Proportion of modular forms with transcendental zeros for general levels

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Abstract: Let Γ be a congruence subgroup such that $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$ for some positive integer N. For a positive integer k, let $M_{k,\mathbf{Z}}(\Gamma)$ be the set of modular forms of weight k on Γ with integral Fourier coefficients. Let $R_k(\Gamma)$ be the set of common zeros in the upper half plane $\mathbf{H}$ of all the modular forms of weight k on Γ. In this note, we prove that the density of modular forms in $M_{k,\mathbf{Z}}(\Gamma)$ with an algebraic zero $z \notin R_k(\Gamma)$ is zero.

Key words: Modular form; transcendental zero; density.

1. Introduction. The zeros of modular forms have been studied in several aspects. Let E_k be the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbf{Z})$. Rankin and Swinnerton-Dyer [7] proved that if z is a zero of E_k in the fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$, then z lies on the unit circle. In [4], Kanou proved that there is a transcendental zero of E_k in the upper half plane $\mathbf{H}$ if $k \geq 16$. Subsequently, Kohnen [6] proved that if z is an algebraic zero of E_k in the fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$, then z is i or $e^{2\pi i/3}$.

Gun, Murty, and Rath [2] proved that if z is an algebraic zero of a modular form on $\Gamma(N)$ with algebraic Fourier coefficients, then z is a CM point. Gun and Saha [3] expanded this result to weakly holomorphic modular forms. Furthermore, they also proved that if f is a weakly holomorphic modular form on $\mathrm{SL}_2(\mathbf{Z})$ with rational Fourier coefficients such that all zeros of f in the fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$ lie on the unit circle, then all such zeros of f except i and $e^{2\pi i/3}$ are transcendental. Recently, the first three authors [1] proved that if k and N are fixed positive integers such that the genus of $\Gamma_0(N)$ is zero, then the set of modular forms f of weight k on $\Gamma_0(N)$ having an algebraic zero except elliptic points of $\Gamma_0(N)$ has density zero. In this note, we extend this result to general $\Gamma_0(N)$.

Let Γ be a congruence subgroup such that $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$ for some positive integer N. For a positive integer k, let $M_k(\Gamma)$ be the space of modular forms of weight k on Γ. We say $z_0 \in \mathbf{H}$ *is a base point of* $M_k(\Gamma)$ if $f(z_0) = 0$ for every $f \in M_k(\Gamma)$.

Let $M_{k,\mathbf{Z}}(\Gamma)$ be the $\mathbf{Z}$-submodule of $M_k(\Gamma)$ consisting of modular forms with integral Fourier coefficients. Let $M^{tran}_{k,\mathbf{Z}}(\Gamma)$ be the set of modular forms f in $M_{k,\mathbf{Z}}(\Gamma)$ such that all zeros of f in $\mathbf{H}$ other than the base points of $M_k(\Gamma)$ are transcendental.

Definition 1.1. Let $e(k,\Gamma)$ be a maximal order of a modular form in $M_k(\Gamma)$ at $i\infty$. For $f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_{k,\mathbf{Z}}(\Gamma)$ with $q := e^{2\pi i z}$, we define

$$\omega(f) := \sum_{n=0}^{e(k,\Gamma)} |a_f(n)|.$$

The following theorem gives the ratio of the number of modular forms in $M^{tran}_{k,\mathbf{Z}}(\Gamma)$ with $\omega(f) \leq X$ to the number of modular forms in $M_{k,\mathbf{Z}}(\Gamma)$ with $\omega(f) \leq X$.

Theorem 1.2. *We have*

$$\frac{\#\{f \in M^{tran}_{k,\mathbf{Z}}(\Gamma) : \omega(f) \leq X\}}{\#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \leq X\}} = 1 - \frac{\alpha_{k,\Gamma}}{X} + O\left(\frac{1}{X^2}\right),$$

where $\alpha_{k,\Gamma}$ is a constant depending on k and Γ.

Remark 1.3.

(a) The constant $\alpha_{k,\Gamma}$ is given by the volumes of certain polytopes defined by the Fourier coefficients of modular forms in a basis of $M_{k,\mathbf{Z}}(\Gamma)$.

(b) If $\dim_{\mathbf{C}} M_k(\Gamma) \geq 2$, then there exists a modular form f in $M_k(\Gamma)$ such that f has a zero in $\mathbf{H}$ which is not a base point of $M_k(\Gamma)$ (for the details, see Section 3).

(c) The following is a list of some cases where

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$\dim_{\mathbf{C}} M_k(\Gamma) \geq 2$:

(i) $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and k is a positive even integer such that $k = 12$ or $k \geq 16$,

(ii) $\Gamma = \Gamma_0(2) = \Gamma_1(2)$ and k is a positive even integer such that $k \neq 2$,

(iii) $\Gamma = \Gamma_1(N)$ for some $N \geq 3$ and $k \geq 2$ except $(N, k) = (3, 2)$,

(iv) $\Gamma = \Gamma_0(N)$ for some $N \geq 3$ and k is a positive even integer such that $k \geq 4$.

The following corollary is immediately implied by Theorem 1.2.

Corollary 1.4. *We have*

$$\lim_{X\to\infty} \frac{\#\{f \in M_{k,\mathbf{Z}}^{tran}(\Gamma) : \omega(f) \leq X\}}{\#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \leq X\}} = 1.$$

If the genus of Γ is zero and j_Γ denotes a Hauptmodul of Γ, then there exists $g_{k,\Gamma} \in M_{k,\mathbf{Z}}(\Gamma)$ such that the first nonzero Fourier coefficient of $g_{k,\Gamma}$ is one and for each $f \in M_{k,\mathbf{Z}}(\Gamma)$ we have $f/g_{k,\Gamma} = P_f(j_\Gamma)$ with $P_f \in \mathbf{Z}[X]$. In [1], the proof of Theorem 1.2 for the case when the genus of Γ is zero was obtained by considering the number of $f \in M_{k,\mathbf{Z}}(\Gamma)$ such that P_f is reducible in $\mathbf{Z}[X]$. To extend the result of [1] to general Γ, we consider a $\mathbf{Z}$-submodule of $M_{k,\mathbf{Z}}(\Gamma)$ consisting of modular forms with a certain fixed zero and study its number elements.

2. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. First, let us fix some notation. Let k be a nonnegative integer. Let Γ be a congruence subgroup such that $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$ for some positive integer N, and $\mathcal{F}_\Gamma$ be a fundamental domain for Γ. For $f \in M_k(\Gamma)$, let $a_f(n)$ denote the nth Fourier coefficient of f.

In the following lemma, we prove that for each fixed k and Γ, there are only finitely many algebraic numbers z in $\mathcal{F}_\Gamma$ such that $f(z) = 0$ for some $f \in M_{k,\mathbf{Z}}(\Gamma)$.

Lemma 2.1. *For each k and Γ, the set*

$$\{z \in \overline{\mathbf{Q}} \cap \mathcal{F}_\Gamma : f(z) = 0 \text{ for some } f \in M_{k,\mathbf{Z}}(\Gamma)\}$$

is a finite set.

Proof. Let $z_1 \in \mathcal{F}_\Gamma$ be an algebraic zero of a modular form f in $M_{k,\mathbf{Z}}(\Gamma)$. There are only finitely many elliptic points of Γ on $\mathcal{F}_\Gamma$. Thus, we may assume that z_1 is not an elliptic point of Γ. Let

$$F := \prod_{\gamma \in \Gamma \backslash \mathrm{SL}_2(\mathbf{Z})} f|_k \gamma.$$

Then $F(z_1) = 0$ and $F \in M_{km}(\mathrm{SL}_2(\mathbf{Z}))$, where $m = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma]$. Note that by the q-expansion principle, there is a finite extension E of $\mathbf{Q}$ in $\mathbf{C}$ such that the Fourier coefficients of f at any cusp are in E (see, [5, Chapter 1.6]). This implies that $F \in E[[q]]$. Let $r_{km} := \dim_{\mathbf{C}} M_{km}(\mathrm{SL}_2(\mathbf{Z})) - 1$. Then, there exists a unique modular form $\phi \in M_{km,\mathbf{Z}}(\mathrm{SL}_2(\mathbf{Z}))$ such that

$$\phi(z) = q^{r_{km}} + \sum_{n=r_{km}+1}^{\infty} a_\phi(n) q^n \in \mathbf{Z}[[q]]$$

(for example, see [1, Section 2.1]). Thus, $\{\phi, \phi j, \ldots, \phi j^{r_{km}}\}$ is a basis of $M_{km}(\mathrm{SL}_2(\mathbf{Z}))$. This implies that there is a unique polynomial $P_F \in E[X]$ of degree less than or equal to r_{km} such that

$$F(z) = \phi(z) P_F(j(z)).$$

Let us note that $\phi = \Delta^{c_1} E_4^{c_2} E_6^{c_3}$ for some nonnegative integers c_1, c_2, and c_3, where Δ denotes the Ramanujan Delta function. Thus, ϕ has a zero only at a cusp of $\mathrm{SL}_2(\mathbf{Z})$ or an elliptic point of $\mathrm{SL}_2(\mathbf{Z})$. Thus, we have $P_F(j(z_1)) = 0$ and $[\mathbf{Q}(j(z_1)) : \mathbf{Q}] \leq r_{km}[E : \mathbf{Q}]$. Since z_1 is algebraic, the result of Schneider [8] implies that z_1 is a CM point. Therefore, there are only finitely many z_1 by [1, Lemma 2.5] (for the details, see Section 2 in [1]). □

Let us note that $M_{k,\mathbf{Z}}(\Gamma)$ is a free $\mathbf{Z}$-module since $M_{k,\mathbf{Z}}(\Gamma)$ is a $\mathbf{Z}$-submodule of $\mathbf{Z}[[q]]$. For a free $\mathbf{Z}$-module M, let $r(M)$ be the rank of M. We have the following lemma about the rank of $M_{k,\mathbf{Z}}(\Gamma)$ as a $\mathbf{Z}$-module. For convenience, let $r := \dim_{\mathbf{C}} M_k(\Gamma)$.

Lemma 2.2. *The rank of $M_{k,\mathbf{Z}}(\Gamma)$ as a $\mathbf{Z}$-module is equal to r.*

Proof. There is a basis $\{f_1, \ldots, f_r\}$ of $M_k(\Gamma)$ such that each f_i has integral Fourier coefficients. Since $\{f_1, \ldots, f_r\} \subset M_{k,\mathbf{Z}}(\Gamma)$, we have $r(M_{k,\mathbf{Z}}(\Gamma)) \geq r$.

By $\mathbf{Q}$-linear combinations of f_i, we obtain another basis $\{g_1, \ldots, g_r\}$ of $M_k(\Gamma)$ and integers $0 = t_1 < \cdots < t_r$ such that for each $i \in \{1, \ldots, r\}$,

$$g_i(z) = q^{t_i} + \sum_{n > t_i} a_{g_i}(n) q^n \in \mathbf{Q}[[q]],$$

and such that $a_{g_i}(n) = 0$ if $n = t_j$ for some $j > i$. Assume that $f \in M_{k,\mathbf{Z}}(\Gamma)$. Then, there is a unique $(c_1, \ldots, c_r) \in \mathbf{C}^r$ such that $f = \sum_{i=1}^r c_i g_i$. We have

$$a_f(t_i) = \sum_{i=1}^r c_i a_{g_i}(t_i) = c_i,$$

and so $(c_1, \ldots, c_r) \in \mathbf{Z}^r$. Thus, $M_{k,\mathbf{Z}}(\Gamma)$ can be considered as a $\mathbf{Z}$-submodule of $\bigoplus_{i=1}^r \mathbf{Z} g_i$. Thus, $r(M_{k,\mathbf{Z}}(\Gamma)) \leq r$. Therefore, we have $r(M_{k,\mathbf{Z}}(\Gamma)) = r$. □

To obtain the number of modular forms g in $M_{k,\mathbf{Z}}^{tran}(\Gamma)$ with $\omega(g) \leq X$, we consider the number of modular forms h in $M_{k,\mathbf{Z}}(\Gamma)$ such that there is an algebraic zero of h that is not a base point of $M_k(\Gamma)$.

Lemma 2.3. *Assume that z_1 is an algebraic number that is not a base point of $M_k(\Gamma)$. Let*

$$N_{k,\Gamma}^{z_1} := \{f \in M_{k,\mathbf{Z}}(\Gamma) : f(z_1) = 0\}.$$

Then, $r(N_{k,\Gamma}^{z_1}) \leq r - 1$.

Proof. Let $\{f_1, \dots, f_r\}$ be a basis of a $\mathbf{Z}$-module $M_{k,\mathbf{Z}}(\Gamma)$. Assume that g is a modular form in $N_{k,\Gamma}^{z_1}$. Then, $g = \sum_{i=1}^{r} c_i f_i$ for some $c_i \in \mathbf{Z}$. Since $\{f_1, \dots, f_r\}$ is also a basis of $M_k(\Gamma)$ and z_1 is not a base point of $M_k(\Gamma)$, we may assume that $f_1(z_1) \neq 0$. Then,

$$c_1 = -\frac{1}{f_1(z_1)} \sum_{i=2}^{r} c_i f_i(z_1),$$

and so $r(N_{k,\Gamma}^{z_1}) \leq r - 1$. □

To prove Theorem 1.2, we need the asymptotic behavior of the number of modular forms in $M_{k,\mathbf{Z}}(\Gamma)$ with $\omega(f) \leq X$ as $X \to \infty$. Recall that $e(k,\Gamma)$ is a maximal order of a modular form in $M_k(\Gamma)$ at $i\infty$. Let t be an integer less than or equal to $e(k,\Gamma)+1$. Let A be a $(e(k,\Gamma)+1) \times t$ matrix. For a subring R of $\mathbf{C}$, let

$$W_{A,X}(R) := \left\{ (x_1, \dots, x_t) \in R^t : \sum_{i=1}^{e(k,\Gamma)+1} \left| \sum_{j=1}^{t} x_j A_{ij} \right| \leq X \right\}.$$

It was proved in [1, Lemma 3.2] that if $X \geq \frac{1}{2}\sum_{i=1}^{e(k,\Gamma)+1} \sum_{j=1}^{t} |A_{ij}|$, then

$$\beta\left(X - \frac{1}{2} \sum_{i=1}^{e(k,\Gamma)+1} \sum_{j=1}^{t} |A_{ij}| \right)^t \leq \#W_{A,X}(\mathbf{Z}) \tag{2.1}$$

and

$$\#W_{A,X}(\mathbf{Z}) \leq \beta\left(X + \frac{1}{2} \sum_{i=1}^{e(k,\Gamma)+1} \sum_{j=1}^{t} |A_{ij}| \right)^t, \tag{2.2}$$

where β denotes the volume of $W_{A,1}(\mathbf{R})$. For a $\mathbf{Z}$-submodule M of $M_{k,\mathbf{Z}}(\Gamma)$, let $M_X := \{f \in M : \omega(f) \leq X\}$. We have the following lemma.

Lemma 2.4. *Suppose that M is a $\mathbf{Z}$-submodule of $M_{k,\mathbf{Z}}(\Gamma)$ with a basis $\{u_1, \dots, u_s\}$. If $X \geq \frac{1}{2}\sum_{i=1}^{e(k,\Gamma)+1} \sum_{j=1}^{s} |a_{u_j}(i-1)|$, then*

$$\beta\left(X - \frac{1}{2} \sum_{i=1}^{e(k,\Gamma)+1} \sum_{j=1}^{s} |a_{u_j}(i-1)| \right)^s \leq \#M_X, \tag{2.3}$$

and

$$\#M_X \leq \beta\left(X + \frac{1}{2} \sum_{i=1}^{e(k,\Gamma)+1} \sum_{j=1}^{s} |a_{u_j}(i-1)| \right)^s. \tag{2.4}$$

Proof. Note that for $f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M$, there is a unique $(c_1, \dots, c_s) \in \mathbf{Z}^s$ such that $f = \sum_{j=1}^{s} c_j u_j$. Let us take A as a $(e(k,\Gamma)+1) \times s$ matrix with $A_{ij} = a_{u_j}(i-1)$. Then, for each $1 \leq i \leq e(k,\Gamma)+1$,

$$a_f(i-1) = \sum_{j=1}^{s} c_j a_{u_j}(i-1) = \sum_{j=1}^{s} c_j A_{ij}.$$

Thus, $\omega(f) = \sum_{i=1}^{e(k,\Gamma)+1} |\sum_{j=1}^{s} c_j A_{ij}|$. This implies that a function $\Psi : M_X \to W_{A,X}(\mathbf{Z})$ defined by $\Psi(\sum_{j=1}^{s} c_j u_j) := (c_1, \dots, c_s)$ is bijective. Thus, by inequalities (2.1) and (2.2), we complete the proof. □

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Let us note that $M_{k,\mathbf{Z}}(\Gamma)$ is a $\mathbf{Z}$-module of rank r. By Lemma 2.4, there exists a positive real constant α such that if $X \geq \alpha$, then

$$\beta(X - \alpha)^r \leq \#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \leq X\},$$

and

$$\#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \leq X\} \leq \beta(X + \alpha)^r.$$

By Lemma 2.1, the subset of $\{z \in \overline{\mathbf{Q}} \cap \mathcal{F}_\Gamma : f(z) = 0 \text{ for some } f \in M_{k,\mathbf{Z}}(\Gamma)\}$ whose elements are not base points of $M_k(\Gamma)$ is a finite set. We denote it by $\{z_1, \dots, z_t\}$. For $1 \leq i_1 < \cdots < i_l \leq t$, let

$$D^{i_1 \cdots i_l} := \{f \in M_{k,\mathbf{Z}}(\Gamma) : f(z_{i_1}) = \cdots = f(z_{i_l}) = 0\}.$$

Thus, $D^{i_1 \cdots i_l}$ is a $\mathbf{Z}$-submodule of $M_{k,\mathbf{Z}}(\Gamma)$ with $s_{i_1 \cdots i_l} := r(D^{i_1 \cdots i_l}) \leq r - 1$ by Lemma 2.3. By the inclusion–exclusion principle, we have

$$\begin{aligned} &\#\{f \in M_{k,\mathbf{Z}}^{tran}(\Gamma) : \omega(f) \leq X\} \\ &\quad = \#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \leq X\} \\ &\qquad + \sum_{l=1}^{t} \sum_{1 \leq i_1 < \cdots < i_l \leq t} (-1)^l \#D_X^{i_1 \cdots i_l}. \end{aligned}$$

By Lemma 2.4, there exist positive real constants $A_{i_1 \cdots i_l}$, $B_{i_1 \cdots i_l}$ and a nonnegative integer $s_{i_1 \cdots i_l} \leq r - 1$ such that if $X \geq B_{i_1 \cdots i_l}$, then

$$A_{i_1 \cdots i_l}(X - B_{i_1 \cdots i_l})^{s_{i_1 \cdots i_l}} \leq \#D_X^{i_1 \cdots i_l}, \tag{2.5}$$

and

$$\#D_X^{i_1 \cdots i_l} \leq A_{i_1 \cdots i_l}(X + B_{i_1 \cdots i_l})^{s_{i_1 \cdots i_l}}. \tag{2.6}$$

Since every $s_{i_1\cdots i_l}$ is less than or equal to $r-1$, we have by (2.3) and (2.4)

$$1 - \frac{\#\{f \in M_{k,\mathbf{Z}}^{tran}(\Gamma) : \omega(f) \le X\}}{\#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \le X\}} = O\left(\frac{1}{X}\right)$$

as $X \to \infty$. By Lemma 2.4, (2.5) and (2.6), we have

$$\lim_{X\to\infty} X\left(1 - \frac{\#\{f \in M_{k,\mathbf{Z}}^{tran}(\Gamma) : \omega(f) \le X\}}{\#\{f \in M_{k,\mathbf{Z}}(\Gamma) : \omega(f) \le X\}}\right) = \sum_{l=1}^{t} \sum_{\substack{1\le i_1<\cdots<i_l\le t \\ s_{i_1\cdots i_l}=r-1}} (-1)^{l-1} \frac{A_{i_1\cdots i_l}}{\beta},$$

which is the desired constant $\alpha_{k,\Gamma}$. □

3. Remarks on a base point of $M_k(\Gamma)$. If $\dim_{\mathbf{C}} M_k(\Gamma) \ge 2$, then there exists a modular form $f \in M_{k,\mathbf{Z}}(\Gamma)$ such that f has a zero in $\mathbf{H}$ which is not a base point of $M_k(\Gamma)$. In this section, we give the details on this statement.

Assume that $\dim_{\mathbf{C}} M_k(\Gamma) \ge 2$. To get a contradiction, assume that all of f in $M_{k,\mathbf{Z}}(\Gamma)$ have the same zeros on $\mathbf{H}$. Let f_1 and f_2 be linearly independent elements in $M_{k,\mathbf{Z}}(\Gamma)$. For $f \in M_k(\Gamma)$ and $z \in \mathbf{H}^* = \mathbf{H} \cup \mathbf{Q} \cup \{i\infty\}$, let $\mathrm{ord}_z(f)$ be the order of f at z. If $\mathrm{ord}_z(f_1) = \mathrm{ord}_z(f_2)$ for every $z \in \mathbf{H}^*$, then f_1/f_2 is a nowhere vanishing holomorphic function on the compact Riemann surface $X(\Gamma)$ and so f_1/f_2 is a constant function on $X(\Gamma)$. This contradicts to that f_1 and f_2 are linearly independent. Thus, there exists $z_1 \in \mathbf{H}^*$ such that $\mathrm{ord}_{z_1}(f_1) \ne \mathrm{ord}_{z_1}(f_2)$.

By reordering f_1 and f_2, we may assume that $\mathrm{ord}_{z_1}(f_1) > \mathrm{ord}_{z_1}(f_2)$. Let $\{z_1, \ldots, z_m\}$ be the set of inequivalent zeros of f_1 including cusps of Γ. For each z_i, there is at most one $a_i \in \mathbf{C}$ such that

$$\mathrm{ord}_{z_i}(f_1 + a_i f_2) > \min\{\mathrm{ord}_{z_i}(f_1), \mathrm{ord}_{z_i}(f_2)\}.$$

For $a \in \mathbf{Z} \setminus \{0, a_1, \ldots, a_m\}$, we have

$$\mathrm{ord}_{z_i}(f_1 + af_2) = \min\{\mathrm{ord}_{z_i}(f_1), \mathrm{ord}_{z_i}(f_2)\} \le \mathrm{ord}_{z_i}(f_1).$$

Since $\mathrm{ord}_{z_1}(f_1 + af_2) = \mathrm{ord}_{z_1}(f_2) < \mathrm{ord}_{z_1}(f_1)$, there exists $w \in \mathcal{F}_\Gamma$ such that $f_1(w) + af_2(w) = 0$ but $f_1(w) \ne 0$, by the valence formula. Thus, we see that if $\dim_{\mathbf{C}} M_k(\Gamma) \ge 2$, then all of f in $M_{k,\mathbf{Z}}(\Gamma)$ do not have the same zeros on $\mathbf{H}$.

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