on this subject we note papers of Coxeter [2], Kneser [12], Milnor [18] and Vinberg [24]. In particular, Schläfli established the formula of volume differential, depending on the differentials of angles in the spherical n-dimensional space. Coxeter and Kneser established the formula in the hyperbolic space.

In three-dimensional space the results of these investigations can be represented by the theorem:

THEOREM 7 (Differential Schläfli formula). *For a convex polyhedron in a space of constant curvature k, the angles of which depends on some set of parameters analytically, the volume V, the lengths ℓ_i of edges and the corresponding dihedral angles α_i are linked to each other by the following relation* [18, 24]:

$$kdV = \sum_i \frac{\ell_i}{2} d\alpha_i\,. \tag{23}$$

We apply Theorem 7 to calculate the volumes of the cone-manifolds $\mathcal{C}(\theta)$ to the polyhedron constructed in the previous Section. First we make some remarks. As mentioned above, it is possible to divide the polyhedron $\mathcal{P}(\theta)$ into the ten tetrahedra $\mathcal{T}_0, \ldots, \mathcal{T}_9$. We note that from the polyhedron's construction, the sum of the dihedral angles $\theta_0, \ldots, \theta_9$ corresponding to the edges $P_0P_1, \ldots, P_9P_0$ is equal to 2π, and the lengths ℓ_{θ_i} of the edges are equal to ℓ_0. Hence, we have:

$$\sum_i \ell_{\theta_i} d\theta_i = 0\,. \tag{24}$$

Actually,

$$\sum_i \ell_{\theta_i} d\theta_i = \sum_i \ell_0 d\theta_i = \ell_0 \sum_i d\theta_i = 0\,.$$

Similar arguments are correct for the other set of edges forming incident cycles (i.e. pairwise equivalent under the identification of the polyhedron's faces and producing as their sum the angle 2π). Thus, the differentials of incident angles are not a part of the final formula. Except for the edges forming incident cycles, the polyhedron has just two non-incident edges P_0P_5 and P_2P_7 with the same length ℓ and dihedral angle θ. So, the differential of volume is equal to

$$kdV = \frac{1}{2}\sum_i \ell_i d\theta_i = \frac{1}{2}\ell d\theta + \frac{1}{2}\ell d\theta = \ell d\theta\,. \tag{25}$$

We find the explicit expression for ℓ and V as functions of θ in the hyperbolic and spherical spaces.

In the hyperbolic case, from formulas (8–12) we have

$$\mathrm{ch}\frac{\ell}{2} = \cos\frac{\theta}{2}\sqrt{4\sin^2\frac{\theta}{2} + 1}\,, \tag{26}$$

and

$$\mathrm{ch}\ell = 1 + \cos\theta - \cos 2\theta\,. \tag{27}$$

Then, taking into account that $k = -1$, we have

$$-dV = \frac{1}{2}\operatorname{arcch}(1 + \cos\theta - \cos 2\theta)d\theta \tag{28}$$

and for all $0 < \theta, \widetilde{\theta} < 2\pi/3$

$$V(\theta) = -\int_{\widetilde{\theta}}^{\theta} \operatorname{arcch}(1 + \cos\theta - \cos 2\theta)d\theta + V(\widetilde{\theta})\,. \tag{29}$$

For angle $\widetilde{\theta} \to 2\pi/3$ from left we have that lenghts of edges $P_i P_{i+1}$, $(i \ mod\ 10)$ and $P_0 P_5$, $P_2 P_7$ of the polyhedron $\mathcal{P}(\theta)$ are equal to zero. Actually, $\mathrm{ch}x \to 1$, $\mathrm{ch}y \to 1$, $\mathrm{ch}z \to 1$. Then $V(\widetilde{\theta}) \to 0$ for $\widetilde{\theta} \to 2\pi/3$.

Finally, for the limiting situation in formula (29) we have the following theorem:

THEOREM 8. *The hyperbolic volume of the cone-manifold $\mathcal{C}(\theta)$, $0 < \theta < \pi/3$ is represented by the formula*:

$$V(\theta) = \int_{\frac{2\pi}{3}}^{\theta} \operatorname{arcch}(1 + \cos\theta - \cos 2\theta)d\theta\,. \tag{30}$$

We note, that a more complicated version of this formula, obtained from (29) as $\theta \to 0$, can be found in [13].

In the spherical case the situation is more specific. Formulas (18)–(22) allow us to calculate $Y = \cos\ell$ in the following way: $Y = 1 + \cos\theta - \cos 2\theta$. Then $\ell = \pm \arccos Y + 2\pi k$, $k \in \mathbf{Z}$. Thus, we need just to define the sign and value of k. Now we show that this choice depends on the angle θ. More exactly, the following lemma holds:

LEMMA 9. *For the fundamental polyhedron $\mathcal{P}(\theta)$*

(i) *for $2\pi/3 < \theta \le \pi$, $\ell = \arccos Y$, and for $\pi < \theta < 4\pi/3$, $\ell = 2\pi - \arccos Y$.*

(ii) *for $\theta = 4\pi/3$ the singular set of the cone-manifold $\mathcal{C}(\theta)$ degenerates to a generalized knot with two vertices, pairwise joined by four arcs (Figure 8).*

PROOF. We observe the evolution of the tetrahedra $\mathcal{T}_1$, $\mathcal{T}_3$ for $2\pi/3 < \theta \le \pi$. The functions $\cos x$, $\cos y$ and $\cos z$ on the interval $2\pi/3 < \theta \le \pi$ are decreasing functions. Thus, the functions x, y and z are expressed by the main branch of the arccosine increase and

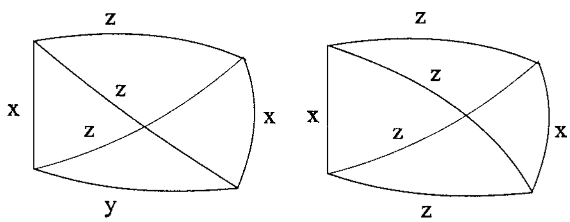

FIGURE 5. The tetrahedra $\mathcal{T}_1$ and $\mathcal{T}_3$ for $2\pi/3 < \theta \le \pi$.

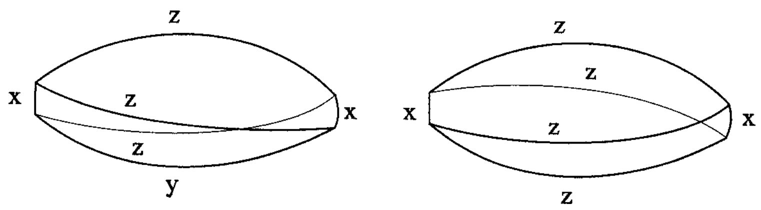

FIGURE 6. The tetrahedra $\mathcal{T}_1$ and $\mathcal{T}_3$ for $\pi < \theta < 4\pi/3$.

$$0 < x \le \frac{\pi}{5}\,, 0 < y \le \frac{\pi}{2}\,, 0 < z \le \frac{\pi}{2}\,. \tag{31}$$

Now we observe the evolution of tetrahedra $\mathcal{T}_1$, $\mathcal{T}_3$ for $\pi < \theta < 4\pi/3$.

In this case, the functions x, y, z are expressed by the main branch of arccosine, functions y and z increase, function x decreases and

$$0 < x \le \frac{\pi}{5}\,, \frac{\pi}{2} < y \le \pi\,, \frac{\pi}{2} < z \le \pi\,. \tag{32}$$

The new parameters

$$x = \arccos\left(\frac{1}{\sqrt{3-2\cos\theta}-1}\right), \tag{33}$$

$$y = \arccos\left(\cos\frac{\theta}{2}\sqrt{3-2\cos\theta}\right), \tag{34}$$

$$z = \arccos\left(\cos\frac{\theta}{2}\left(1+\frac{1}{\sqrt{3-2\cos\theta}-1}\right)\right) \tag{35}$$

are continuous and depend on the parameter θ analytically. Let us calculate the length of ℓ depending on θ. We have:

$$\ell = 2y,\ \text{where } y = \arccos\left(\cos\frac{\theta}{2}\sqrt{3-2\cos\theta}\right).$$

Due to formulae

$$2\arccos y = \arccos(2y^2-1)\,, \quad \textit{if } y \ge 0\,,$$

$$2\arccos y = 2\pi - \arccos(2y^2-1)\,, \quad \textit{if } y < 0\,, \tag{36}$$

in case of $2\pi/3 \le \theta \le \pi$ we obtain

$$\cos\frac{\theta}{2}\sqrt{3-2\cos\theta} \ge 0\,, \tag{37}$$

$$\ell = 2\arccos y = \arccos(1+\cos\theta-\cos 2\theta)\,. \tag{38}$$

If $\pi < \theta < 4\pi/3$,

$$\cos\frac{\theta}{2}\sqrt{3-2\cos\theta} < 0\,, \tag{39}$$

$$\ell = 2\arccos y = 2\pi - \arccos(1+\cos\theta-\cos 2\theta)\,. \tag{40}$$

Now we show, that the singular set of the cone-manifold $\mathcal{C}(\theta)$ degenerates for $\theta = 4\pi/3$. Actually, for $\theta \to 4\pi/3$, $\cos x \to 0$. Therefore, the distance between the points Q_0, Q_1 goes to zero and for $\theta = 4\pi/3$ becomes the vertex of valency four. Thus, the figure-eight knot turns into a singular knot in Birman's terminology [1]. The detailed structure of the singular set of the cone-manifolds $\mathcal{C}(\theta)$ for $\theta = 4\pi/3$ will be described in the fourth Section.

Using Schläfli formula we calculate the volume of cone-manifolds in the spherical space. As before, taking into consideration that $k = +1$, we have

$$V(\theta) = \int_{\widetilde{\theta}}^{\theta} \arccos(\ell)d\theta + V(\widetilde{\theta})\,, \tag{41}$$

where ℓ is defined by formulae (38) and (40), $\widetilde{\theta}$ and θ belong to the interval $(2\pi/3, 4\pi/3)$. We note, that for $\widetilde{\theta} \to 2\pi/3$ from formulae (18)–(22) $x, y, z \to 0$, and, consequently, $V(\widetilde{\theta}) \to 0$. Hence,

$$V(\theta) = \int_{\frac{2\pi}{3}}^{\theta} \arccos(\ell)d\theta\,. \tag{42}$$

Consider the two cases.

If $2\pi/3 < \theta \le \pi$, then $\ell = \arccos(1+\cos\theta-\cos 2\theta)$ and

$$V(\theta) = \int_{\frac{2\pi}{3}}^{\theta} \arccos(1+\cos\theta-\cos 2\theta)d\theta\,. \tag{43}$$

In particular, for $\theta = \pi$ we have $V(\pi) = \pi^2/5$. It is an obvious consequence of the fact that the two-fold covering of the π-orbifold is the Lens space $L(5,2)$, which is the 5-fold covering over the three-dimensional sphere. Recall, that the volume of the three-dimensional sphere is equal to $Vol(\mathbf{S}^3) = 2\pi^2$.

If $\pi < \theta < 4\pi/3$, then $\ell = 2\pi - \arccos(1+\cos\theta-\cos 2\theta)$, and

$$\begin{aligned} V(\theta) &= \int_{\frac{2\pi}{3}}^{\theta} \ell d\theta \int_{\frac{2\pi}{3}}^{\pi} \ell d\theta + \int_{\pi}^{\theta} \ell d\theta \\ &= \frac{\pi^2}{5} + \int_{\pi}^{\theta} (2\pi - \arccos(1+\cos\theta-\cos 2\theta))d\theta \\ &= 2\pi(\theta - 0.9\pi) - \int_{\pi}^{\theta} \arccos(1+\cos\theta-\cos 2\theta)d\theta\,. \end{aligned} \tag{44}$$

Thus we have the following theorem:

THEOREM 9. *The spherical volume of the cone-manifold $\mathcal{C}(\theta)$ is given by the following formulae*:

$$V(\theta) = \int_{\frac{2\pi}{3}}^{\theta} \arccos(1 + \cos\theta - \cos 2\theta) d\theta\,, \quad \textit{if}\ 2\pi/3 < \theta \leq \pi\,, \tag{45}$$

$$V(\theta) = 2\pi(\theta - 0.9\pi) - \int_{\pi}^{\theta} \arccos(1 + \cos\theta - \cos 2\theta) d\theta\,, \quad \textit{if}\ \pi < \theta < 4\pi/3\,. \tag{46}$$

We note, that as obtained in [8], the spherical structure exists on the cone-manifold $\mathcal{C}(\theta)$ only for $2\pi/3 < \theta \leq \pi$.

4. The cone–manifold structure's degeneration

We start this section from the following remark. Consider the two edges P_0P_1 and Q_0Q_1 of the tetrahedron $\mathcal{T}_0$. They trace the shortest distance between the axes of the elliptic elements S, T and S, $TSTS^{-1}T^{-1}$. The method of identifying the faces of the polyhedron $\mathcal{P}(\theta)$ allows us to conclude that these edges correspond to the two tunnels of the figure-eight knot, shown in Figure 7. Note that the arcs Q_0P_1, Q_0P_0, Q_1P_0 and Q_1P_1 divide the singular set of the cone-manifold into four parts with lengths $\ell/2$.

Due to the results of the previous sections, the lengths P_0P_1 and Q_0Q_1 are equal to x and can be obtained from the formula:

$$\cos x = \frac{1}{\sqrt{3 - 2\cos\theta} - 1}$$

for θ from interval $2\pi/3 < \theta < 4\pi/3$. As it was mentioned above, for $\theta \to 4\pi/3$, we have $\cos x \to 1$, and then $x \to 0$. It means, that the singular set of the cone-manifold $\mathcal{C}(\theta)$ for $\theta \to 4\pi/3$ becomes the degenerate knot shown in Figure 8.

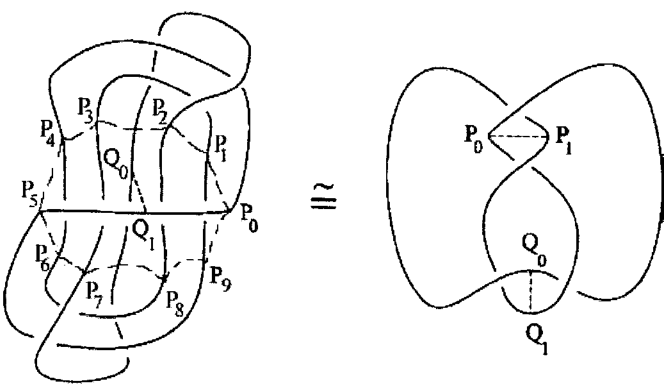

FIGURE 7. The tunnels P_0P_1 and Q_0Q_1 of the figure-eight knot.

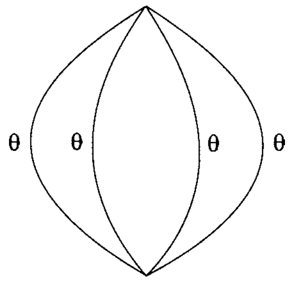

FIGURE 8. The singular set of $G(\theta)$.

The singular knot is a graph with two vertices, pairwise joined by four edges. All the edges have the same length $\ell/2 = \pi$. The cone angle around each edge is equal to $\theta = 4\pi/3$. Denote by $G(\theta)$ the cone-manifold with singular set shown in Figure 8 and cone angle θ. Then we have the following statement:

LEMMA 10. *The cone-manifold $G(\theta)$ is realized in spherical space for all $\pi < \theta \leq 2\pi$ and has the spherical volume equal to $2(k-1)\pi^2$, where k is defined by equality $\theta = k\pi$.*

PROOF. Consider the sphere $\mathbf{S}^2(\theta, \theta, \theta, \theta)$ with four conic points. It can be obtained by gluing together two spherical squares $Q(\theta/2)$ with angles $\theta/2$ along their common boundary. Consequently, for all $\pi/2 < \theta/2 \leq \pi$ it is realized in two-dimensional spherical geometry $\mathbf{S}^2$. Consider cone $K(\theta)$ over $\mathbf{S}^2(\theta, \theta, \theta, \theta)$. Obviously it can be realized in spherical geometry $\mathbf{S}^3$ and represents the half of the cone-manifold $G(\theta)$. More precisely, $G(\theta)$ is doubled $K(\theta)$, obtained by mirror reflection in the boundary $\partial K(\theta) = \mathbf{S}^2(\theta, \theta, \theta, \theta)$.

We calculate the volume of the cone-manifold $G(\theta)$. For this purpose we imagine $K(\theta)$ as a union of two cones over the square $Q(\theta/2)$. Since the ratio of the square's area to the area of the sphere $\mathbf{S}^2$ is equal to $(2\theta - 2\pi)/4\pi = (k-1)/2$, we have

$$Vol(K(\theta)) = \frac{(k-1)}{2} Vol(\mathbf{S}^3) = (k-1)\pi^2 \,, \tag{47}$$

then

$$Vol(G(\theta)) = 2Vol(K(\theta)) = 2(k-1)\pi^2 \,. \tag{48}$$

Note, that for $\theta = \pi$ the considered cone-manifold $G(\theta)$ is realized in Euclidean geometry $\mathbf{E}^3$ and has an infinite volume. For $\theta = 2\pi$, $G(\theta)$ coincides with the three-dimensional sphere $\mathbf{S}^3$ and has volume $2\pi^2$.

To conclude this Section, we will prove the following theorem:

THEOREM 11. *The limit of the volumes of the cone-manifolds $\mathcal{C}(\theta)$ for $\theta \to 4\pi/3$ is equal to the volume of the cone-manifold $G(4\pi/3)$:*

$$\lim_{\theta \to 4\pi/3} Vol(\mathcal{C}(\theta)) = Vol(G(4\pi/3)) \,.$$

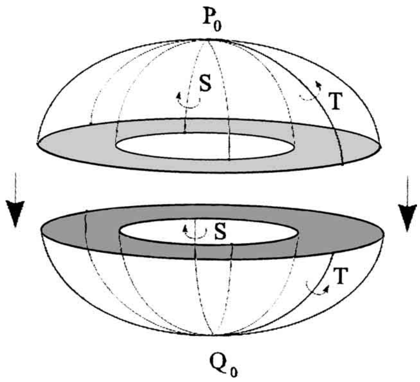

FIGURE 9. The fundamental polyhedron $\mathcal{P}(4\pi/3)$.

To prove the theorem it is sufficient to prove the convergence of fundamental polyhedra. Show, that we can choose as the fundamental polyhedron for the cone-manifold $G(4\pi/3)$ the polyhedron obtained from $\mathcal{P}(\theta)$ for $\theta \to 4\pi/3$. Actually, for $\theta = 4\pi/3$ the points Q_0, Q_1 and $P_i, i = 0, \ldots, 9$ belong to just two points: $Q = Q_0 = Q_1$ and $P = P_0 = P_1 = \cdots = P_9$. Moreover, the axes of elements S and T belong to circles, crossing P and Q. The fundamental set is bounded by two interior and two exterior semiplanes. The two exterior planes intersect in axis of element S, the two interior planes intersect in the axis of element T. In Figure 9 the fundamental polyhedron $\mathcal{P}(4\pi/3)$ of the cone-manifold $G(4\pi/3)$ is shown bisected by a plane into two parts.

Thus, the singular set of the limit cone-manifold consists of two vertices, pairwise joined by four arcs. After identifying the faces of the limit polyhedron under the action of the isometries S and T we have the whole space $\mathbf{S}^3$. Consequently, we have the coincidence of the geometrical parameters of the polyhedron $\mathcal{P}(\theta)$ with the geometrical parameters of the cone-manifold $G(4\pi/3)$ fundamental polyhedron.

Now the statement of the theorem follows immediately from the equation:

$$\begin{aligned}\lim_{\theta\to 4\pi/3} Vol(\mathcal{C}(\theta)) &= \lim_{\theta\to 4\pi/3}\left(2\pi(\theta - 0.9\pi) - \int_{\pi}^{\theta} \arccos(1+\cos\theta - \cos 2\theta)d\theta\right)\\ &= \frac{13\pi^2}{15} - \int_{2\pi/3}^{\pi} \arccos(1+\cos\theta-\cos 2\theta)d\theta = 2\pi^2/3\\ &= Vol(G(4\pi/3))\,.\end{aligned}$$

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the existence of spherical structure similar to Theorem 6 and its degeneration in Theorem 11 for any two bridge knot were obtained by Joan Porti [19], after the first version [16] of this paper was completed.

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