Positivity of the exterior power of the tangent bundles

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(Communicated by Shigefumi MORI, M.J.A., Nov. 13, 2023)

Abstract: Let X be a complex smooth projective variety such that the exterior power of the tangent bundle $\bigwedge^r T_X$ is nef for some $1 \leq r < \dim X$. We prove that, up to a finite étale cover, X is a Fano fiber space over an Abelian variety. This gives a generalization of the structure theorem of varieties with nef tangent bundle by Demailly, Peternell and Schneider [5] and that of varieties with nef $\bigwedge^2 T_X$ by the author [20]. Our result also gives an answer to a question raised by Li, Ou and Yang [15] for varieties with strictly nef $\bigwedge^r T_X$ when $r < \dim X$.

Key words: Tangent bundle; exterior power; nef.

1. Introduction. Positivity for vector bundles such as ampleness and nefness has left its mark on the study of algebraic geometry. Let X be a complex smooth projective variety of dimension n; we focus on the positivity of the tangent bundle T_X, which reflects the global geometry of X. As a generalization of the Hartshorne-Frankel conjecture solved by Mori [17] (see also [18] by Siu and Yau), Campana and Peternell [2] studied the structure of smooth projective varieties with nef tangent bundle, paying special attention to 3-folds. In higher dimensional case, Demailly, Peternell and Schneider obtained the following structure theorem:

Theorem 1.1 ([5, Main Theorem]). *If T_X is nef, then there exists a finite étale cover $X' \to X$ such that X' is a locally trivial fibration $\varphi : X' \to \mathrm{Alb}(X')$ whose fibers are Fano varieties.*

Remark that [5] proved something more than Theorem 1.1; they proved the above theorem holds for any compact Kähler manifold with nef tangent bundle. Moreover the local triviality of the above Albanese map follows from [5, 3.D]. On the other hand, some years ago, Cao and Höring extended Theorem 1.1 to a more general setting:

Theorem 1.2 ([4, Theorem 1.3]). *If the anticanonical divisor $-K_X$ is nef, then there exists a finite étale cover $X' \to X$ such that $X' \cong Y \times Z$ where K_Y is trivial and Z is a locally trivial fibration $\varphi : Z \to \mathrm{Alb}(Z)$ with a rationally connected fiber.*

In general a fiber of φ in Theorem 1.2 is not a Fano variety, because there exists a lot of rationally connected projective varieties with nef anticanonical divisor which is not Fano (for instance, consider the blow up of $\mathbf{P}^2$ in nine points). Theorem 1.2 can be seen as an extension of the classical Beauville-Bogomolov decomposition [1]. The main result of this paper is a generalization of Theorem 1.1:

Theorem 1.3. *Let X be a smooth projective variety of dimension n. Assume that $\bigwedge^r T_X$ is nef for some $1 \leq r < n$. Then if we take a suitable finite étale cover $\tilde{X} \to X$, there exists a locally trivial fibration $\varphi : \tilde{X} \to A$ such that the fiber F is a Fano variety and A is an Abelian variety. Moreover, if $\dim A \geq r - 1$, then T_X is nef; otherwise $\bigwedge^{r-\dim A} T_{\tilde{X}/A}$ is nef.*

This theorem reduces the study of smooth projective varieties with nef $\bigwedge^r T_X$ ($r < \dim X$) to that of Fano varieties. For $r = 1$, Theorem 1.3 is nothing but Theorem 1.1; for $r = 2$ this was obtained in [20, Theorem 1.5]. The proof of [20, Theorem 1.5] involves the deformation theory of rational curves and some complicated arguments. On the other hand, in this short paper, we give a simple proof of Theorem 1.3. Our proof relies on two key ingredients; one is Theorem 1.2; the other is a recent result of Gachet [6]. In [6, Theorem 1.2], she proved that for a smooth rationally connected projective variety X of dimension n if $\bigwedge^{n-1} T_X$ is nef, then X is a Fano variety. Moreover by using a result by Laytimi and Nahm [12], we see that if $\bigwedge^r T_X$ is nef for some $r < n$, then so is $\bigwedge^{n-1} T_X$. Thus we have the following

2020 Mathematics Subject Classification. Primary 14J40, 14J45, 14M22.

doi: 10.3792/pjaa.99.015

Proposition 1.4. *Let X be a smooth projective variety of dimension n. Assume that X is rationally connected and $\bigwedge^r T_X$ is nef for some $1 \leq r < n$. Then X is a Fano variety.*

Remark that, combining with [15, Theorem 1.2], Proposition 1.4 gives an affirmative answer to the following question by Li, Ou and Yang when $r < \dim X$:

Question 1.5 ([15, Remark 5.3], [16, Conjecture 4.9], [6, Question in Section 1]). Assume that $\bigwedge^r T_X$ is strictly nef for some $1 \leq r \leq n$. Then is X a Fano variety?

Finally, Theorem 1.3 follows from Theorem 1.2, Proposition 1.4 and standard arguments. We remark that the results of this paper will be extended to compact Kähler manifolds if Theorem 1.2 is also valid for those manifolds.

2. Preliminaries.

2.1. Notation and conventions. We will use the basic notation and definitions in [8], [10], [13], [14] and [11]. Along this paper, we work over the complex number field.

- A *curve* means a projective variety of dimension one.
- Let X be a smooth projective variety. A line bundle L on X is said to be *strictly nef* (resp. *nef*) if the intersection number $L \cdot C$ is positive (resp. non-negative) for any curve $C \subset X$. In general, we say that a vector bundle $\mathcal{E}$ is *strictly nef* (resp. *nef*) if the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is strictly nef (resp. *nef*) on $\mathbf{P}(\mathcal{E})$.
- For a non-constant morphism $f : \mathbf{P}^1 \to X$ from a projective line $\mathbf{P}^1$ to a smooth projective variety X, f is said to be *free* if f^*T_X is nef.

Throughout this section, we always assume the following

Assumption 2.1. Assume X is a smooth projective variety of dimension n such that the exterior power $\bigwedge^r T_X$ is nef for some $1 \leq r < n$.

Proposition 2.2. *The following hold*:

(*i*) *The anticanonical divisor $-K_X$ is nef.*

(*ii*) *If the Kodaira dimension $\kappa(X) = 0$, then there exists a finite étale cover $f : \tilde{X} \to X$ such that $\tilde{X}$ is an Abelian variety.*

Proof. The first part follows from

$$\det\left(\bigwedge^r T_X\right) \cong \mathcal{O}_X\left(\binom{n-1}{r-1}(-K_X)\right).$$

The second part follows from [22, Theorem 1.1] (see also [3, Proposition 1.2]). □

Lemma 2.3 ([3, Lemma 1.3], [21, Lemma 2.9]). *Let $f : \mathbf{P}^1 \to X$ be a non-free rational curve, that is, f^*T_X is not nef. Then we have*

$$-K_X \cdot f_*(\mathbf{P}^1) \geq n - r + 1.$$

Proof. Assume that the splitting type of f^*T_X is $(a_1, a_2, \dots, a_n)$, that is,

$$f^*T_X \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^1}(a_i) \quad (a_1 \geq a_2 \geq \dots \geq a_n,\ a_1 \geq 2).$$

The r-th exterior power

$$\bigwedge^r f^*T_X \cong \bigoplus_{1 \leq i_1 < i_2 < \dots < i_r \leq n}^{n} \mathcal{O}_{\mathbf{P}^1}(a_{i_1} + a_{i_2} + \dots + a_{i_r})$$

is nef; this yields

$$a_{n-r+1} + a_{n-r+2} + \dots + a_n \geq 0.$$

Since f is not free, a_n is negative. These imply that

$$\begin{aligned}(r-1)a_{n-r+1} &\geq a_{n-r+1} + a_{n-r+2} + \dots + a_{n-1} \\ &\geq -a_n \geq 1.\end{aligned}$$

Thus a_{n-r+1} is positive. As a consequence, we have the inequality

$$\begin{aligned}&-K_X \cdot f_*(\mathbf{P}^1) \\ &\quad = a_1 + (a_2 + \dots + a_{n-r}) + (a_{n-r+1} + \dots + a_n) \\ &\quad \geq 2 + (n - r - 1) + 0 = n - r + 1.\end{aligned}$$

□

Proposition 2.4 ([19, Proposition 3.3]). *Let $\varphi : X \to A$ be a smooth morphism onto an Abelian variety with connected fibers. Then the following hold*:

(*i*) *If $\dim A \geq r - 1$, then T_X is nef.*

(*ii*) *If $\dim A < r - 1$, then $\bigwedge^{r-\dim A} T_{X/A}$ is nef.*

Proof. We have an exact sequence

$$0 \to T_{X/A} \to T_X \to \varphi^*T_A \to 0. \tag{1}$$

By [8, Chapter II, Exercise 5.16 (d)], we have a filtration of $\bigwedge^r T_X$:

$$\bigwedge^r T_X = E^0 \supset E^1 \supset E^2 \supset \dots \supset E^{r+1} = 0$$

such that

$$E^p/E^{p+1} \cong \left(\bigwedge^p T_{X/A}\right) \bigotimes \left(\bigwedge^{r-p} \varphi^*T_A\right)$$

for any p. In particular, we have the following exact sequences:

$$(2) \qquad 0 \to E^1 \to \bigwedge^r T_X \to \bigwedge^r \varphi^* T_A \to 0,$$

$$(3) \quad 0 \to E^2 \to E^1 \to T_{X/A} \bigotimes \left(\bigwedge^{r-1} \varphi^* T_A \right) \to 0.$$

To prove (i), assume $\dim A \geq r-1$. Remark that $T_A \cong \mathcal{O}_A^{\oplus \dim A}$. We claim that E^1 is nef. If $\dim A \geq r$, then it follows from the sequence (2) and [2, Proposition 1.2 (8)] that E^1 is nef. If $\dim A = r-1$, then the sequence (2) yields $E^1 \cong \bigwedge^r T_X$; this implies that E^1 is nef. By the sequence (3), $T_{X/A} \bigotimes \left(\bigwedge^{r-1} \varphi^* T_A \right)$ is nef. Since $\bigwedge^{r-1}(\varphi^* T_A)$ is trivial bundle, we conclude that the relative tangent bundle $T_{X/A}$ is nef. Finally our assertion follows from the sequence (1).

To prove (ii), assume that $\dim A < r-1$. Since $\bigwedge^p \varphi^* T_A = 0$ for any $p > \dim A$, we have

$$\bigwedge^r T_X = E^0 = E^1 = \cdots = E^{r-\dim A}.$$

Thus we have a surjection

$$\bigwedge^r T_X = E^{r-\dim A} \to \bigwedge^{r-\dim A} T_{X/A};$$

this implies that $\bigwedge^{r-\dim A} T_{X/A}$ is nef. $\square$

3. Proof of the Main Theorem. The following is due to Gachet:

Proposition 3.1 ([6, Theorem 1.2])**.** *Let X be a smooth projective variety of dimension n. Assume that X is rationally connected and $\bigwedge^{n-1} T_X$ is nef. Then $-K_X$ is ample, that is, X is a Fano variety.*

Remark 3.2. Although Proposition 3.1 was not written explicitly in the first draft of [6], Gachet introduced this statement holds at Algebraic Geometry seminar of the University of Tokyo (see Acknowledgements below).

Theorem 3.3 ([12, Theorem 3.3], [9])**.** *Let X be a smooth projective variety of dimension n. For a vector bundle E of rank r, assume that its exterior power $\bigwedge^m E$ is nef for some positive integer m. Then the vector bundle $\bigwedge^{m+k} E$ is also nef for any $0 \leq k \leq r-m$.*

Remark 3.4. In general, if a vector bundle E is strictly nef, it is not necessarily true that its exterior power $\bigwedge^r E$ is strictly nef. For instance, see ([7, Section 10 in Chapter I] and [16, Example 2.1]). This means that an analogue of Theorem 3.3 does not hold if we replace nefness of $\bigwedge^m E$ by strict nefness.

Proof of Proposition 1.4. Assume that X is rationally connected and $\bigwedge^r T_X$ is nef for some $1 \leq r < n$. Then Theorem 3.3 implies that $\bigwedge^{n-1} T_X$ is nef. Applying Proposition 2.4, we see that X is a Fano variety. $\square$

Proof of Theorem 1.3. By Proposition 2.2 (i), $-K_X$ is nef; according to Theorem 1.2, this turns out that there exists a finite étale cover $X' \to X$ such that $X' \cong Y \times Z$ where K_Y is trivial and Z is a locally trivial fibration $Z \to \mathrm{Alb}(Z)$ with a rationally connected fiber. Since we have $X' \to X$ is étale, $\bigwedge^r T_{X'}$ is also nef; then by Theorem 3.3, $\bigwedge^{n-1} T_{X'}$ is also nef. Let $p_1 : X' \to Y$ (resp. $p_2 : X' \to Z$) be the first projection (resp. the second projection). We denote by ℓ the dimension of Y. Since $\bigwedge^{n-1} T_{X'}$ is isomorphic to the direct sum of

$$p_1^* \left(\bigwedge^{\ell} T_Y \right) \bigotimes p_2^* \left(\bigwedge^{n-\ell-1} T_Z \right)$$

and

$$p_1^* \left(\bigwedge^{\ell-1} T_Y \right) \bigotimes p_2^* \left(\bigwedge^{n-\ell} T_Z \right).$$

The direct summand $p_1^*(\bigwedge^{\ell-1} T_Y) \bigotimes p_2^*(\bigwedge^{n-\ell} T_Z)$ is nef; restricting this bundle to a fiber of the projection p_2, we see that $\bigwedge^{\ell-1} T_Y$ is also nef provided that $\ell > 0$. If $\ell = 1$, then Y is an elliptic curve. Furthermore if $\ell > 1$, then Proposition 2.2 (ii) implies that Y is a finite étale quotient of an Abelian variety $\tilde{Y}$. Hence, in any case, there exists a finite étale cover $\tilde{X} \to X'$ such that $\tilde{X}$ is a locally trivial fibration $\varphi : \tilde{X} \to A$ onto an Abelian variety A with a rationally connected fiber. Then our assertion follows from Proposition 2.4 and Proposition 1.4. $\square$

Acknowledgements. The author is partially supported by JSPS KAKENHI Grant Number 21K03170 and the Sumitomo Foundation Grant Number 190170. On August the 1st 2022, Cécile Gachet gave a talk on the result of [6] in Algebraic Geometry seminar of the University of Tokyo; the author knew her excellent result [6, Theorem 1.2] at the time. The author would like to thank Cécile Gachet for sending him her preprint [6] and answering his questions. The author would like to thank Kento Fujita for sending him his private note [9], which states that for a vector bundle E of rank r if $\bigwedge^\ell E$ is nef, then so is $\bigwedge^m E$ for any $\ell \leq m \leq r$. On the other hand, while preparing this

paper, the author noticed the result of [9] was contained in [12].

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