Gosper's strange series: A new, simplified proof and generalizations

By John CAMPBELL

Department of Mathematics, Toronto Metropolitan University, 350 Victoria Street, Toronto, ON, M5B 2K3, Canada

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Abstract: In 1977, Gosper introduced a conjectural evaluation for a hypergeometric series that has been described as strange by a number of authors. In 2013, Ebisu proved Gosper's conjecture using contiguity operators. Subsequently, in 2017, Chu provided another proof of Gosper's conjecture, using a telescoping argument together with Pfaff's transformation. In this article, we present a new and simplified proof of Gosper's conjecture that is inequivalent to the previous proofs due to Ebisu and Chu. Our proof relies on an evaluation technique that was previously given by Campbell and Cantarini and that involves the modified Abel lemma on summation by parts. We also show how this method may be applied to prove generalizations and variants of Gosper's summation.

Key words: Gosper's series; hypergeometric series; symbolic evaluation.

1. Introduction. As in [3,4], our article is based on the following identity that had been conjectured to hold true by Gosper in 1977:

$$(1)\qquad {}_2F_1\left[\begin{matrix}1-a, b\ ;\ \dfrac{b}{a+b}\\ b+2\ ;\end{matrix}\right] = (b+1)\left(\frac{a}{a+b}\right)^a.$$

We are letting generalized hypergeometric series be defined and denoted so that

$${}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p\ ;\\ \beta_1,\dots,\beta_q\ ;\end{matrix} x\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{x^n}{n!}.$$

We are letting the Pochhammer symbol be such that $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, and we recall that the Γ-function is such that $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$ for $\Re(x) > 0$. In this article, we introduce a new, simplified proof of Gosper's conjecture that is inequivalent to the previous proofs due to Ebisu [4] and Chu [3], and we apply a method related to our new proof to determine new generalizations and variants of (1).

1.1. Background. Ebisu's proof [4] of Gosper's conjecture heavily relied on contiguity operators. The main feature of this proof by Ebisu [4] is given by the following result. If $\ell \in \mathbf{N}$, $a \in \mathbf{C}$, and c is a non-integer complex number, then, for any root λ of

$${}_2F_1\left[\begin{matrix}1-a, -\ell\ ;\\ 2-c\ ;\end{matrix} x\right],$$

we have that

$${}_2F_1\left[\begin{matrix}a, 1+\ell\ ;\\ c\ ;\end{matrix} \lambda\right] = -\frac{(1-c)q_0(\lambda)}{\ell!(1-\lambda)^\ell}$$

and that

$${}_2F_1\left[\begin{matrix}c-a, c-1-\ell\ ;\\ c\ ;\end{matrix} \lambda\right] = -\frac{1-c}{\ell!}(1-\lambda)^{a+1-c}q_0(\lambda),$$

where $q_0(x)$ is a polynomial given by hypergeometric expressions evaluated by Ebisu in [4]. Chu's proof in [3] of Gosper's conjecture relied on the application of a telescoping sum to produce an identity for an infinite series with four free parameters, together with the application of the Pfaff transformation. More specifically, writing

$$T_k = \frac{(a)_k(b)_k}{(c)_k(d)_k},$$

as in [3], a telescoping argument based on the use of the difference operator $\Delta T_k = T_{k+1} - T_k$ was used to evaluate the ${}_4F_3(1)$-series

$${}_4F_3\left[\begin{matrix}1, a, b, 1+\frac{ab-cd}{a+b-c-d}\ ;\\ c+1, d+1, \frac{ab-cd}{a+b-c-d}\ ;\end{matrix} 1\right] = \frac{cd}{cd-ab},$$

and this, in turn, was used to reprove a similar ${}_4F_3(1)$-series evaluation given by Gessel and Stanton [5] in 1982. A limiting case then produced a rational function evaluation for a ${}_3F_2(x)$-series, and the Pfaff transformation

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$$ {}_2F_1\left[\begin{matrix} a, b\ ; \\ c\ ; \end{matrix} x\right] = (1-x)^{-a}{}_2F_1\left[\begin{matrix} a, c-b\ ; \\ c\ ; \end{matrix} \frac{x}{x-1}\right] $$

was then applied by Chu to prove Gosper's conjecture [3]. From our proof in Section 2, it should be clear that this is completely different compared to the past proofs in [3,4]. Notably, our proof does not involve the Pfaff transformation, telescoping, ${}_4F_3(1)$-series, polynomial roots, etc.

Let us write

$$ \Gamma\left[\begin{matrix} a, b, \dots, c \\ A, B, \dots, C \end{matrix}\right] = \frac{\Gamma(a)\Gamma(b)\cdots\Gamma(c)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}. $$

The key to our new proof is the following hypergeometric identity introduced by Campbell and Cantarini in 2022 [1]: For free parameters α, β, and γ, the identity

$$ (2) \quad {}_3F_2\left[\begin{matrix} \alpha, \beta, \gamma\ ; \\ \alpha+\beta-\frac{(\alpha-1)(\beta-1)}{\gamma}-1, \gamma+2\ ; \end{matrix} 1\right] = (\gamma+1)\cdot \Gamma\left[\begin{matrix} \alpha+\beta-\frac{(\alpha-1)(\beta-1)}{\gamma}-1, 1-\frac{(\alpha-1)(\beta-1)}{\gamma} \\ \alpha-\frac{(\alpha-1)(\beta-1)}{\gamma}, \beta-\frac{(\alpha-1)(\beta-1)}{\gamma} \end{matrix}\right] $$

holds true. Actually, as we are to demonstrate in Section 2, Gosper's summation (1) can be shown to hold in a direct way by applying the limiting operator $\lim_{\alpha\to\infty}\cdot$ to both sides of the equality in (2). This approach is considerably simpler relative to [3,4].

The ${}_3F_2(1)$-identity (2) was proved in [1] via an evaluation technique introduced in [1] that may, informally, be described as being given by something of a combination of the modified Abel lemma on summation by parts and a method of undetermined coefficients. Letting ∇ and Δ be such that $\nabla\tau_n = \tau_n - \tau_{n-1}$ and $\Delta\tau_n = \tau_n - \tau_{n+1}$ for a sequence $(\tau_n : n \in \mathbf{N})$, the aforementioned summation lemma gives us that

$$ (3) \quad \sum_{n=1}^{\infty} B_n \nabla A_n = (\lim_{m\to\infty} A_m B_{m+1}) - A_0 B_1 + \sum_{n=1}^{\infty} A_n \Delta B_n $$

if this limit exists and one of the two series given above converges, referring to [1] and the references therein. We may obtain (2) by combining (3) and Gauss's famous evaluation for ${}_2F_1(1)$-series. This evaluation technique from [1] may be used to produce families of generalizations and variants of Gosper's sum (1), which will be obtained in Section 3.

An interesting property concerning (2) is given by how the Wilf–Zeilberger method and/or Zeilberger's algorithm [6] cannot be used directly to prove (2), in contrast to classically known ${}_3F_2(1)$-identities. Also, it is quite remarkable that current or recent Computer Algbera Systems such as the 2022 version of Mathematica and the 2023 version of Maple cannot evaluate the ${}_3F_2(1)$-series due to Campbell and Cantarini [1] for free parameters α, β, and γ, which adds to the interest in the material in Section 2 below.

2. A proof of Gosper's summation.

Theorem 2.1. *The evaluation for Gosper's series* (1) *holds.*

Proof. We apply the operation $\lim_{\alpha\to\infty}\cdot$ to both sides of (2). Elementary real analysis may then be used to justify our interchanging the operation of $\lim_{\alpha\to\infty}\cdot$ and the infinite summation operator, for parameters β and γ such that the resultant series converges. We may verify the evaluation

$$ \lim_{\alpha\to\infty} \frac{(\alpha)_n}{\left(\alpha+\beta-\frac{(\alpha-1)(\beta-1)}{\gamma}-1\right)_n} = \left(\frac{\gamma}{\gamma-\beta+1}\right)^n $$

using the Euler asymptotic identity

$$ \Gamma(\lambda+n) \sim (n-1)!n^{\lambda}. $$

By again using the Euler asymptotic identity, the right-hand side of (2) tends to

$$ (4) \quad (\gamma+1)\exp\left((\beta-1)\left(\log\left(\frac{\gamma-\beta+1}{\gamma}\right) - \log\left(\frac{1-\beta}{\gamma}\right)\right)\right) = (\gamma+1)\left(\frac{\beta-\gamma-1}{\beta-1}\right)^{\beta-1} $$

as $\alpha \to \infty$. So, we have shown that (4) is equal to the following:

$$ \sum_{n=0}^{\infty} \frac{\left(\frac{\gamma}{\gamma-\beta+1}\right)^n (\beta)_n(\gamma)_n}{n!(\gamma+2)_n}. $$

This equality is equivalent to Gosper's identity (1), since we may rewrite the above series as

$$ {}_2F_1\left[\begin{matrix}\beta,\gamma\,; & \dfrac{\gamma}{\gamma-\beta+1}\\ \gamma+2\,; & \end{matrix}\right] $$

and we may thus proceed to set $\gamma = b$ and $\beta = 1-\alpha$. □

3. Further applications of an Abel-type lemma. Informally, the evaluation technique from [1] is given by setting the B-sequence in (3) to be hypergeometric and by then setting the corresponding A-sequence to be a rational function such that the coefficients are solved for in such a way so as to ensure that the summand $A_n \Delta B_n$ simplifies in a certain way described in [1]. In addition to our application of this method, via the ${}_3F_2(1)$-identity (2) proved via this method in [1], to determine a new proof of (1), we have applied this same method to determine many generalizations and variants of (1). For example, as we are to demonstrate below, the Gosper-type identity in (6) can be shown to be equivalent to (1).

Proof #2 of (1). Setting

$$A_n = \frac{1}{n - \frac{1-\alpha x}{x-1}}$$

and

$$B_n = \frac{x^n (\alpha)_n}{n!} \tag{5}$$

in the modified Abel lemma (3), we then write $x = \frac{b}{a+b}$ and $\alpha = -a$. It then follows from (3) that

$$ {}_2F_1\left[\begin{matrix}-a, b+\frac{b}{a}\,; & \dfrac{b}{a+b}\\ b+2+\frac{b}{a}\,; & \end{matrix}\right] = \left(\frac{a}{a+b}\right)^a \frac{a+b+ab}{a+b}. \tag{6}$$

By replacing replacing (a,b) with $(a+1, b+\frac{b}{a})$ in (1), we find that (1) and (6) are equivalent. □

The new proof for Gosper's strange series (1) given above inspires the application of variants of the B-sequence indicated in (5), as below.

Example 1. Setting

$$A_n = \frac{1}{n + \frac{-\sqrt{\alpha^2x^2-4\alpha x^2+4x^2-4x+4}+\alpha x+2x-4}{2(x-1)}}$$

and

$$B_n = \frac{x^n(\alpha)_n}{(n+2)n!}$$

in the modified Abel lemma and in accordance with the technique from [1], this gives us a way of determining a closed form for

$$\sum_{n=0}^{\infty} \frac{x^n(\alpha)_n}{n!(n+2)} \times \frac{1}{(2+2n-2nx-x\alpha+\beta)} \times \frac{1}{(4+2n-2x-2nx-x\alpha+\beta)},$$

writing $\beta = \sqrt{\alpha^2x^2-4\alpha x^2+4x^2-4x+4}$.

Example 2. Setting

$$A_n = \frac{1}{n - \frac{1-\alpha x}{x-1}}$$

and

$$B_n = \frac{(-1)^n x^n (\alpha)_n}{n!}$$

in the modified Abel lemma and in accordance with the technique from [1], we can show that

$$\sum_{n=0}^{\infty} \frac{(-x)^n(\alpha)_n}{n!} \times \frac{2nx-2n+2\alpha x-x-1}{(nx-n+\alpha x-1)(nx-n+\alpha x-x)} = \frac{(x+1)^{1-\alpha}}{(\alpha-1)x}.$$

Example 3. Setting

$$A_n = \frac{1}{n - \frac{1-\alpha x}{x-1}+1}$$

and

$$B_n = \frac{x^n(\alpha)_n}{n!}$$

in the modified Abel lemma and in accordance with the technique from [1], we can show that

$$\sum_{n=0}^{\infty} \frac{x^n(\alpha)_n}{n!} \times \frac{nx^2-2nx+n+\alpha x^2-2\alpha x-x+2}{(nx-n+\alpha x-1)(nx-n+\alpha x+x-2)} = (1-x)^{-\alpha}.$$

We may produce many similar results, by altering the B-sequences indicated above.

4. Conclusion. In much the same way that the modified Abel lemma was applied in [1] to

Gauss' ${}_2F_1(1)$-identity so as to obtain the ${}_3F_2(1)$-identity (2), the Abel-based technique from [1] was applied in [1] to Kummer's ${}_2F_1(-1)$-identity so as to obtain a new ${}_3F_2(-1)$-identity, and such results could, in view of the material from Section 2, perhaps be applied to obtain ${}_2F_1(-1)$-variants of Gosper's sum. We encourage the exploration of this area.

Recall that the ${}_3F_2(1)$-identity (2) was proved via a direct application of the technique from [1] using Gauss' ${}_2F_1$-identity. If instead a ${}_pF_q(1)$-variant were to be applied using the technique from [1], how could we mimic our proof of Theorem 2.1?

Campbell and Cantarini's technique from [1] was also applied to prove and generalize the Ramanujan-like formula

$$\sum_{n=0}^{\infty}\left(-\frac{1}{64}\right)^n\binom{2n}{n}^3\frac{(4n+1)^2}{(4n-1)(4n+3)} = -\frac{32(2+\sqrt{2})\Gamma^2\left(\frac{1}{4}\right)}{\Gamma^4\left(\frac{1}{8}\right)}$$

introduced by Cantarini in [2] in the context of the study of the Clebsch–Gordan integral. In view of our applications of the technique from [1], perhaps the proof in [1] of Cantarini's Ramanujan-like formula could be altered in some way by analogy with the proof of Theorem 2.1.

References

[1] J. M. Campbell and M. Cantarini, A series evaluation technique based on a modified Abel lemma, Turkish J. Math. **46** (2022), no. 4, 1520–1537.

[2] M. Cantarini, A note on Clebsch-Gordan integral, Fourier-Legendre expansions and closed form for hypergeometric series, Ramanujan J. **59** (2022), no. 2, 549–557.

[3] W. Chu, Telescopic approach to a formula of ${}_2F_1$-series by Gosper and Ebisu, Proc. Japan Acad. Ser. A Math. Sci. **93** (2017), no. 3, 13–15.

[4] A. Ebisu, On a strange evaluation of the hypergeometric series by Gosper, Ramanujan J. **32** (2013), no. 1, 101–108.

[5] I. M. Gessel and D. W. Stanton, Strange evaluations of hypergeometric series, SIAM J. Math. Anal. **13** (1982), no. 2, 295–308.

[6] M. Petkovšek, H. S. Wilf, and D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA, 1996.