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$\mathcal{D}$-Modules and Arrangements of Hyperplanes

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Abstract. Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\boldsymbol{C}^n$ defined by the homogeneous polynomial $d_{\mathcal{A}}$. Let D_n be the Weyl algebra of rank n over $\boldsymbol{C}$ and let $P = \boldsymbol{C}[x_1, \dots, x_n, d_{\mathcal{A}}^{-1}]$ be the algebra of rational functions on the variety $Y_{\mathcal{A}} = \boldsymbol{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Studying the structure of P as a D_n-module we obtain a sequence of new D_n-modules. These modules allow us to define useful complexes that determine the De Rham cohomology of $Y_{\mathcal{A}} = \boldsymbol{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Finally we compute the Poincaré series of P.

1. Introduction

Let $\mathcal{A} = \{H_1, \dots, H_k\}$ be a finite central arrangement of hyperplanes in $\boldsymbol{C}^n$, i.e., every hyperplane contains the origin. For each $H \in \mathcal{A}$, fix a linear form α_H whose kernel is H. The arrangement $\mathcal{A}$ is also defined by the homogeneous polynomial $d_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_H$.

Let $D_n = \boldsymbol{C}\langle x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n \rangle$ be the Weyl algebra of rank n over $\boldsymbol{C}$ and let $P = P(\mathcal{A}) = \boldsymbol{C}[x_1, \dots, x_n, d_{\mathcal{A}}^{-1}]$ be the algebra of rational functions on $Y_{\mathcal{A}} = \boldsymbol{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$. In the present work we construct a sequence of D_n-submodules of P and direct sum decompositions of the associated quotient modules. Furthermore, using this decomposition, we compute the cohomology ring $H^*(Y_{\mathcal{A}})$ (Section 3), and the Poincaré series of P (Section 4). All D_n-modules mentioned here are left D_n-modules. Denote the poset of intersections of elements of $\mathcal{A}$ by $L = L(\mathcal{A})$ ordered by reversed inclusion, and with a rank function defined by $r(X) = \mathrm{codim} X$, $X \in L$. Let $r = r(\mathcal{A}) = r(\bigcap_{H \in \mathcal{A}} H)$ be the rank of the maximal element of $L(\mathcal{A})$, namely, the cardinality of a maximal linearly independent subset of $\mathcal{A}^* = \{\alpha_H \mid H \in \mathcal{A}\}$. Then each element of P can be written as a finite sum of quotients of the form $f/\prod_{j=1}^{h} \alpha_{i_j}^{m_j}$, where $0 \le h \le r$, $\{\alpha_{i_1}, \dots, \alpha_{i_h}\}$ is a linearly independent subset of $\mathcal{A}^*$, $m_j \in \boldsymbol{N}$, $f \in \boldsymbol{C}[\mathbf{x}] = \boldsymbol{C}[x_1, \dots, x_n]$ and $\prod_{j=1}^{0} \alpha_{i_j}^{m_j} := 1$. This allows us to obtain the following sequence of holonomic D_n-submodules of $P : 0 = P_{-1} \subset \boldsymbol{C}[\mathbf{x}] = P_0 \subset P_1 \subset$

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$\cdots \subset P_r = P$, where

$$P_h = \left\{ \sum \frac{f_{i_1 \cdots i_t}^{m_1 \cdots m_t}}{\alpha_{i_1}^{m_1} \cdots \alpha_{i_t}^{m_t}} \;\middle|\; 0 \leq t \leq h,\ f_{i_1 \cdots i_t}^{m_1 \cdots m_t} \in \boldsymbol{C}[\mathbf{x}],\ m_1, \ldots, m_t \in \boldsymbol{N} \right\}.$$

For each $X \in L_h = L_h(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid r(X) = h\}$ consider its dual subspace X^* of $(\boldsymbol{C}^n)^*$ of dimension h. Let $\mathcal{B}_{X^*}$ be the set of all possible bases of X^* constituted with elements of $\mathcal{A}^*$. For each X^* and basis $B = \{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \in \mathcal{B}_{X^*}$, we define the following holonomic D_n-submodule of P_h/P_{h-1}

$$V_{X^*}^B = \left\{ \sum \left(\frac{f_{i_1 \cdots i_h}^{m_1 \cdots m_h}}{\alpha_{i_1}^{m_1} \cdots \alpha_{i_h}^{m_h}} \bmod P_{h-1} \right) \;\middle|\; f_{i_1 \cdots i_h}^{m_1 \cdots m_h} \in \boldsymbol{C}[\mathbf{x}],\ m_1, \ldots, m_h \in \boldsymbol{N} \right\}.$$

We show in Proposition 2.9 that for each basis $B \in \mathcal{B}_{X^*}$ the D_n-modules $V_{X^*}^B$ are isomorphic, and after a linear change of coordinates in $(\boldsymbol{C}^n)^*$ such that $X^* = \langle y_1, \ldots, y_h \rangle$, $V_{X^*}^B$ is isomorphic, as a D_n-module, to $M_{X^*} = \boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ where $\partial_{y_j} = \partial/\partial y_j$. Now let $V_{X^*}^{\text{mod}}$ be the $\boldsymbol{C}$-subspace of P_h/P_{h-1} generated by all $[1/\prod_{\alpha \in B} \alpha]$, $B \in \mathcal{B}_{X^*}$, then the holonomic D_n-module P_h/P_{h-1} has the following decomposition:

$$P_h/P_{h-1} = \bigoplus_{X \in L_h} \sum_{B \in \mathcal{B}_{X^*}} V_{X^*}^B = \bigoplus_{X \in L_h} M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*}^{\text{mod}}.$$

It is possible to determine a basis of $V_{X^*}^{\text{mod}}$ applying the notion of "no broken circuit" nbc (cf. [7]) to $\mathcal{B}_{X^*}$. Let V_{X^*} be the $\boldsymbol{C}$-vector space generated by the set of rational forms $\{1/\prod_{\alpha \in B} \alpha \mid B \in \mathcal{B}_{X^*}\}$, then the set $\{1/\prod_{\alpha \in B} \alpha \mid B \in \mathcal{B}_{X^*}$ and B is a nbc$\}$ is a basis of V_{X^*}, cf. Lemma 2.14, and we have:

THEOREM 2.19. *For $1 \leq h \leq r$, we have $P_h = \bigoplus_{j=0}^h \bigoplus_{X \in L_j(\mathcal{A})} M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*}$. In particular, since $P = P_r$, we have $P = \bigoplus_{X \in L(\mathcal{A})} M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*}$.*

THEOREM 2.20. *For $0 \leq h \leq r$, the natural map $\psi : \bigoplus_{X \in L_h} M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*} \to P_h/P_{h-1}$ is an isomorphism of D_n-modules.*

This allows us to decompose the De Rham complex for $Y_{\mathcal{A}}$ as a direct sum of complexes with cohomology just in one degree and 1-dimensional. Define the following cochain complex $(\mathcal{L}_h^*, \delta_{\mathcal{L}_h^*})$:

$$\mathcal{L}_h^s = \mathcal{L}_h^s(\{y_1, \ldots, y_h\}) = \left\{ \sum_{1 \leq i_1 < \cdots < i_s \leq n} f_{i_1 \cdots i_s} \bullet \frac{1}{y_1 \cdots y_h} dy_{i_1} \cdots dy_{i_s} \right\}$$

with $\delta_{\mathcal{L}_h^*} : \mathcal{L}_h^* \to \mathcal{L}_h^*$ the usual differential, and $f_{i_1 \cdots i_s} \in \boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$. Thus, cf. Corollary 3.4, the cohomology groups $H^*(\mathcal{L}_h^*)$ are $\boldsymbol{C} \cdot \dfrac{1}{y_1 \cdots y_h} dy_1 \cdots dy_h$ in dimension

h and 0 elsewhere. Now for each $X \in L_h(\mathcal{A})$ we associate the following complex

$$\mathcal{L}_h(X) = \bigoplus_{\substack{\langle \alpha_{j_1},\dots,\alpha_{j_h}\rangle = X^* \\ (j_1,\dots,j_h)\ \mathbf{nbc}}} \mathcal{L}_h(\{\alpha_{j_1},\dots,\alpha_{j_h}\})$$

where $\mathcal{L}_h(\{\alpha_{j_1},\dots,\alpha_{j_h}\})$ is the same complex $\mathcal{L}_h^*$ but it is just defined for $\{\alpha_{j_1},\dots,\alpha_{j_h}\}$. Finally, associated to the D_n-module $\mathcal{P}_h = P_h/P_{h-1}$, the complex $\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X\in L_h} \mathcal{L}_h(X)$ allows us to calculate the h-th cohomology of $Y_{\mathcal{A}}$.

THEOREM 3.6. *For $1 \le h \le r$, there exists an isomorphism between $H^h_{DR}(Y_{\mathcal{A}})$ and $H^h(\mathcal{L}(\mathcal{P}_h))$:*

$$H^h_{DR}(Y_{\mathcal{A}}) \cong H^h(\mathcal{L}(\mathcal{P}_h)) = \bigoplus_{X\in L_h} \bigoplus_{\substack{\langle \alpha_{j_1},\dots,\alpha_{j_h}\rangle = X^* \\ (j_1,\dots,j_h)\ \mathbf{nbc}}} \boldsymbol{C} \cdot \frac{1}{\alpha_{j_1}\cdots\alpha_{j_h}}\, d\alpha_{j_1} \wedge \cdots \wedge d\alpha_{j_h}\,.$$

Finally, we compute the Poincaré series of $P(\mathcal{A})$ as a function of the Poincaré polynomial of $\mathcal{A}$:

THEOREM 4.4. *The Poincaré series Poin$(P(\mathcal{A}), t)$ of the graded D_n-module $P(\mathcal{A})$ is equal to $(1-t)^{-n}$Poin$(\mathcal{A}, t)$.*

Throughout the paper we follow notation, definitions and results of [8], [9] on arrangements, and of [1], [3] for the Weyl algebra and its left modules.

2. The D_n-module $\boldsymbol{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}]$

This section is dedicated to the algebraic properties of the left D_n-module $P = P(\mathcal{A}) = \boldsymbol{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}]$. The main results are Theorem 2.19 and Theorem 2.20.

Due to [3, 3.2 Theorem (p. 92)], the D_n-module $P(\mathcal{A})$ is holonomic.

Recall that the rank $r = r(\mathcal{A})$ of $\mathcal{A}$ is the cardinality of a maximal linearly independent subset of $\mathcal{A}^*$. The following Lemma is straightforward and allows us to write in a very convenient way every element of P.

LEMMA 2.1. *It is possible to write every element in P as a finite sum of quotients of the form $f/\prod_{j=1}^h \alpha_{i_j}^{m_j}$, where $0 \le h \le r$, $\{\alpha_{i_1},\dots,\alpha_{i_h}\}$ is a linearly independent subset of $\mathcal{A}^*$, $m_1,\dots,m_h \in \boldsymbol{N}$, $f \in \boldsymbol{C}[\mathbf{x}]$ and $\prod_{j=1}^0 \alpha_{i_j}^{m_j} := 1$.*

This Lemma inspires the following definition.

DEFINITION 2.2. For $h = 0, 1, \dots, r$, define the D_n-submodule of P by

$$P_h = \left\{ \sum \frac{f_{i_1\cdots i_t}^{m_1\cdots m_t}}{\prod_{j=1}^t \alpha_{i_j}^{m_i}} \;\middle|\; 0 \le t \le h,\ f_{i_1\cdots i_t}^{m_1\cdots m_t} \in \boldsymbol{C}[\mathbf{x}],\ m_1,\dots,m_t \in \boldsymbol{N} \right\},$$

where $\{\alpha_{i_1},\dots,\alpha_{i_t}\}$ varies over all the linearly independent subsets of $\mathcal{A}^*$ of cardinality t.

Hence, by Lemma 2.1, we have the following finite ascending chain of D_n-submodules of P:

$$0 =: P_{-1} \subseteq \boldsymbol{C}[\mathbf{x}] = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_r = P\,. \tag{2.1}$$

Note that every module P_h and every quotient P_h/P_{h-1} is holonomic since P is, cf. [3, p. 86].

Our next aim is to get a decomposition of P_h/P_{h-1} as a direct sum of isotropy component D_n-modules associated to each $X \in L_h(\mathcal{A})$, cf. Proposition 2.10.

For each $X \in L(\mathcal{A})$, consider the dual subspace X^* of $(\boldsymbol{C}^n)^*$ of dimension $r(X)$.

DEFINITION 2.3. For each X in $L_h(\mathcal{A})$, $1 \leq h \leq r$, let $\mathcal{B}_{X^*}$ be the set of all possible bases of X^* constituted with elements of $\mathcal{A}^*$, and for each basis $B = \{\alpha_{i_1}, \dots, \alpha_{i_h}\}$ in $\mathcal{B}_{X^*}$ define the holonomic D_n-submodule of P_h/P_{h-1}

$$V_{X^*}^B = \left\{ \sum \left(\frac{f_{i_1\cdots i_h}^{m_1\cdots m_h}}{\alpha_{i_1}^{m_1}\cdots\alpha_{i_h}^{m_h}} \bmod P_{h-1} \right) \;\middle|\; f_{i_1\cdots i_h}^{m_1\cdots m_h} \in \boldsymbol{C}[\mathbf{x}],\ m_1, \dots, m_h \in \boldsymbol{N} \right\}.$$

From the definition it follows that $V_{X^*}^B$ is an irreducible D_n-module. Then it is cyclic (this is also a consequence of its holonomicity). A generator for $V_{X^*}^B$, as a D_n-module, is the class of $1/\prod_{\alpha\in B}\alpha$, cf. Proposition 2.5.

Let $X \in L_h$ and let $B = \{\alpha_{i_1}, \dots, \alpha_{i_h}\}$ be a basis for X^*. Then there exists a basis $\{y_1 := \alpha_{i_1}, \dots, y_h := \alpha_{i_h}, \dots, y_r := \alpha_{i_r}, y_{r+1}, \dots, y_n\}$ of $(\boldsymbol{C}^n)^*$, where $\{y_1, \dots, y_r\}$ is a maximal linearly independent subset of $\mathcal{A}^*$. The element $[1/y_1\cdots y_h]$ in $V_{X^*}^{\{y_1,\dots,y_h\}}$ is annihilated by the linear operators $y_1, \dots, y_h, \partial_{y_{h+1}}, \dots, \partial_{y_n}$, i.e. by the left D_n-ideal $I_B = D_n(y_1, \dots, y_h, \partial_{y_{h+1}}, \dots, \partial_{y_n})$. Actually it is very easy to see that:

LEMMA 2.4. *With the previous definitions we have*

$$\operatorname{Ann}_{D_n}([1/y_1\cdots y_h]) = I_B\,.$$

The ideal I_B plays an important role in what follows.

PROPOSITION 2.5. *Let M_B be the D_n-module D_n/I_B. Then we have the isomorphism of D_n-modules*

$$V_{X^*}^{\{y_1,\dots,y_h\}} \cong M_B \cong D_n \bullet [1/y_1\cdots y_h]\,. \tag{2.2}$$

PROOF. The first isomorphism follows from Lemma 2.4 and [3, p. 36]. The second isomorphism follows from the exact sequence $0 \to I_B \to D_n \to D_n \bullet [1/y_1\cdots y_h] \to 0$. □

COROLLARY 2.6. *Consider two different elements X_1, X_2 in L_h, $1 \leq h \leq r$. Then $V_{X_1^*}^{B_1} \cap V_{X_2^*}^{B_2} = \{[0]\}$ for each B_1 in $\mathcal{B}_{X_1^*}$ and for each B_2 in $\mathcal{B}_{X_2^*}$.*

PROPOSITION 2.7. *There exists an isomorphism of D_n-modules between M_B and the ring of polynomials $\boldsymbol{C}[y_{h+1}, \dots, y_n, \partial_{y_1}, \dots, \partial_{y_h}]$. This last one is an irreducible, holonomic D_n-module, and its characteristic variety is the conormal space defined by the system of* *equations* $\xi_1 = \cdots = \xi_h = \xi_{n+h+1} = \cdots = \xi_{2n} = 0$, *where for* $i = 1, \ldots, h$ $\xi_i = \sigma_1(y_i)$, *and for* $i = 1, \ldots, n-h$, $\xi_{n+h+i} = \sigma_1(\partial_{y_{h+i}})$ (σ_1 *is the symbol map of order 1, cf.* [3, p. 57]).

PROOF. Let $\mathcal{T}$ be the automorphism of D_n defined by

$$\begin{array}{lll} \mathcal{T}(y_i) = \partial_{y_i}\ , & \mathcal{T}(\partial_{y_i}) = -y_i & \text{for} \quad 1 \le i \le h \\ \mathcal{T}(y_i) = y_i\ , & \mathcal{T}(\partial_{y_i}) = \partial_{y_i} & \text{for} \quad h+1 \le i \le n\ . \end{array}$$

Recall that $\boldsymbol{C}[y_1, \ldots, y_n] \cong D_n/J$, where $J = \sum_1^n D_n \cdot \partial_{y_i}$, then it is easy to see that

$$\mathcal{T}^{-1}(J) = \sum_1^h D_n \cdot y_i + \sum_{h+1}^n D_n \cdot \partial_{y_i} = I_B\ ,$$

and, by [3, p. 38], we get

$$M_B = D_n/\mathcal{T}^{-1}(J) \cong \boldsymbol{C}[y_1, \ldots, y_n]_{\mathcal{T}} \cong \boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]\ .$$

Thus, by [3, p. 38, p. 86], $\boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ is an irreducible, holonomic D_n-module isomorphic to M_B.

Let $\mathcal{B}$ be the Bernstein filtration of D_n, cf. [1], [3]. Recall that the graded algebra $\mathrm{gr}^{\mathcal{B}} D_n$ is isomorphic to the polynomial ring in $2n$ variables $\boldsymbol{C}[\xi] = \boldsymbol{C}[\xi_1, \ldots, \xi_{2n}]$, cf. [3, p. 58]. Let Γ be a good filtration of $\boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ with respect to $\mathcal{B}$, for example, the induced one by $\mathcal{B}$. The exact sequence $0 \to I_B \to D_n \to M_B \to 0$ in turns implies the following exact sequence of $\boldsymbol{C}[\xi]$-modules

$$0 \to \mathrm{gr}^{\Gamma'} I_B \to \mathrm{gr}^{\mathcal{B}} D_n \to \mathrm{gr}^{\Gamma} M_B \to 0\,,$$

where Γ' is the filtration induced by $\mathcal{B}$ on I_B. Then $\mathrm{gr}^{\Gamma} M_B \cong \dfrac{\boldsymbol{C}[\xi]}{\mathrm{gr}^{\Gamma'} I_B}$ and

$$\begin{aligned} \mathrm{Ann}(\boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}], \Gamma) &= \mathrm{Ann}_{\boldsymbol{C}[\xi]}(\mathrm{gr}^{\Gamma} \boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]) \\ &= \mathrm{Ann}_{\boldsymbol{C}[\xi]}(\mathrm{gr}^{\Gamma} M_B) = \mathrm{gr}^{\Gamma'} I_B \\ &= \boldsymbol{C}[\xi](\xi_1, \ldots, \xi_h, \xi_{n+h+1}, \ldots, \xi_{2n})\ . \end{aligned}$$

Thus, by definition (cf. [1], [3]), the characteristic variety of $\boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ is the zero set of the ideal $\boldsymbol{C}[\xi](\xi_1, \ldots, \xi_h, \xi_{n+h+1}, \ldots, \xi_{2n})$. □

By the isomorphism (2.2) and Proposition 2.7, we have the following Corollary.

COROLLARY 2.8. *There exists an isomorphism of irreducible* D_n*-modules*

$$V_{X^*}^{\{y_1, \ldots, y_h\}} \cong \boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]\ . \tag{2.3}$$

PROPOSITION 2.9. *For each* X *in* L_h, $1 \le h \le r$, *and each basis* B *in* $\mathcal{B}_{X^*}$

(1) *The vector spaces* $V_{X^*}^B$ *are isomorphic to each other as* D_n*-modules.*

(2) *The ideal* $I_{X^*} := I_B$ *is independent of* B.

(3) *The canonical holonomic* D_n*-module* $M_{X^*} := D_n/I_{X^*}$ *is isomorphic to* $V_{X^*}^B$.

PROOF. Fix a basis $B = \{\alpha_{i_1}, \dots, \alpha_{i_h}\}$ of X^*. There exists a basis $\{y_1 := \alpha_{i_1}, \dots, y_h := \alpha_{i_h}, \dots, y_r := \alpha_{i_r}, y_{r+1}, \dots, y_n\}$ of $(\boldsymbol{C}^n)^*$, where $\{y_1, \dots, y_r\}$ is a maximal linearly independent subset of $\mathcal{A}^*$. Every other basis $B' = \{\alpha_{j_1}, \dots, \alpha_{j_h}\}$ of X^* satisfies $B' \subset \mathrm{Span}\{y_1, \dots, y_h\}$ and $\{y'_1 := \alpha_{j_1}, \dots, y'_h := \alpha_{j_h}, y'_{h+1} := y_{h+1}, \dots, y'_n := y_n\}$ is a basis of $(\boldsymbol{C}^n)^*$. Associated to the bases B and B' we have the change of bases matrix

$$C^B_{B'} = \begin{pmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-h} \end{pmatrix}$$

where $D \in GL_h(\boldsymbol{C})$ and $\mathbf{I}_s$ is the unit matrix of rank s, such that the corresponding bases of $(\boldsymbol{C}^n)^*$ change linearly by means of

$$^t(y'_1, \dots, y'_h, y'_{h+1}, \dots, y'_n) = C^B_{B'}\ ^t(y_1, \dots, y_h, y_{h+1}, \dots, y_n)\,, \tag{2.4}$$

and the partial derivatives by

$$^t(\partial_{y'_1}, \dots, \partial_{y'_h}, \partial_{y'_{h+1}}, \dots, \partial_{y'_n}) = (\,^tC^B_{B'})^{-1}\ ^t(\partial_{y_1}, \dots, \partial_{y_h}, \partial_{y_{h+1}}, \dots, \partial_{y_n})\,. \tag{2.5}$$

Then we obtain $\boldsymbol{C}[\partial_{y'_1}, \dots, \partial_{y'_h}, y'_{h+1}, \dots, y'_n] = \boldsymbol{C}[\partial_{y_1}, \dots, \partial_{y_h}, y_{h+1}, \dots, y_n]$ and (1) follows by Corollary 2.8. Moreover, (2) follows from (2.4) and (2.5); and (3) follows from (2) and Proposition 2.5. □

PROPOSITION 2.10. *For $1 \le h \le r$, the quotient of two consecutive D_n-modules of sequence* (2.1) *has the following decomposition*

$$P_h/P_{h-1} = \bigoplus_{X \in L_h} \sum_{B \in \mathcal{B}_{X^*}} V^B_{X^*} = \bigoplus_{X \in L_h} \left(\bigoplus_{B \in \widetilde{\mathcal{B}}_{X^*}} V^B_{X^*} \right) \tag{2.6}$$

where $\widetilde{\mathcal{B}}_{X^*}$ is a convenient subset of $\mathcal{B}_{X^*}$, that is, a subset of $\mathcal{B}_{X^*}$ such that $\sum_{B \in \mathcal{B}_{X^*}} V^B_{X^*} = \bigoplus_{B \in \widetilde{\mathcal{B}}_{X^*}} V^B_{X^*}$.

PROOF. By Propositions 2.5 and 2.9(1), there exists a subset $\widetilde{\mathcal{B}}_{X^*}$ of $\mathcal{B}_{X^*}$ such that the vector space $\sum_{B \in \mathcal{B}_{X^*}} V^B_{X^*}$ associated to X is equal to $\bigoplus_{B \in \widetilde{\mathcal{B}}_{X^*}} V^B_{X^*}$. Thus the last equality in (2.6) holds. For two different elements X_1, X_2 in L_h, it follows by Corollary 2.6 that $\sum_{B \in \mathcal{B}_{X_1^*}} V^B_{X_1^*} \cap \sum_{B \in \mathcal{B}_{X_2^*}} V^B_{X_2^*} = \{[0]\}$. Then $\bigoplus_{X \in L_h} \sum_{B \in \mathcal{B}_{X^*}} V^B_{X^*} \subset P_h/P_{h-1}$. Actually, by Lemma 2.1, $P_h/P_{h-1} = \bigoplus_{X \in L_h} \sum_{B \in \mathcal{B}_{X^*}} V^B_{X^*}$. □

PROPOSITION 2.11. *Let X in L_h, $1 \le h \le r$, and let $V^{\mathrm{mod}}_{X^*}$ be the $\boldsymbol{C}$-subspace of P_h/P_{h-1} annihilated by I_{X^*}. Then $V^{\mathrm{mod}}_{X^*}$ is generated by*

$$\mathcal{U}^{\mathrm{mod}}_{X^*} = \left\{ \frac{1}{\prod_{\alpha \in B} \alpha} \bmod P_{h-1} \;\middle|\; B \in \mathcal{B}_{X^*} \right\}.$$

PROOF. By Proposition 2.9(2), $\mathrm{Ann}_{D_n}([1/\prod_{\alpha \in B} \alpha]) = I_{X^*}$ if and only if B is a basis of X^*. Then the space $V^{\mathrm{mod}}_{X^*}$ maps into a unique component in the decomposition of P_h/P_{h-1} as in (2.6) and is generated by $\mathcal{U}^{\mathrm{mod}}_{X^*}$. □

PROPOSITION 2.12. *With the above notation we have*

$$P_h/P_{h-1} \cong \bigoplus_{X \in L_h} M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*}^{\mathrm{mod}} . \tag{2.7}$$

PROOF. This follows from Propositions 2.10, 2.9(3) and 2.11. □

Our next aim is to choose a basis for $V_{X^*}^{\mathrm{mod}}$. This is possible using the notion of "no broken circuit" (nbc) for the set $\mathcal{B}_{X^*}$, consequently for $\mathcal{U}_{X^*}^{\mathrm{mod}}$.

Fix a total order on $\mathcal{A}^*$ by $\mathcal{A}^* = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. A subset of $\mathcal{A}^*$ is a circuit if it is a minimally dependent set. A *no broken circuit* (*nbc*) *subset* is a subset of elements containing no circuit with its smallest element deleted.

Throughout the rest of the paper we identify each ordered set $\{\alpha_{l_1}, \ldots, \alpha_{l_s}\}$ with its s-tuple $(l_1, \ldots, l_s)$.

DEFINITION 2.13. For every X in $L_h(\mathcal{A})$, $1 \le h \le r$, define the D_n-module

$$R_{X^*} = M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*}$$

where V_{X^*} is the $\boldsymbol{C}$-vector space generated by $\mathcal{U}_{X^*} = \{1/\prod_{\alpha \in B} \alpha \mid B \in \mathcal{B}_{X^*}\}$.

For $\boldsymbol{C}^n$ in $L_0(\mathcal{A})$, define $V_{(\boldsymbol{C}^n)^*} = \boldsymbol{C}$ and $R_{(\boldsymbol{C}^n)^*} = \boldsymbol{C}[x_1, \ldots, x_n]$.

LEMMA 2.14. *For $X \in L_h(\mathcal{A})$, define*

$$\mathcal{B}_{X^*}^{\mathrm{nbc}} = \{\{\alpha_{j_1}, \ldots, \alpha_{j_h}\} \in \mathcal{B}_{X^*} \mid (j_1, \ldots, j_h) \text{ is a nbc}\} .$$

The corresponding set $\mathcal{U}_{X^*}^{\mathrm{nbc}} = \{1/\prod_{\alpha \in B} \alpha \mid B \in \mathcal{B}_{X^*}^{\mathrm{nbc}}\}$ is a basis of V_{X^*}.

PROOF. The set $\mathcal{U}_{X^*}^{\mathrm{nbc}}$ generates V_{X^*}: For each basis $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ of X^*, there exist two possibilities: If $(i_1, \ldots, i_h)$ is a nbc, then $\dfrac{1}{\alpha_{i_1} \cdots \alpha_{i_h}} \in \mathcal{U}_{X^*}^{\mathrm{nbc}}$. Otherwise, there exists an m-subtuple $(j_1, \ldots, j_m)$ of $(i_1, \ldots, i_h)$, $1 < m < h$, such that $(j_1, \ldots, j_m)$ is a broken circuit. Thus there exists $1 \le l < i_1$, such that $(l, j_1, \ldots, j_m)$ is a circuit. Equivalently we have the following relation $a_1\alpha_{j_1} + \cdots + a_m\alpha_{j_m} = \alpha_l$, for some $a_1, \ldots, a_m \in \boldsymbol{C}$. This implies that

$$\sum_{u=1}^{m} \frac{a_u}{\alpha_l \alpha_{j_1} \cdots \widehat{\alpha_{j_u}} \cdots \alpha_{j_m}} = \frac{1}{\alpha_{j_1} \cdots \alpha_{j_m}} . \tag{2.8}$$

Note that, for each $1 \le u \le m$, the set $\{\alpha_l, \alpha_{j_1}, \ldots, \widehat{\alpha_{j_u}}, \ldots, \alpha_{j_m}\}$ is linearly independent and $B_u = (\{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \setminus \{\alpha_{j_u}\}) \cup \{\alpha_l\}$ is another basis of X^*. From (2.8) we get

$$\frac{a_1}{\alpha_l \alpha_{i_1} \cdots \widehat{\alpha_{j_1}} \cdots \alpha_{i_h}} + \cdots + \frac{a_m}{\alpha_l \alpha_{i_1} \cdots \widehat{\alpha_{j_m}} \cdots \alpha_{i_h}} = \frac{1}{\alpha_{i_1} \cdots \alpha_{i_h}} . \tag{2.9}$$

If each basis B_u is a nbc, then $\dfrac{1}{\alpha_{i_1} \cdots \alpha_{i_h}}$ is in $\langle \mathcal{U}_{X^*}^{\mathrm{nbc}} \rangle$. Otherwise, there exists at least one h-tuple $(l_1, \ldots, l_h)$ that is not a nbc. Then for each such $(l_1, \ldots, l_h)$ we can repeat the initial process, as with $(i_1, \dots, i_h)$. This procedure ends after a finite number of steps because the cardinality of $\mathcal{U}_{X^*}$ is finite. Finally we obtain $\dfrac{1}{\alpha_{i_1}\cdots\alpha_{i_h}} \in \langle \mathcal{U}^{\mathrm{nbc}}_{X^*}\rangle$.

The set $\mathcal{U}^{\mathrm{nbc}}_{X^*}$ is $\boldsymbol{C}$-linearly independent: Suppose that

$$\sum_{(i_1,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}} \frac{c_{i_1\cdots i_h}}{\alpha_{i_1}\cdots\alpha_{i_h}} = 0$$

with $c_{i_1\cdots i_h} \in \boldsymbol{C}$. Let l_X be the smallest among all the first entries of the h-tuples in $\mathcal{B}^{\mathrm{nbc}}_{X^*}$. Thus we can divide the last sum as

$$\frac{1}{\alpha_{l_X}} \cdot \sum_{(l_X,i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}} \frac{c_{l_X i_2\cdots i_h}}{\alpha_{i_2}\cdots\alpha_{i_h}} + \underbrace{\sum_{\substack{(i_1,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}\\ i_1\neq l_X}} \frac{c_{i_1\cdots i_h}}{\alpha_{i_1}\cdots\alpha_{i_h}}}_{T_X} = 0$$

or $\sum_{(l_X,i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}} \dfrac{c_{l_X i_2\cdots i_h}}{\alpha_{i_2}\cdots\alpha_{i_h}} + \alpha_{l_X}\cdot T_X = 0$. So $\sum_{(l_X,i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}} \dfrac{c_{l_X i_2\cdots i_h}}{\alpha_{i_2}\cdots\alpha_{i_h}} = 0$ within $\ker(\alpha_{l_X})$. Note that $\{\alpha_{i_2},\dots,\alpha_{i_h}\}$ is linearly independent modulo α_{l_X}, and $(i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{Y^*}$ for some subspace $Y^* = \langle\alpha_{i_2},\dots,\alpha_{i_h}\rangle$ of X^* obtained after removing α_{l_X} from every basis $\{\alpha_{l_X},\alpha_{i_2},\dots,\alpha_{i_h}\}$ in $\mathcal{B}^{\mathrm{nbc}}_{X^*}$. Thus we have

$$\sum_{(l_X,i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}} \frac{c_{l_X i_2\cdots i_h}}{\alpha_{i_2}\cdots\alpha_{i_h}} = 0\,.$$

By induction on $\dim X^*$, we shall prove $c_{l_X i_2\cdots i_h} = 0$ for all $(l_X, i_2,\dots,i_h)$ in $\mathcal{B}^{\mathrm{nbc}}_{X^*}$ and $T_X = 0$. In fact, let $\mathcal{Z}_{X^*} = \{Y^*\subset X^* \mid Y^* = \langle\alpha_{i_2},\dots,\alpha_{i_h}\rangle \text{ if } (l_X,i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{X^*}\}$ and fix one Y^* in $\mathcal{Z}_{X^*}$. Then we may divide the last sum to get

$$\sum_{(l_Y,i_3,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{Y^*}} \frac{c_{l_X l_Y i_3\cdots i_h}}{\alpha_{i_3}\cdots\alpha_{i_h}} + \alpha_{l_Y}\Bigg(\underbrace{\sum_{\substack{(i_2,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{Y^*}\\ i_2\neq l_Y}} \frac{c_{l_X i_2\cdots i_h}}{\alpha_{i_2}\cdots\alpha_{i_h}}}_{T_Y} + \sum_{\substack{\langle\alpha_{j_2},\dots,\alpha_{j_h}\rangle = Z^*\\ Z^*\in\mathcal{Z}_{X^*}\setminus\{Y^*\}}} \frac{c_{l_X j_2\cdots j_h}}{\alpha_{j_2}\cdots\alpha_{j_h}}\Bigg) = 0\,.$$

Then $\sum_{(l_Y,i_3,\dots,i_h)\in\mathcal{B}^{\mathrm{nbc}}_{Y^*}} \dfrac{c_{l_X l_Y i_3\cdots i_h}}{\alpha_{i_3}\cdots\alpha_{i_h}} = 0$ within $\ker(\alpha_{l_Y})$. By induction on $\dim X^*$, since $\dim Y^* < \dim X^*$, $c_{l_X l_Y i_3\cdots i_h} = 0$ for all $(l_Y, i_3,\dots,i_h)$ in $\mathcal{B}^{\mathrm{nbc}}_{Y^*}$, and $T_Y = 0$. But this is true for every Y^* in $\mathcal{Z}_{X^*}$. Thus, $c_{l_X i_2\cdots i_h} = 0$ for all $(l_X, i_2,\dots,i_h)$ in $\mathcal{B}^{\mathrm{nbc}}_{X^*}$. This implies that $T_X = 0$. Thus α_{l_X} appears in every basis in $\mathcal{B}^{\mathrm{nbc}}_{X^*}$ and $\mathcal{U}^{\mathrm{nbc}}_{X^*}$ is linearly independent. □

COROLLARY 2.15. *Let $X\in L_h$, $1\le h\le r$, and let l_X be the smallest among all the first entries of h-tuples $(i_1,\dots,i_h)$ such that $\{\alpha_{i_1},\dots,\alpha_{i_h}\}\in\mathcal{B}_{X^*}$. Then $B\in\mathcal{B}^{\mathrm{nbc}}_{X^*}$ if and only if $\alpha_{l_X}\in B$.*

LEMMA 2.16. *Let X, Y be two elements in L_h, $1 \leq h \leq r$. Then $X \neq Y$ if and only if* $V_{X^*} \cap V_{Y^*} = \{0\}$.

PROOF. Assume $X^* \neq Y^*$. Suppose that there exists a non-zero element υ in $V_{X^*} \cap V_{Y^*}$. Since $[\upsilon] = \upsilon \bmod P_{h-1}$ belongs to $M_{X^*} \otimes V_{X^*}^{\text{mod}} \cap M_{Y^*} \otimes V_{Y^*}^{\text{mod}} = \{[0]\}$, then υ belongs to P_{h-1}. This is a contradiction. □

The next two lemmas enable us to write the D_n-module P as a direct sum of the R_{X^*}.

LEMMA 2.17. *Fix $I = (i_1, \ldots, i_h)$ and $J = (j_1, \ldots, j_s)$ such that $h + s = n$ and consider a polynomial f in $\boldsymbol{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \partial_{y_{j_s}}]$. Then*

(a) *If f is such that $f \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$, then $f \equiv 0$.*

(b) *If $f \cdot \partial_{y_{j_l}} \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ for some $1 \leq l \leq s$, then $f \equiv 0$.*

More generally, if the subset $\{\alpha_1, \ldots, \alpha_s\}$ of $\operatorname{Span}\{y_{j_1}, \ldots, y_{j_s}\}$ is linearly independent, then (a) and (b) hold with $\dfrac{1}{\alpha_1 \cdots \alpha_s}$ instead of $\dfrac{1}{y_{j_1} \cdots y_{j_s}}$.

PROOF. We start to show (a) by induction on s: If $f \in \boldsymbol{C}[y_1, \ldots, y_n]$ ($s = 0$), then it is clear that $f \equiv 0$. Now let $s > 0$. If there is no $1 \leq u \leq s$ such that $\deg_{\partial_{y_{j_u}}} f = m > 0$, then it is also clear that $f \equiv 0$, otherwise f can be written as

$$Q_m \partial_{y_{j_u}}^m + Q_{m-1} \partial_{y_{j_u}}^{m-1} + \cdots + Q_1 \partial_{y_{j_u}} + Q_0$$

where $Q_m, \ldots, Q_0 \in \boldsymbol{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$ and $Q_m \neq 0$. Thus $f \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ is equivalent to

$$\left(\frac{(-1)^m m!}{y_{j_u}^{m+1}} Q_m + \frac{(-1)^{m-1}(m-1)!}{y_{j_u}^m} Q_{m-1} + \cdots + \frac{1}{y_{j_u}} Q_0 \right) \bullet \frac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}} = 0$$

or

$$((-1)^m m! Q_m + (-1)^{m-1}(m-1)! y_{j_u} Q_{m-1} + \cdots + y_{j_u}^m Q_0) \bullet \frac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}} = 0\,.$$

Denote by $\tilde{f}$ the operator that acts on $\dfrac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}}$ in the last equation. Note that $\tilde{f}$ belongs to $\boldsymbol{C}[y_{i_1}, \ldots y_{i_h}, y_{j_u}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$. By induction on s we have $\tilde{f} \equiv 0$. Then $Q_m = 0$ and $f \equiv 0$.

In order to prove (b), note that $f \cdot \partial_{y_{j_l}} = \partial_{y_{j_l}} \cdot f$. Again, by induction on s, if $s = 0$ then $f = 0$. For $s > 0$, if there is no $1 \leq u \leq s$ such that $\deg_{\partial_{y_{j_u}}} f = m > 0$, then it is also clear that $f \cdot \partial_{y_{j_l}} \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ implies $f = 0$, otherwise $f \cdot \partial_{y_{j_l}}$ can be written as

$$(Q_m \partial_{y_{j_l}}) \partial^m_{y_{j_u}} + (Q_{m-1} \partial_{y_{j_l}}) \partial^{m-1}_{y_{j_u}} + \cdots + (Q_1 \partial_{y_{j_l}}) \partial_{y_{j_u}} + (Q_0 \partial_{y_{j_l}})$$

where $Q_m, \ldots, Q_0 \in \boldsymbol{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$ and $Q_m \neq 0$. If $l \neq u$ then again $Q'_p = Q_p \partial_{y_{j_l}} \in \boldsymbol{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$ for $p = 0, 1, \ldots, m$, and the result follows from (a). Otherwise $f \cdot \partial_{y_{j_l}} \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ is equivalent to

$$((-1)^{m+1}(m+1)! Q_m + (-1)^m m! y_{j_u} Q_{m-1} + \cdots - y^m_{j_u} Q_0) \bullet \frac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}} = 0$$

and again the result follows from (a) and induction on s.

The general case follows by induction on s and from relations (2.4) and (2.5). □

LEMMA 2.18. *Let X in L_h, $1 \leq h \leq r$. The natural map of D_n-modules $\phi_X : R_{X^*} = M_{X^*} \otimes_{\boldsymbol{C}} V_{X^*} \to P$, $m \otimes v \mapsto m \bullet v$, is injective.*

PROOF. Let $\{y_1, \ldots, y_n\}$ be a basis of $(\boldsymbol{C}^n)^*$ such that $X^* = \langle y_1, \ldots, y_h \rangle$. By Lemma 2.14, R_{X^*} can be written as $\boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}] \otimes_{\boldsymbol{C}} \langle \mathcal{U}^{\text{nbc}}_{X^*} \rangle = \bigoplus_{B \in \mathcal{B}^{\text{nbc}}_{X^*}} M_{X^*} \otimes_{\boldsymbol{C}} (1/\prod_{\alpha \in B} \alpha)$. Then the map ϕ_X is injective if and only if for each $B \in \mathcal{B}^{\text{nbc}}_{X^*}$ the map $\phi^B_X : M_{X^*} \otimes_{\boldsymbol{C}} (1/\prod_{\alpha \in B} \alpha) \to P$ is injective, i.e., if $Q \bullet (1/\prod_{\alpha \in B} \alpha) = 0$, where $Q \in \boldsymbol{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ and $B \in \mathcal{B}^{\text{nbc}}_{X^*}$, then $Q = 0$. This follows from Lemma 2.17. □

THEOREM 2.19. *For $1 \leq h \leq r$, we have*

$$P_h = \bigoplus_{j=0}^{h} \bigoplus_{X \in L_j(\mathcal{A})} R_{X^*}.$$

In particular, since $P = P_r$, we have $P = \bigoplus_{X \in L(\mathcal{A})} R_{X^*}$.

PROOF. This is an immediate consequence of Lemma 2.18 and the definition of P_h. □

REMARK. M. Brion and M. Vergne [2], and H. Terao [10], have studied the action of $\boldsymbol{C}[\partial]$ on P. Horiuchi and Terao [5] have also studied the naturally double filtration of P by the degrees of the denominators and numerators.

THEOREM 2.20. *For $0 \leq h \leq r$, the natural map induced by ϕ_X, $\psi : \bigoplus_{X \in L_h} R_{X^*} \to P_h/P_{h-1}$, $m \otimes v \mapsto [m \bullet v]$, is an isomorphism of D_n-modules.*

PROOF. It follows from Proposition 2.12 that the D_n-morphism ψ is surjective. In order to see that ψ is injective, it is sufficient to show that $\psi_X : R_{X^*} \to P_h/P_{h-1}$ is injective for each $X \in L_h$. Recall that $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \subseteq X\}$. Let $d_{\mathcal{A}_X} = \prod_{\alpha \in \mathcal{A}^*_X} \alpha$ be the homogeneous polynomial that defines the subarrangement $\mathcal{A}_X$. Define the D_n-submodule P^X of P by $\mathbf{C}[\mathbf{x}, d^{-1}_{\mathcal{A}_X}]$. By Lemma 2.1, P^X admits a finite ascending chain similar to one of

(2.1) to P. Then, the map ψ_X is injective if and only if the map $\overline{\psi}_X : R_{X^*} \to P_h^X/P_{h-1}^X$ is injective, i.e., $V_{X^*} \cap P_{h-1}^X = \{0\}$. Suppose that there exists a non-zero element υ in $V_{X^*} \cap P_{h-1}^X$. Let $\{y_1, \dots, y_n\}$ be a basis of $(\boldsymbol{C}^n)^*$ such that $X^* = \langle y_1, \dots, y_h \rangle$, then υ can be written as

$$\upsilon = \sum_{B \in \mathcal{B}_{X^*}^{\text{nbc}}} \frac{c_B}{\prod_{\alpha \in B} \alpha} = \sum \frac{a_{j_1 \cdots j_s}}{\alpha_{j_1}^{m_1} \cdots \alpha_{j_s}^{m_s}} ,$$

where the first sum belongs to V_{X^*}, the second to P_{h-1}^X, $c_B \in \boldsymbol{C}$, $0 \le s \le h-1$, $a_{j_1 \cdots j_s} \in \boldsymbol{C}[y_1, \dots, y_n]$, $\{\alpha_{j_1}, \dots, \alpha_{j_s}\}$ is a linearly independent subset of $\mathrm{Span}\{y_1, \dots, y_h\} \cap \mathcal{A}_X^*$ and $m_1, \dots, m_s \in \boldsymbol{N}$. It is clear that $\sum_{B \in \mathcal{B}_{X^*}^{\text{nbc}}} (c_B / \prod_{\alpha \in B} \alpha) \bmod P_{h-1} \neq [0]$ and $\sum (a_{j_1 \cdots j_s} / \alpha_{j_1}^{m_1} \cdots \alpha_{j_s}^{m_s}) \bmod P_{h-1} = [0]$. This is a contradiction. □

COROLLARY 2.21. *If $X \in L_h$, $1 \le h \le r$, then the set of cosets $\{1/\prod_{\alpha \in B} \alpha \bmod P_{h-1} \mid B \in \mathcal{B}_{X^*}^{\text{nbc}}\}$ is a $\boldsymbol{C}$-basis of $V_{X^*}^{\text{mod}}$.*

DEFINITION 2.22. Let $\mathcal{A}$ be an arrangement in $\boldsymbol{C}^n$ of rank r. Define the holonomic D_n-module $\mathcal{P} = \mathcal{P}(\mathcal{A}) = \bigoplus_{h=0}^r \mathcal{P}_h$, associated to the arrangement $\mathcal{A}$ and isomorphic to $P(\mathcal{A})$, as follows: let $\mathcal{P}_0 = P_0 = \boldsymbol{C}[x_1, \dots, x_n]$, and for $1 \le h \le r$

$$\mathcal{P}_h = P_h/P_{h-1} \cong \bigoplus_{X \in L_h} R_{X^*} = \bigoplus_{X \in L_h} M_{X^*} \otimes_{\boldsymbol{C}} \langle \mathcal{U}_{X^*}^{\text{nbc}} \rangle \cong \bigoplus_{X \in L_h} M_{X^*}^{a(X^*)}$$

where $a(X^*) := \dim V_{X^*}$ is equal to $|\mathcal{U}_{X^*}^{\text{nbc}}|$, the multiplicity of M_{X^*}.

3. Complexes and cohomology of $Y_{\mathcal{A}}$

We begin by defining some useful cochain complexes $\mathcal{L}_h^*$, $\mathcal{G}_h^*$ and $\mathcal{H}_h^*$. The first complex $\mathcal{L}_h$, cf. (3.1), is associated to every basis $B \in \mathcal{B}_{X^*}$, $X \in L_h$, and then we get a complex $\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X \in L_h} \bigoplus_{B \in \mathcal{B}_{X^*}^{\text{nbc}}} \mathcal{L}_h(B)$ associated to $\mathcal{P}_h$. The cohomology of $\mathcal{L}(\mathcal{P}_h)$ is the h-th De Rham cohomology of $Y_{\mathcal{A}}$, cf. Theorem 3.6 (see also [9], Theorem 3.26, Theorem 3.43 and Theorem 5.90).

Fix h, $0 \le h \le n$, we define the following cochain complexes (3.1), (3.2) and (3.3).

The complex of rational differential forms on $Y_{\mathcal{A}}$:

$$\mathcal{L}_h^* = \mathcal{L}_h^*(\{y_1, \dots, y_h\}) :\ 0 \longrightarrow \mathcal{L}_h^0 \xrightarrow{\delta_{\mathcal{L}}^0} \mathcal{L}_h^1 \xrightarrow{\delta_{\mathcal{L}}^1} \mathcal{L}_h^2 \longrightarrow \ \cdots \ \longrightarrow \mathcal{L}_h^{n-1} \xrightarrow{\delta_{\mathcal{L}}^{n-1}} \mathcal{L}_h^n \xrightarrow{\delta_{\mathcal{L}}^n} 0 \tag{3.1}$$

where

$$\mathcal{L}_h^0 = \boldsymbol{C}[y_{h+1}, \dots, y_n, \partial_{y_1}, \dots, \partial_{y_h}] \bullet \frac{1}{y_1 \cdots y_h} ,$$

$$\mathcal{L}_h^s = \left\{ \sum_{1 \le i_1 < \cdots < i_s \le n} f_{i_1 \cdots i_s} \bullet \frac{1}{y_1 \cdots y_h} dy_{i_1} \wedge \cdots \wedge dy_{i_s} \right\} , \quad s = 1, \dots, n ,$$

$f_{i_1\cdots i_s} \in \boldsymbol{C}[y_{h+1}, \dots, y_n, \partial_{y_1}, \dots, \partial_{y_h}]$, and the differential $\delta_{\mathcal{L}} : \mathcal{L}_h \to \mathcal{L}_h$ is the usual differential.

A subcomplex of $\mathcal{L}_h$:

$$\mathcal{G}_h^* :\ 0 \longrightarrow \mathcal{G}_h^0 \xrightarrow{\delta_{\mathcal{G}}^0} \mathcal{G}_h^1 \xrightarrow{\delta_{\mathcal{G}}^1} \mathcal{G}_h^2 \longrightarrow \ \cdots \ \longrightarrow \mathcal{G}_h^{h-1} \xrightarrow{\delta_{\mathcal{G}}^{h-1}} \mathcal{G}_h^h \xrightarrow{\delta_{\mathcal{G}}^h} 0 \tag{3.2}$$

where

$$\mathcal{G}_h^0 = \boldsymbol{C}[\partial_{y_1}, \dots, \partial_{y_h}] \bullet \frac{1}{y_1 \cdots y_h} ,$$

$$\mathcal{G}_h^r = \left\{ \sum_{1 \le i_1 < \cdots < i_r \le h} f_{i_1\cdots i_r} \bullet \frac{1}{y_1 \cdots y_h} dy_{i_1} \wedge \cdots \wedge dy_{i_r} \right\} , \quad r = 1, \dots, h ,$$

$f_{i_1\cdots i_r} \in \boldsymbol{C}[\partial_{y_1}, \dots, \partial_{y_h}]$, and the differential $\delta_{\mathcal{G}} : \mathcal{G}_h \to \mathcal{G}_h$ is the usual differential.

Finally, the De Rham subcomplex on $\boldsymbol{C}^{n-h}$:

$$\mathcal{H}_h^* :\ 0 \longrightarrow \mathcal{H}_h^0 \xrightarrow{\delta_{\mathcal{H}}^0} \mathcal{H}_h^1 \xrightarrow{\delta_{\mathcal{H}}^1} \mathcal{H}_h^2 \longrightarrow \ \cdots \ \longrightarrow \mathcal{H}_h^{n-h-1} \xrightarrow{\delta_{\mathcal{H}}^{n-h-1}} \mathcal{H}_h^{n-h} \xrightarrow{\delta_{\mathcal{H}}^{n-h}} 0 \tag{3.3}$$

where

$$\mathcal{H}_h^0 = \boldsymbol{C}[y_{h+1}, \dots, y_n] ,$$

$$\mathcal{H}_h^t = \left\{ \sum_{h+1 \le i_1 < \cdots < i_t \le n} f_{i_1\cdots i_t} dy_{i_1} \wedge \cdots \wedge dy_{i_t} \right\} , \quad t = 1, \dots, n-h ,$$

$f_{i_1\cdots i_t} \in \boldsymbol{C}[y_{h+1}, \dots, y_n]$, and the differential $\delta_{\mathcal{H}} : \mathcal{H}_h \to \mathcal{H}_h$ is the usual differential.

LEMMA 3.1. *The complex $\mathcal{G}_h$ has cohomology*

$$H^*(\mathcal{G}_h) = \begin{cases} \boldsymbol{C} \cdot \dfrac{1}{y_1 \cdots y_h} dy_1 \wedge \cdots \wedge dy_h & \textit{in dimension } h , \\ 0 & \textit{elsewhere} . \end{cases}$$

PROOF. For $r = 0$: Let $\omega = f \bullet \dfrac{1}{y_1 \cdots y_h} \in \mathcal{G}_h^0$. If $\delta_{\mathcal{G}}^0 \omega = \sum_{i=1}^h (f \cdot \partial_{y_i}) \bullet \dfrac{1}{y_1 \cdots y_h} dy_i = 0$, then we have $\delta_{\mathcal{G}}^0 \omega \wedge (dy_1 \cdots \widehat{dy_i} \cdots dy_h) = (-1)^{i-1} (f \cdot \partial_{y_i}) \bullet \dfrac{1}{y_1 \cdots y_h} dy_1 \cdots dy_h = 0$ for all $1 \le i \le h$. It is possible if and only if $(f \cdot \partial_{y_i}) \bullet \dfrac{1}{y_1 \cdots y_h} = 0$. By Lemma 2.17 (b), we have $f = 0$. Thus, we have $\ker(\delta_{\mathcal{G}}^0) = \{0\}$ and $H^0(\mathcal{G}_h) = 0$.

For $0 < r < h$: Let $\omega = \sum_{1 \le i_1 < \cdots < i_r \le h} f_{i_1\cdots i_r} \bullet \dfrac{1}{y_1 \cdots y_h} dy_{i_1} \cdots dy_{i_r}$ be an element in $\mathcal{G}_h^r$. If $\delta_{\mathcal{G}}^r \omega = \sum_{1 \le l_1 < \cdots < l_r < l_{r+1} \le h} (\sum_{j=1}^{r+1} (-1)^{j-1} f_{l_1 \cdots \widehat{l_j} \cdots l_{r+1}} \cdot \partial_{y_j}) \bullet \dfrac{1}{y_1 \cdots y_h} dy_{l_1} \cdots dy_{l_{r+1}} = 0$, where $\{l_1, \dots, \widehat{l_j}, \dots, l_{r+1}\}$ is equal to some $\{i_1, \dots, i_r\}$, then, as for the case $r = 0$, we have $(\sum_{j=1}^{r+1} (-1)^{j-1} f_{l_1 \cdots \widehat{l_j} \cdots l_{r+1}} \cdot \partial_{y_j}) \bullet \dfrac{1}{y_1 \cdots y_h} = 0$ for all $1 \le l_1 < \cdots < l_r < l_{r+1} \le h$.

By Lemma 2.17, this is possible if and only if $\sum_{j=1}^{r+1}(-1)^{j-1} f_{l_1\cdots\widehat{l_j}\cdots l_{r+1}} \cdot \partial_{y_j} = 0$. This last equality is true if and only if $f_{i_1\cdots i_r} = 0$ for all $1 \leq i_1 < \cdots < i_r \leq h$. Thus we have again that $\ker(\delta^r_{\mathcal{G}}) = \{0\}$ and $H^r(\mathcal{G}_h) = 0$ for $0 < r < h$.

Finally, for $r = h$, $\delta^h_{\mathcal{G}}(\omega) = 0$ for all $\omega \in \mathcal{G}^h_h$. Thus $\ker(\delta^h_{\mathcal{G}}) = \mathcal{G}^h_h$. Since

$$Im(\delta^{h-1}_{\mathcal{G}}) = \left\{(f_1 \cdot \partial_{y_1} - f_2 \cdot \partial_{y_2} + \cdots + (-1)^{h-1} f_h \cdot \partial_{y_h}) \bullet \frac{1}{y_1 \cdots y_h} dy_1 \cdots dy_h\right\},$$

we obtain $H^h(\mathcal{G}_h) = \boldsymbol{C} \cdot \dfrac{1}{y_1 \cdots y_h} dy_1 \cdots dy_h$. □

LEMMA 3.2. *The complex $\mathcal{H}_h$ has cohomology*

$$H^*(\mathcal{H}_h) = \begin{cases} \boldsymbol{C} & \text{in dimension } 0\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

PROOF. This is a consequence of the fact that $\mathcal{H}_h$ is a subcomplex of the De Rham complex $\Omega_{DR}(\boldsymbol{C}^{n-h})$ on $\boldsymbol{C}^{n-h}$. □

PROPOSITION 3.3. *There exists the following relation between the complexes $\mathcal{L}_h$, $\mathcal{G}_h$ and $\mathcal{H}_h$:*

$$\mathcal{L}_h = \mathcal{G}_h \otimes_{\boldsymbol{C}} \mathcal{H}_h\,.$$

PROOF. We will prove, cf. [4], that:

(1) $\mathcal{L}^s_h = \bigoplus_{r+t=s} \mathcal{G}^r_h \otimes_{\boldsymbol{C}} \mathcal{H}^t_h$ $(= (\mathcal{G}_h \otimes_{\boldsymbol{C}} \mathcal{H}_h)^s)$, and

(2) $\delta^s_{\mathcal{L}} = \delta^s_{\mathcal{G}\otimes\mathcal{H}} : (\mathcal{G}_h \otimes_{\boldsymbol{C}} \mathcal{H}_h)^s \to (\mathcal{G}_h \otimes_{\boldsymbol{C}} \mathcal{H}_h)^{s+1}$.

To prove (1), note that every monomial of $f_{i_1\cdots i_s}(y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}) \bullet \dfrac{1}{y_1 \cdots y_h} dy_{i_1} \cdots dy_{i_s} \in \mathcal{L}^s_h$: $c_{j_1\cdots j_n} y^{j_{h+1}}_{h+1} \cdots y^{j_n}_n \partial^{j_1}_{y_1} \cdots \partial^{j_h}_{y_h} \bullet \dfrac{1}{y_1 \cdots y_h} dy_{i_1} \cdots dy_{i_r} dy_{i_{r+1}} \cdots dy_{i_s}$, $c_{j_1\cdots j_n} \in \boldsymbol{C}$, can be written as $\left(\partial^{j_1}_{y_1} \cdots \partial^{j_h}_{y_h} \bullet \dfrac{1}{y_1 \cdots y_h} dy_{i_1} \cdots dy_{i_r}\right) \otimes_{\boldsymbol{C}} (c_{j_1\cdots j_n} y^{j_{h+1}}_{h+1} \cdots y^{j_n}_n dy_{i_{r+1}} \cdots dy_{i_s})$, where the first factor belong to $\mathcal{G}^r_h$ and the second to $\mathcal{H}^{s-r}_h$. So $\mathcal{L}^s_h \subseteq \bigoplus_{r+t=s} \mathcal{G}^r_h \otimes_{\boldsymbol{C}} \mathcal{H}^t_h$. The second inclusion is obvious.

In order to show (2), we will show that if $s = r + t$ for some $0 \leq r \leq h$ then $\delta^s_{\mathcal{G}\otimes\mathcal{H}}|_{\mathcal{G}^r\otimes\mathcal{H}^t} = \delta^s_{\mathcal{L}}|_{\mathcal{G}^r\otimes\mathcal{H}^t}$. It follows from the respective definition of $\delta_{\mathcal{G}\otimes\mathcal{H}}$, $\delta_{\mathcal{L}}$, $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}}$. □

COROLLARY 3.4. *The complex $\mathcal{L}_h = \mathcal{L}_h(\{y_1, \ldots, y_h\})$ has cohomology*

$$H^*(\mathcal{L}_h(\{y_1, \ldots, y_h\})) = \begin{cases} \boldsymbol{C} \cdot \dfrac{1}{y_1 \cdots y_h} dy_1 \cdots dy_h & \text{in dimension } h, \\ 0 & \text{elsewhere.} \end{cases}$$

PROOF. Thanks to Proposition 3.3 and the algebraic Künneth formula for the cohomology of a tensor product of two complexes, we have that $H^s(\mathcal{L}_h) = \bigoplus_{r+t=s} H^r(\mathcal{G}_h) \otimes_{\boldsymbol{C}} H^t(\mathcal{H}_h)$. Hence, the result follows from Lemmas 3.1 and 3.2. □

DEFINITION 3.5. For each subspace X in L_h, define the following complex:

$$\mathcal{L}_h(X) = \bigoplus_{\{\alpha_{j_1},\dots,\alpha_{j_h}\}\in\mathcal{B}^{\text{nbc}}_{X^*}} \mathcal{L}_h(\{\alpha_{j_1},\dots,\alpha_{j_h}\})$$

where $\mathcal{L}_h(\{\alpha_{j_1},\dots,\alpha_{j_h}\})$ is the same complex $\mathcal{L}^*_h$ defined in (3.1) but for the set of generators $\{\alpha_{j_1},\dots,\alpha_{j_h}\}$ of X^*. Associated to the D_n-module $\mathcal{P}_h \cong \bigoplus_{X\in L_h} R_{X^*}$, define the complex

$$\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X\in L_h} \mathcal{L}_h(X)\,.$$

Finally define the complex $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}(\mathcal{A})) = \bigoplus_{h=0}^{r} \mathcal{L}(\mathcal{P}_h)$ associated to the D_n-module $\mathcal{P}$, cf. Definition 2.22.

Notice that $\mathcal{L}(\mathcal{P})$ is the algebraic De Rham complex of $Y_{\mathcal{A}}$.

THEOREM 3.6. *For $1 \le h \le r$, there exists an isomorphism between $H^h_{DR}(Y_{\mathcal{A}})$ and $H^h(\mathcal{L}(\mathcal{P}_h))$:*

$$H^h_{DR}(Y_{\mathcal{A}}) \cong H^h(\mathcal{L}(\mathcal{P}_h)) = \bigoplus_{X\in L_h} \bigoplus_{\{\alpha_{j_1},\dots,\alpha_{j_h}\}\in\mathcal{B}^{\text{nbc}}_{X^*}} \boldsymbol{C}\cdot\frac{1}{\alpha_{j_1}\cdots\alpha_{j_h}} d\alpha_{j_1}\wedge\cdots\wedge d\alpha_{j_h}\,.$$

PROOF. Fix a subspace $X \in L_h(\mathcal{A})$. By Corollary 3.4, the associated complex $\mathcal{L}_h(X)$ has cohomology non-null only in dimension h. It is

$$H^h(\mathcal{L}_h(X)) = \bigoplus_{\{\alpha_{j_1},\dots,\alpha_{j_h}\}\in\mathcal{B}^{\text{nbc}}_{X^*}} \boldsymbol{C}\cdot\frac{1}{\alpha_{j_1}\cdots\alpha_{j_h}} d\alpha_{j_1}\wedge\cdots\wedge d\alpha_{j_h}\,.$$

Therefore, the complex $\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X\in L_h} \mathcal{L}_h(X)$ has nonzero cohomology only in dimension h. Since $Y_{\mathcal{A}}$ is a smooth affine variety it follows, by [6, Theorem 1], that $H^h_{DR}(Y_{\mathcal{A}}) \cong H^h(\mathcal{L}(\mathcal{P}_h))$. □

COROLLARY 3.7. *Let $b_h(Y_{\mathcal{A}})$ be the Betti numbers of $Y_{\mathcal{A}}$, $1 \le h \le r$. Then we have*

$$b_h(Y_{\mathcal{A}}) = \sum_{X\in L_h} a(X^*)\,.$$

PROOF. It is a consequence of Theorem 3.6 that

$$rank\, H^h_{DR}(Y_{\mathcal{A}}) = rank\, H^h(\mathcal{L}(\mathcal{P}_h)) = \sum_{X\in L_h} |\mathcal{B}^{\text{nbc}}_{X^*}| = \sum_{X\in L_h} |\mathcal{U}^{\text{nbc}}_{X^*}| = \sum_{X\in L_h} a(X^*)\,,$$

where the last equality holds by Definition 2.22. □

4. The Poincaré series of $P(\mathcal{A})$

In this last section we compute the Poincaré series of the D_n-module $P(\mathcal{A})$.

DEFINITION 4.1. If $M = \bigoplus_{i \geq 0} M_i$ is a graded vector space with $\dim M_i < +\infty$ for all $i \geq 0$, we define the Poincaré series of M by

$$Poin(M, t) = \sum_{i=0}^{\infty} (\dim M_i) t^i \, .$$

From Definition 2.13 and Lemma 2.16 and 2.14, we have the following Lemma.

LEMMA 4.2. *Let $\mathcal{A}$ be an arrangement of hyperplanes. Define the finite dimensional graded **C**-vector space*

$$V(\mathcal{A}) = \bigoplus_{h=0}^{r} \bigoplus_{X \in L_h} V_{X^*} \, .$$

Then the set

$$\{1\} \cup \bigcup_{h=1}^{r} \bigcup_{X \in L_h} \mathcal{U}_{X^*}^{\text{nbc}}$$

is a basis of $V(\mathcal{A})$.

We must express the dimension of V_{X^*} $(= |\mathcal{U}_{X^*}^{\text{nbc}}|)$ by using the Möbius function in one variable $\mu(X)$ defined in [9]. Recall that the Poincaré polynomial of $\mathcal{A}$ is combinatorially defined by using μ: $Poin(\mathcal{A}, t) = \sum_{X \in L} (-1)^{r(X)} \mu(X) t^{r(X)}$.

THEOREM 4.3. (*see* [7], [5]) *For $X \in L$, we have* $\dim V_{X^*} = (-1)^{r(X)} \mu(X)$, *and the Poincaré series Poin$(V(\mathcal{A}), t)$ of the space $V(\mathcal{A})$ is equal to Poin$(\mathcal{A}, t)$.*

By Theorem 2.19, the dimension of the graded D_n-module $P(\mathcal{A})$ is infinite. Then its Poincaré series is a formal power series. The following theorem gives a combinatorial formula for it.

THEOREM 4.4. *The Poincaré series Poin$(P(\mathcal{A}), t)$ of the graded D_n-module $P(\mathcal{A})$ is equal to* $(1 - t)^{-n}$*Poin*$(\mathcal{A}, t)$.

PROOF. According to Theorem 2.19, we have

$$Poin(P(\mathcal{A}), t) = \sum_{X \in L} Poin(R_{X^*}, t) = \sum_{X \in L} Poin(M_{X^*}, t) Poin(V_{X^*}, t) \, .$$

Since the **C**-algebra M_{X^*} is isomorphic to the polynomial algebra with n variables, we have $Poin(M_{X^*}, t) = (1 - t)^{-n}$. Moreover, by the definition of $Poin(V_{X^*}, t) = \dim V_{X^*} t^{r(X)}$ and by Theorem 4.3, we have $Poin(V_{X^*}, t) = (-1)^{r(X)}\mu(X)t^{r(X)}$. Thus

$$\begin{aligned} Poin(P(\mathcal{A}), t) &= \sum_{X\in L}(1-t)^{-n}(-1)^{r(X)}\mu(X)t^{r(X)} \\ &= (1-t)^{-n}Poin(\mathcal{A}, t)\,. \end{aligned}$$

□

By Theorem 2.20, we have the following Corollary.

COROLLARY 4.5. *The Poincaré series* $Poin(\mathcal{P}(\mathcal{A}), t)$ *of* $\mathcal{P}(\mathcal{A})$ *is equal to* $Poin(P(\mathcal{A}), t)$.

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