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On the Iwasawa Invariants of the Cyclotomic Z_2-Extensions of Certain Real Quadratic Fields

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Abstract. We study some conditions that the Iwasawa λ-, μ-invariants of the the cyclotomic $\mathbf{Z}_2$-extension of $k = \mathbf{Q}(\sqrt{pq})$ with $p \equiv 7 \pmod 8$, $q \equiv 1 \pmod 8$, $(\frac{p}{q}) = -1$ are zero.

1. Introduction

Let k be a finite extension of the field $\mathbf{Q}$ of rational numbers, l any prime number, and k_∞ the cyclotomic $\mathbf{Z}_l$-extension of k, where $\mathbf{Z}_l$ is the ring of l-adic integers. Then k_∞ has the unique subfield k_n which is a cyclic extension of degree l^n over k for any integer $n \geq 0$. Let e_n be the highest power of l dividing the class number of k_n. The following theorem about e_n is well-known as Iwasawa's class number formula.

THEOREM 1 (Iwasawa) (cf. [4], [9]). *There exist integers $\lambda_l(k)$, $\mu_l(k) \geq 0$, $\nu_l(k)$, all independent of n, and an integer n_0 such that*

$$e_n = \lambda_l(k)n + \mu_l(k)l^n + \nu_l(k)$$

for all $n \geq n_0$.

$\lambda_l(k)$, $\mu_l(k)$, and $\nu_l(k)$ are called Iwasawa λ-, μ-, and ν-invariants of k_∞ , respectively.

Greenberg conjectured that if k is a totally real number field, then $\lambda_l(k) = \mu_l(k) = 0$ for any prime number l (cf. [2]).Many authors have studied the conditions that Iwasawa λ-, μ-invariants are zero. In this paper, we prove the following theorem related to the Iwasawa λ-, μ-invariants of the cyclotomic $\mathbf{Z}_2$-extensions of certain real quadratic fields.

THEOREM 2. *Let p, q be prime numbers such that*

$$p \equiv 7 \pmod 8\,, \quad q \equiv 1 \pmod 8\,, \quad \left(\frac{p}{q}\right) = -1\,,$$

where $(\frac{*}{*})$ is Legendre's symbol. Let $k = \mathbf{Q}(\sqrt{pq})$ or $\mathbf{Q}(\sqrt{2pq})$, and $\lambda_2(k)$, $\mu_2(k)$, the Iwasawa λ-, μ-invariants of the cyclotomic $\mathbf{Z}_2$-extension k_∞ of k, respectively.

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(1) *If* $q \equiv 9 \pmod{16}$, *then* $\lambda_2(k) = \mu_2(k) = 0$.

(2) *If* $q \equiv 1 \pmod{16}$, $p \equiv 7 \pmod{16}$, *and* $2^{\frac{q-1}{4}} \equiv -1 \pmod{q}$, *then* $\lambda_2(k) = \mu_2(k) = 0$.

2. Known results

There are many results about the Iwasawa invariants of the cyclotomic $\mathbf{Z}_2$-extensions of real quadratic fields. We refer to some of them in this section.

Let n be a non-negative integer, $a_n = 2\cos(\frac{2\pi}{2^{n+2}})$ and $\mathbf{Q}_n = \mathbf{Q}(a_n)$.Then $\mathbf{Q}_n \subset \mathbf{Q}_{n+1}$ by $a_{n+1} = \sqrt{2+a_n}$. $\mathbf{Q}_n$ is a cyclic extension of $\mathbf{Q}$ of degree 2^n and $\mathbf{Q}_\infty = \cup_{n=0}^{\infty}\mathbf{Q}_n$ is the unique $\mathbf{Z}_2$-extension of $\mathbf{Q}$. Weber proved that $\lambda_2(\mathbf{Q}) = \mu_2(\mathbf{Q}) = \nu_2(\mathbf{Q}) = 0$ (cf. [3], Satz 6, p.29).

Let m be a positive square-free integer, let $k = \mathbf{Q}(\sqrt{m})$, and $k_n = k\mathbf{Q}_n$. Then the cyclotomic $\mathbf{Z}_2$-extension k_∞ of k is given by $\cup_{n=0}^{\infty}k_n = k\mathbf{Q}_\infty$. If $m > 2$, $k_1 = \mathbf{Q}(\sqrt{2}, \sqrt{m})$ contains just three real quadratic subfields $\mathbf{Q}_1$, k, $k' = \mathbf{Q}(\sqrt{2m})$. Hence k and k' have the same cyclotomic $\mathbf{Z}_2$-extension, which means the Iwasawa invariants are also the same.

Iwasawa proved that for each prime number l, if a Galois l-extension K/k of number fields has at most one (finite or infinite) ramified prime and the class number of k is not divisible by l, then the class number of K is also not divisible by l (cf. [5]). This implies if a real quadratic field k with odd class number has only one prime ideal above the prime number 2, then the class number of k_n is also odd for each $n \geq 0$, i.e., $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$. Moreover, by genus theory and the theorem of Rédei and Reichardt (cf. [8]), we can determine the real quadratic fields which have odd class number and only one prime ideal above the prime number 2. Hence we obtain the following:

THEOREM 3. *Let* $k = \mathbf{Q}(\sqrt{m})$ *or* $\mathbf{Q}(\sqrt{2m})$ *and let* $\lambda_2(k)$, $\mu_2(k)$, $\nu_2(k)$ *be the Iwasawa* λ-, μ-, *and* ν-*invariants of the cyclotomic* $\mathbf{Z}_2$-*extension* k_∞ *of* k, *respectively. Suppose that* m *is one of the following*:

(1) $m = 2$,
(2) $m = p \quad p \equiv 5 \pmod{8}$,
(3) $m = q \quad q \equiv 3 \pmod{4}$,
(4) $m = pq \quad p \equiv 3, q \equiv 7 \pmod{8}$,

where p *and* q *are prime numbers. Then we have* $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$.

These cases are often called trivial cases.

On the other hand, Ozaki and Taya, Fukuda and Komatsu proved the following theorems which are non-trivial.

THEOREM 4 (Ozaki-Taya) (cf. [7]). *Let* $k = \mathbf{Q}(\sqrt{m})$ *or* $\mathbf{Q}(\sqrt{2m})$ *and let* $\lambda_2(k)$, $\mu_2(k)$ *be the Iwasawa* λ-, μ-*invariants of the cyclotomic* $\mathbf{Z}_2$-*extension* k_∞ *of* k, *respectively. Suppose that* m *is one of the following*:

(1) $m = p \quad p \equiv 1 \pmod 8$ *and* $2^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{8}} \pmod p$,
(2) $m = pq \quad p \equiv q \equiv 3 \pmod 8$,
(3) $m = pq \quad p \equiv 3, q \equiv 5 \pmod 8$,
(4) $m = pq \quad p \equiv 5, q \equiv 7 \pmod 8$,
(5) $m = pq \quad p \equiv q \equiv 5 \pmod 8$,

where p and q are distinct prime numbers. Then we have $\lambda_2(k) = \mu_2(k) = 0$.

THEOREM 5 (Fukuda-Komatsu) (cf. [1]). *Let $k = \mathbf{Q}(\sqrt{m})$ or $\mathbf{Q}(\sqrt{2m})$ and let $\lambda_2(k)$, $\mu_2(k)$ be the Iwasawa λ-, μ-invariants of the cyclotomic $\mathbf{Z}_2$-extension k_∞ of k, respectively. Suppose that*

$$m = pq \quad p \equiv 3\,, \quad q \equiv 1 \pmod 8\,, \quad \left(\frac{p}{q}\right) = -1 \quad and \quad 2^{\frac{q-1}{4}} \equiv -1 \pmod q\,,$$

where p and q are prime numbers and $(\frac{*}{*})$ is Legendre's symbol. Then we have $\lambda_2(k) = \mu_2(k) = 0$.

Theorem 2 deals with non-trivial cases. We prove it according to the idea of Theorem 5.

3. Preparation

To prove Theorem 2 we need some preparation which were also used in the proof of Theorem 5.

Let p and q be prime numbers such that $p \equiv 7 \pmod 8, q \equiv 1 \pmod 8, (\frac{p}{q}) = -1$, and $k = \mathbf{Q}(\sqrt{pq}), k_n = k\mathbf{Q}_n, k_\infty = \cup_{n=0}^{\infty} k_n$.
Since $k_n = k(a_n) = k_{n-1}(\sqrt{2 + a_{n-1}})$, we have $N_{k_n/k_{n-1}}(2 - a_n) = (2 - a_n)(2 + a_n) = 2 - a_{n-1}$, where $N_{k_n/k_{n-1}}$ is the norm. Thus $N_{k_n/k}(2 - a_n) = 2$. Since a_n is an algebraic integer of $\mathbf{Q}_n$, it means $2\mathfrak{O}_{\mathbf{Q}_n} = (2 - a_n)^{2^n}\mathfrak{O}_{\mathbf{Q}_n} = (2 + a_n)^{2^n}\mathfrak{O}_{\mathbf{Q}_n}$, where $\mathfrak{O}_{\mathbf{Q}_n}$ is the integer ring of $\mathbf{Q}_n$. So the ideal $(2 - a_n)\mathfrak{O}_{\mathbf{Q}_n} = (2 + a_n)\mathfrak{O}_{\mathbf{Q}_n}$ is the unique prime ideal of $\mathbf{Q}_n$ lying above 2. Therefore the square of the unique prime ideal $\mathfrak{L}_n$ of k_n lying above 2 is $(2 - a_n)\mathfrak{O}_{k_n}$, where $\mathfrak{O}_{k_n}$ is the integer ring of k_n.
First, we show the following important proposition.

PROPOSITION 1. *Let k be as above and $\lambda_2(k)$, $\mu_2(k)$ the Iwasawa λ-, μ-invariants of the cyclotomic $\mathbf{Z}_2$-extension k_∞ of k, respectively. If there exists a non-negative integer n_0 such that $\mathfrak{L}_{n_0}$ is non-principal in k_{n_0}, then $\lambda_2(k) = \mu_2(k) = 0$.*

PROOF. Let A_n be the 2-Sylow subgroup of the ideal class group of k_n, B_n the subgroup of A_n consisting of ideal classes invariant under the action of $Gal(k_n/k)$ and B'_n the subgroup of B_n consisting of ideal classes containing ideals invariant under the action of $Gal(k_n/k)$. Then by genus formula, we have

$$o(B_n) = 2\text{-}part \ \ of \ \ h_k/(E_k : E_k \cap N_{k_n/k}(k_n^\times))\,,$$

$$o(B'_n) = 2\text{-}part \ \ of \ \ h_k/(E_k : N_{k_n/k}(E_{k_n}))\,,$$

where $o(B_n)$ is the order of B_n, h_k the class number of k, E_k the unit group of k, $k_n^\times$ the group of invertible elements of k_n, $(E_k : E_k \cap N_{k_n/k}(k_n^\times))$ the index of $E_k \cap N_{k_n/k}(k_n^\times)$ in E_k, $o(B_n')$ the order of B_n', E_{k_n} the unit group of k_n, $(E_k : N_{k_n/k}(E_{k_n}))$ the index of $N_{k_n/k}(E_{k_n})$ in E_k. By genus fomula, we can also show that $k(\sqrt{q})$ is the 2-genus field of $k/\mathbf{Q}$. Let G be $Gal(k/\mathbf{Q})$, σ a generator of G, A_0^G the subgroup of A_0 consisting of ideal classes invariant under the action of G. Then $A_0/A_0^{1-\sigma} \cong Gal(k(\sqrt{q})/k)$ by Artin map. Since $(\frac{p}{q}) = -1$, we have $A_0 = A_0^G A_0^{1-\sigma}$, which shows $A_0 = A_0^G$. It follows that the 2-Hilbelt class field of k is $k(\sqrt{q})$ and we obtain $o(B_n) = 2/(E_k : E_k \cap N_{k_n/k}(k_n^\times))$, $o(B_n') = 2/(E_k : N_{k_n/k}(E_{k_n}))$. Hence by the assumption, we have $B_n = B_n' = \langle cl(\mathfrak{L}_n)\rangle \cong \mathbf{Z}/2\mathbf{Z}$ for all $n \geq n_0$, where $cl(\mathfrak{L}_n)$ is the ideal class of k_n containing $\mathfrak{L}_n$, $\langle cl(\mathfrak{L}_n)\rangle$ the group generated by $cl(\mathfrak{L}_n)$, and $\mathbf{Z}$ the ring of rational integers. Since $N_{k_n/k_{n_0}}(\mathfrak{L}_n) = \mathfrak{L}_{n_0}$, the norm map $N_{k_n/k_{n_0}}$ of B_n to B_{n_0} is an isomorphism, which shows that the intersection of B_n and the kernel C_n of the norm map A_n to A_{n_0} is trivial. It means C_n is also trivial. Therefore, since $N_{k_n/k_{n_0}}(A_n) = A_{n_0}$, A_n is isomorphic to A_{n_0}, which implies $\lambda_2(k) = \mu_2(k) = 0$. □

REMARK 1. Since the 2-Hilbelt class field of k is $k(\sqrt{q})$ and $q \equiv 1 \pmod 8$, $\mathfrak{L}_0$ is principal in k.

Since $q \equiv 1 \pmod 8$, q splits completely in $\mathbf{Q}_1$. Moreover, the class number of $\mathbf{Q}_1$ is 1 and $N_{\mathbf{Q}_1/\mathbf{Q}}(1+\sqrt{2}) = -1$. Hence there exist positive integers r,s such that $q = (r + s\sqrt{2})(r - s\sqrt{2})$. Let $q_1 = r + s\sqrt{2}$, $q_2 = r - s\sqrt{2}$ (Note that q_1, q_2 are totally positive.). Then there exist integers a, b, c, d with $q_1 = a + b\sqrt{2} + 4\sqrt{2}(c + d\sqrt{2})$, $0 \leq a \leq 8$, $0 \leq b \leq 3$ and we have $q = q_1q_2 \equiv a^2 - 2b^2 \pmod{16}$. Thus if $q \equiv 1 \pmod{16}$, then

$$q_i \equiv \pm 1, \pm(1+\sqrt{2})^2 \quad \pmod{4\sqrt{2}} \quad -\text{(i)}$$

and if $q \equiv 9 \pmod{16}$, then

$$q_i \equiv \pm 3, \pm(1+2\sqrt{2}) \quad \pmod{4\sqrt{2}}. \quad -\text{(ii)}$$

On the other hand, since $p \equiv 7 \pmod 8$, p also splits completely in $\mathbf{Q}_1$. So there exist positive integers t,u such that $p = (t + u\sqrt{2})(t - u\sqrt{2})$. Let $p_1 = t + u\sqrt{2}$, $p_2 = t - u\sqrt{2}$ (Note that p_1, p_2 are also totally positive.). In the same way as above, we can show that if $p \equiv 7 \pmod{16}$, then

$$p_i \equiv 3 \pm \sqrt{2}, -3 \pm \sqrt{2} \quad \pmod{4\sqrt{2}}. \quad -\text{(iii)}$$

By class field theory, we can show the following lemma.

LEMMA 1. (1) *Suppose that* $q \equiv 1 \pmod{16}$.
If $2^{\frac{q-1}{4}} \equiv -1 \pmod q$, *then the ray class field* $\mathbf{Q}_1(mod q_i)$ *of* $\mathbf{Q}_1$ *mod* q_i *does not contain any quadratic extension of* $\mathbf{Q}_1$. *If* $2^{\frac{q-1}{4}} \equiv 1 \pmod q$, *then* $\mathbf{Q}_1(mod q_i)$ *contains a quadratic extension of* $\mathbf{Q}_1$.

(2) *Suppose that* $q \equiv 9 \pmod{16}$.
If $2^{\frac{q-1}{4}} \equiv -1 \pmod{q}$, *then* $\mathbf{Q}_1(mod q_i)$ *contains a quadratic extension of* $\mathbf{Q}_1$. *If* $2^{\frac{q-1}{4}} \equiv 1 \pmod{q}$, *then* $\mathbf{Q}_1(mod q_i)$ *does not contain any quadratic extension of* $\mathbf{Q}_1$.

(3) *Suppose that* $p \equiv 7 \pmod{8}$. *Then the ray class field* $\mathbf{Q}_1(mod p_i)$ *of* $\mathbf{Q}_1$ *mod* p_i *does not contain any quadratic extension of* $\mathbf{Q}_1$.

PROOF. At first we show (1), (2). Note that

$$(2+\sqrt{2})^{\frac{q-1}{2}} = (\sqrt{2}(1+\sqrt{2}))^{\frac{q-1}{2}} = 2^{\frac{q-1}{4}}(1+\sqrt{2})^{\frac{q-1}{2}}.$$

If $q \equiv 1 \pmod{16}$, then q splits completely in $\mathbf{Q}_2/\mathbf{Q}_1$, which implies $(2+\sqrt{2})^{\frac{q-1}{2}} \equiv 1 \pmod{q}$. Hence if $2^{\frac{q-1}{4}} \equiv -1 \pmod{q}$, then $(1+\sqrt{2})^{\frac{q-1}{2}} \equiv -1 \pmod{q}$, and if $2^{\frac{q-1}{4}} \equiv 1 \pmod{q}$, then $(1+\sqrt{2})^{\frac{q-1}{2}} \equiv 1 \pmod{q}$.

If $q \equiv 9 \pmod{16}$, then $(2+\sqrt{2})^{\frac{q-1}{2}} \equiv -1 \pmod{q}$. Hence if $2^{\frac{q-1}{4}} \equiv -1 \pmod{q}$, then $(1+\sqrt{2})^{\frac{q-1}{2}} \equiv 1 \pmod{q}$, and if $2^{\frac{q-1}{4}} \equiv 1 \pmod{q}$, then $(1+\sqrt{2})^{\frac{q-1}{2}} \equiv -1 \pmod{q}$. Let $J_{\mathbf{Q}_1}^{q_i} = \{\mathfrak{a} : ideal\ \ of\ \ \mathbf{Q}_1 \mid \mathfrak{a}\ \ is\ \ relatively\ \ prime\ \ to\ \ q_i\}$, and $P_{\mathbf{Q}_1}^{q_i} = \{(\alpha) : principal\ \ ideal\ \ of\ \ \mathbf{Q}_1 \mid \alpha \equiv 1 \pmod{q_i}\}$. Then we have $J_{\mathbf{Q}_1}^{q_i}/P_{\mathbf{Q}_1}^{q_i} \cong Gal(\mathbf{Q}_1(mod q_i)/\mathbf{Q}_1)$ by class field theory. There is a surjection such that

$$(\mathbf{Z}[\sqrt{2}]/q_i\mathbf{Z}[\sqrt{2}])^\times \to J_{\mathbf{Q}_1}^{q_i}/P_{\mathbf{Q}_1}^{q_i}$$

$$\alpha \mod q_i \mapsto (\alpha) \mod P_{\mathbf{Q}_1}^{q_i},$$

Since the kernel of this morphism is $\langle -1 \mod q_i, 1+\sqrt{2} \mod q_i\rangle$ and -1 is a quadratic residue mod q_i, we obtain (1) and (2).

Similarly, let $J_{\mathbf{Q}_1}^{p_i} = \{\mathfrak{a} : ideal\ \ of\ \ \mathbf{Q}_1 \mid \mathfrak{a}\ \ is\ \ relatively\ \ prime\ \ to\ \ p_i\}$, $P_{\mathbf{Q}_1}^{p_i} = \{(\alpha) : principal\ \ ideal\ \ of\ \ \mathbf{Q}_1 \mid \alpha \equiv 1 \pmod{p_i}\}$. Then we also have $J_{\mathbf{Q}_1}^{p_i}/P_{\mathbf{Q}_1}^{p_i} \cong Gal(\mathbf{Q}_1(mod p_i)/\mathbf{Q}_1)$ and $\langle -1 \mod p_i, 1+\sqrt{2} \mod p_i\rangle$ is the kernel of the surjection

$$(\mathbf{Z}[\sqrt{2}]/p_i\mathbf{Z}[\sqrt{2}])^\times \to J_{\mathbf{Q}_1}^{p_i}/P_{\mathbf{Q}_1}^{p_i}$$

$$\alpha \mod p_i \mapsto (\alpha) \mod P_{\mathbf{Q}_1}^{p_i},$$

Since $p \equiv 7 \pmod{8}$, $2 \mid p-1$ and $2^2 \nmid p-1$. Furthermore, the order of $-1 \mod p_i$ is 2, which implies the order of the kernel is even. Hence we have (3). □

4. Proof of Theorem 2

We use the following well-known fact to prove Theorem 2.

LEMMA 2 (cf. [9], p. 183). *Let a be an element of* $\mathbf{Q}_1$ *which is prime to* 2. *Then,*

(1) *there exists an element* α *of* $\mathbf{Q}_1$ *such that* $\alpha^2 \equiv a \pmod 4$ *if and only if* $\mathbf{Q}_1(\sqrt{a})/\mathbf{Q}_1$ *is unramified at all primes of* $\mathbf{Q}_1$ *above* 2.

(2) *there exists an element* α *of* $\mathbf{Q}_1$ *such that* $\alpha^2 \equiv a \pmod{4\sqrt{2}}$ *if and only if all primes of* $\mathbf{Q}_1$ *above* 2 *split in* $\mathbf{Q}_1(\sqrt{a})/\mathbf{Q}_1$.

PROOF OF THEOREM 2. Note that for any element α in $\mathfrak{O}_{\mathbf{Q}_1}$ which is prime to 2, we have

$$\alpha^2 \equiv 1, 3+2\sqrt{2} \pmod{4\sqrt{2}}. \quad -(iv)$$

(1) Suppose that $2^{\frac{q-1}{4}} \equiv -1 \pmod q$. If $q \equiv 9 \pmod{16}$, $\mathbf{Q}_1(mod q_i)/\mathbf{Q}_1$ has a quadratic subextension by Lemma 1 (2). First we show the quadratic extension of $\mathbf{Q}_1$ must be $\mathbf{Q}_1(\sqrt{q_i})/\mathbf{Q}_1$. Let $\mathbf{Q}_1(\sqrt{m})/\mathbf{Q}_1$ be the quadratic subextension, where $m \in \mathfrak{O}_{\mathbf{Q}_1}$. Since $\mathbf{Q}_1(\sqrt{m})/\mathbf{Q}_1$ is unramified at the infinite primes, we have $m > 0$. Note that we can assume $v_{\mathfrak{p}}(m) = 0$ or 1 for any prime $\mathfrak{p}$ of $\mathbf{Q}_1$, where $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic additive valuation. If $v_{\mathfrak{p}}(m) = 1$, then X^2-m is an Eisenstein polynomial with regard to $\mathfrak{p}$, which implies $\mathfrak{p}$ is totally ramified in $\mathbf{Q}_1(\sqrt{m})/\mathbf{Q}_1$. Furthermore, since the relative discriminant of $\mathbf{Q}_1(\sqrt{m})/\mathbf{Q}_1$ divides $4m\mathfrak{O}_{\mathbf{Q}_1}$, any prime $\mathfrak{p}$ with $\mathfrak{p} \nmid 4m\mathfrak{O}_{\mathbf{Q}_1}$ is unramified in $\mathbf{Q}_1(\sqrt{m})/\mathbf{Q}_1$. Hence m must be q_i or $q_i\varepsilon$, where $\varepsilon = 1+\sqrt{2}$. By (ii), (iv) and Lemma 2 (1), $\mathbf{Q}_1(\sqrt{q_i\varepsilon})/\mathbf{Q}_1$ is ramified at a prime of $\mathbf{Q}_1$ above 2. Therefore $\mathbf{Q}_1(\sqrt{m})$ must be $\mathbf{Q}_1(\sqrt{q_i})$ as desired.

It follows that all primes of $\mathbf{Q}_1$ above 2 are unramified in $\mathbf{Q}_1(\sqrt{q_i})/\mathbf{Q}_1$. Hence we have $q_i \equiv 1\ ,\ 3+2\sqrt{2} \pmod 4$ by Lemma 2 and (iv), which shows $q_i \equiv -3, -1+2\sqrt{2} \pmod{4\sqrt{2}}$ by (ii). On the other hand, $k_1(\sqrt{q_i})$ is an unramified extension of k_1. Since $\mathfrak{L}_1$ does not split in $k_1(\sqrt{q_i})$ by Lemma 2, $\mathfrak{L}_1$ is non-principal in k_1. Therefore we have $\lambda_2(k) = \mu_2(k) = 0$ by Proposition 1.

Secondly, suppose that $2^{\frac{q-1}{4}} \equiv 1 \pmod q$. If $q \equiv 9 \pmod{16}$, then $\mathbf{Q}_1(\sqrt{q_i})$ is not contained in $\mathbf{Q}_1(mod q_i)$ by Lemma 1 (2), which shows $q_i \equiv 3\ ,\ 1+2\sqrt{2} \pmod{4\sqrt{2}}$ by Lemma 2 and (ii), (iv). Hence we have $pq_i \equiv -3, -1+2\sqrt{2} \pmod{4\sqrt{2}}$. Since $\mathfrak{L}_1$ does not split in an unramified extension $k_1(\sqrt{pq_i})/k_1$, $\mathfrak{L}_1$ is non-principal in k_1. Therefore we also have $\lambda_2(k) = \mu_2(k) = 0$ by Proposition 1.
This completes the proof of Theorem 2 (1).

(2) Suppose that $q \equiv 1 \pmod{16}$, $p \equiv 7 \pmod{16}$, and $2^{\frac{q-1}{4}} \equiv -1 \pmod q$. By Lemma 1 (1), Lemma 2, (i) and (iv), we have $q_i \equiv -1, -3+2\sqrt{2} \pmod{4\sqrt{2}}$. By (iii) we have $p_i\varepsilon \equiv \pm 3, \pm 1+2\sqrt{2} \pmod{4\sqrt{2}}$. Lemma 1 (3) implies that all primes of $\mathbf{Q}_1$ above 2 are ramified in $\mathbf{Q}_1(\sqrt{p_i\varepsilon})/\mathbf{Q}_1$, which shows $p_i\varepsilon \equiv 3, 1+2\sqrt{2} \pmod{4\sqrt{2}}$ by Lemma 2 and (iv). Hence we have $p_iq_j\varepsilon \equiv -3, -1+2\sqrt{2} \pmod{4\sqrt{2}}$. Since $\mathfrak{L}_1$ does not split in an unramified extension $k_1(\sqrt{p_iq_j\varepsilon})/k_1$, $\mathfrak{L}_1$ is non-principal. Therefore we have $\lambda_2(k) = \mu_2(k) = 0$ by Proposition 1.

REMARK 2. Suppose that $q \equiv 1 \pmod{16}$, $p \equiv -1 \pmod{16}$, and $2^{\frac{q-1}{4}} \equiv -1 \pmod{q}$. Then we can show that $\mathfrak{L}_1$ splits in an unramified extension $k_1(\sqrt{p_i q_j \varepsilon})/k_1$. But Kuroda's class number formula (cf. [6]) shows that the 2-Hilbelt class field of k_1 is $k_1(\sqrt{p_1 q_1 \varepsilon}, \sqrt{p_1 q_2 \varepsilon})$. Hence $\mathfrak{L}_1$ is principal in k_1, i.e., we can not decide $\lambda_2(k) = \mu_2(k) = 0$ by using Proposition 1.

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