Rational period functions for $\Gamma_0^+(2)$ with poles only in $\mathbf{Q}\cup\{\infty\}$

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Abstract: We extend results of Knopp in [9] to the higher level case. In precise, we characterize a rational period function $q(z)$ for $\Gamma_0^+(2)$ of which poles lie only in $\mathbf{Q}\cup\{\infty\}$. We prove that the Mellin transform $\Phi_F(s)$ of an entire modular integral F of weight $2k$ for such a rational period function $q(z)$ has an analytic continuation to the entire s-plane, except for possible simple poles at some rational integers, satisfies the functional equation $\Phi_F(2k-s) = (-1)^k 2^{s-k}\Phi_F(s)$, and is bounded on each "truncated strip" of the from $\sigma_1 \le \mathrm{Re}(s) \le \sigma_2$ and $|\mathrm{Im}(s)| \ge t_0 > 0$. We also show that the converse is true. The case for $\Gamma_0^+(3)$ is addressed similarly.

Key words: Period functions; modular integrals; Mellin transforms.

1. Introduction and statement of results. For $p \in \{1,2,3\}$, let $\Gamma_0^+(p)$ be the group generated by the congruence group $\Gamma_0(p)$ and the Fricke involution $W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. Knopp [8,9] introduced the idea of a rational period function of an automorphic integral of weight $2k \in 2\mathbf{Z}$ on any Fuchsian group and studied the rational period functions of a modular integral of weight $2k$ for the modular group $\Gamma_0^+(1) = SL_2(\mathbf{Z})$. Since then, explicit charaterizations of the rational period functions for $SL_2(\mathbf{Z})$ have been given in [4,5]. Recently, Choi and Kim [2,3] generalized some results given by Knopp to the rational period functions for $\Gamma_0^+(p)$ when $p \in \{2,3\}$. These rational functions $q(z)$ for $\Gamma_0^+(p)$ when $p \in \{1,2,3\}$ occur in functional equations of the form

$$F(z+1) = F(z), \quad \text{and} \tag{1}$$
$$(\sqrt{p}z)^{-2k} F\left(-\frac{1}{pz}\right) = F(z) + q(z),$$

where $k \in \mathbf{Z}$ and F is a meromorphic function in the upper half plane $\mathbf{H}$ and has a Fourier expansion of the form

$$F(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z}, \quad y = \mathrm{Im}\, z > y_0 \ge 0 \tag{2}$$

with some $n_0 \in \mathbf{Z}$. If (1) and (2) hold, then we call F a modular integral of weight $2k$ for $q(z)$. In fact, F can be taken to be holomorphic in $\mathbf{H}$. Furthermore, if $k > 0$, then we can choose a modular integral F of weight $2k$ satisfying the following: a modular integral F of weight $2k$ has a Fourier expansion of the form

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \quad y = \mathrm{Im}\, z > 0$$

and

$$|F(z)| \le K(|z|^{\alpha} + y^{-\beta}), \quad z \in \mathbf{H} \tag{3}$$

for some positive real numbers K, α, β ([7, pp. 622–623]). Using the integral presentation of a_n and (3), we can get

$$a_n = O(n^{\nu}), \quad n \to \infty \tag{4}$$

for some $\nu > 0$. A modular integral F of weight $2k$ which is holomorphic in $\mathbf{H}$ and has the Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \quad y = \mathrm{Im}\, z > 0$$

is called an entire modular integral if F satisfies (4) or equivalently (3).

In [9, Theorems 1 and 2], Knopp showed that the finite poles of any rational period function $q(z)$ for $SL_2(\mathbf{Z})$ must lie in $\mathbf{Q}(\sqrt{n})$, $n \in \mathbf{Z}^+$, and determined the explicit form of a rational period function $q(z)$ for $SL_2(\mathbf{Z})$ with the only possible finite poles in $\mathbf{Q}$. Choi and Kim [3, Theorems 1.2] showed that the finite poles of any rational period function $q(z)$ for $\Gamma_0^+(p)$ must lie in $\mathbf{Q}(\sqrt{n}), n \in \mathbf{Z}^+$.

For an entire modular integral F of weight $2k$ for $SL_2(\mathbf{Z})$ with its associated rational period function $q(z)$, Knopp [9, Theorem 3] showed that if

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the poles of $q(z)$ lies in $\mathbf{Q}\cup\{\infty\}$, then the Mellin transform $\Phi_F(s)$ of F has an analytic continuation to the entire s-plane, except for possible simple poles at some rational integers, satisfies the functional equation $\Phi_F(2k-s)=(-1)^k\Phi_F(s)$, and is bounded on each "truncated strip" of the from $\sigma_1\le \mathrm{Re}(s)\le\sigma_2$ and $|\mathrm{Im}(s)|\ge t_0>0$. Knopp [9, Theorem 4] also showed that the converse is true.

In this paper, we investigate rational period functions $q(z)$ for $\Gamma_0^+(2)$ with the finite poles only in $\mathbf{Q}$ to extend results of Knopp [9] on rational period functions for $\Gamma_0^+(1)=SL_2(\mathbf{Z})$ to the higher level case. More precisely, we determine the explicit formula of a rational period function $q(z)$ for $\Gamma_0^+(2)$ of which poles lie only in $\mathbf{Q}\cup\{\infty\}$ (Theorem 1.2). We also show that if the poles of $q(z)$ lies in $\mathbf{Q}\cup\{\infty\}$, then the Mellin transform $\Phi_F(s)$ of an entire modular integral $F(z)$ of weight $2k$ for such a rational period function $q(z)$ has an analytic continuation to the entire s-plane, except for possible simple poles at some rational integers, satisfies the functional equation $\Phi_F(2k-s)=(-1)^k2^{s-k}\Phi_F(s)$, and is bounded on each "truncated strip" of the from $\sigma_1\le\mathrm{Re}(s)\le\sigma_2$ and $|\mathrm{Im}(s)|\ge t_0>0$ (Theorem 1.4) and the converse is true (Theorem 1.5).

Since $\Gamma_0^+(2)$ is generated by $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and W_2, we have the following

Proposition 1.1. *Suppose that $q(z)$ is a rational function for $\Gamma_0^+(2)$ as in (1) for some modular integral F. Let $U:=TW_2=\begin{pmatrix}\sqrt{2}&-1/\sqrt{2}\\\sqrt{2}&0\end{pmatrix}$. Then*

$$q|_{2k}W_2+q=0 \tag{5}$$

and

$$q|_{2k}U^3+q|_{2k}U^2+q|_{2k}U+q=0, \tag{6}$$

where $|_{2k}$ is the usual slash operator.

Proof. See [3, Theorem 1.3]. □

It follows from [3, Theorem 1.3] that if a rational function $q(z)$ satisfies (5) and (6), then there exists a modular integral F, which is holomorphic on $\mathbf{H}$, of weight $2k$ for $q(z)$. In particular, when $k>0$, we can construct F that is an entire modular integral of weight $2k$ for $q(z)$. We now present the explicit form of $q(z)$ for $\Gamma_0^+(2)$ with poles only in $\mathbf{Q}\cup\{\infty\}$.

Theorem 1.2. *Let $q(z)$ be any rational period function of weight $2k$ for $\Gamma_0^+(2)$. If the ploes of $q(z)$ lie in $\mathbf{Q}\cup\{\infty\}$, then*

$$q(z)=\begin{cases} b_0(1-(\sqrt{2}z)^{-2k}), & \text{if } k>1,\\ b_0(1-(\sqrt{2}z)^{-2})+b_1z^{-1}, & \text{if } k=1,\\ b_0(2^{k-1}z^{-1}+z^{-2k+1})+p_k(z), & \text{if } k\le 0,\end{cases} \tag{7}$$

where b_0,b_1 are complex numbers and p_k is a polynomial in z of degree at most $-2k$.

We now consider the Mellin transforms of entire modular integrals. Suppose that F is an entire modular integral of weight $2k$ associated to some rational period function for $\Gamma_0^+(2)$ and it has a Fourier expansion of the form

$$F(z)=\sum_{n=0}^{\infty}a_ne^{2\pi inz},\quad y=\mathrm{Im}\,z>0. \tag{8}$$

We then consider the Mellin transform:

$$\Phi_F(s)=\int_0^{\infty}(F(iy)-a_0)y^{s-1}dy. \tag{9}$$

Proposition 1.3. *With the assumptions and notations as above, we have the following.*

For sufficiently large $\sigma=\mathrm{Re}(s)$, the Dirichlet series

$$\Phi(s)=(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}a_nn^{-s} \tag{10}$$

is related to the Mellin transform Φ_F as $\Phi_F=\Phi$, where $\Gamma(s)$ is the gamma-function.

We now present the results on the Mellin transform of the corresponding entire modular integral for a rational period function $q(z)$ for $\Gamma_0^+(2)$.

Theorem 1.4. *Let F be an entire modular integral of weight $2k$ such that its associated rational period function for $\Gamma_0^+(2)$ has poles only in $\mathbf{Q}\cup\{\infty\}$. Then*

(1) *$\Phi_F(s)$ has an analytic continuation to the entire s-plane, except for possible simple poles at*

$$s=0 \quad\text{and}\quad s=2k,\quad \text{when } k>1;$$
$$s=0,1 \quad\text{and}\quad 2,\quad\text{when } k=1;$$
$$s=2k-1,2k,\dots,0,1,\quad\text{when } k\le 0,$$

(2) *for every case,*

$$\Phi_F(2k-s)=(-1)^k2^{s-k}\Phi_F(s)$$

and $\Phi_F(s)$ is bounded in each "truncated strip" of the form $\sigma_1\le Re(s)\le\sigma_2$ and $|Im(s)|\ge t_0>0$.

A converse of Theorem 1.4 is

Theorem 1.5. *Let $\{a_n\}$ be a sequence of complex numbers satisfying*

$$a_n = O(n^{\nu}), \quad n\to\infty$$

for some $\nu > 0$. Suppose also that $\Phi(s) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} a_n n^{-s}$ can be extended to a function meromorphic in the entire s-plane, holomorphic except possibly for simple poles at the rational integers and bounded in every truncated strip of the form $\sigma_1 \le Re(s) \le \sigma_2$, $|Im(s)| \ge t_0 > 0$. If $\Phi(s)$ satisfies the functional equation $\Phi(2k-s) = (-1)^k 2^{s-k}\Phi(s)$ for some k, then

$$(11) \qquad F(z) = F_{\Phi}(z) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is an entire modular integral of weight $2k$, with rational period function for $\Gamma_0^+(2)$ with poles only in $\mathbf{Q}\cup\{\infty\}$. Here a_0 is an arbitrary complex number.

Remark 1.6. By the same arguments in Sections 2 and 3, we can obtain similar results for entire modular integral for $\Gamma_0^+(3)$. By replacing $q(z)$ and $\Phi_F(2k-s) = (-1)^k 2^{s-k}\Phi_F(s)$ by

$$q(z) = \begin{cases} b_0(1-(\sqrt{3}z)^{-2k}), & \text{if } k>1,\\ b_0(1-(\sqrt{3}z)^{-2}) + b_1 z^{-1}, & \text{if } k=1,\\ b_0(3^{k-1}z^{-1} + z^{-2k+1}) + p_k(z), & \text{if } k\le 0,\end{cases}$$

and $\Phi_F(2k-s) = (-1)^k 3^{s-k}\Phi_F(s)$, corresponding statements can be obtained.

This paper is organized as follows: In Section 2, we prove Theorem 1.2. Section 3 is devoted to proofs of Proposition 1.3, Theorems 1.4 and 1.5. Our proof of theorems is build on the main ideas of Knopp in [9].

2. Proof of Theorem 1.2. (I). The case $k>0$. It follows from [3, Theorem 1.2(*b*)] that if z_0 is a finite rational pole of $q(z)$, then $z_0 = 0$. Thus we may write

$$(12) \qquad q(z) = a_l z^{-l} + \cdots + a_1 z^{-1} + b_0 + b_1 z + \cdots + b_m z^m \ (a_l \ne 0, b_m \ne 0),$$

with $l \ge 1, m\ge 0$. Applying (5) to $q(z)$, we have

$$(13) \qquad -q(z) = (\sqrt{2}z)^{-2k} q\left(\frac{-1}{2z}\right).$$

Comparing the lowest term in (13), we have $l = m+2k$. From (5) and (6), we have

$$(14)\ q(z) = (2z+1)^{-2k} q\left(\frac{z}{2z+1}\right) + (\sqrt{2}z+\sqrt{2})^{-2k} q\left(\frac{\sqrt{2}z + \frac{1}{\sqrt{2}}}{\sqrt{2}z+\sqrt{2}}\right) + q(z+1).$$

Comparing the principal part at ∞ in (12) and (14), we get

$$b_0 + b_1 z + \cdots + b_m z^m = b_0 + b_1(z+1) + \cdots + b_m(z+1)^m,$$

which gives $m=0$, hence

$$(15) \qquad l = 2k \text{ and } q(z) = a_l z^{-l} + \cdots + a_1 z^{-1} + b_0.$$

By applying (5) to (15) and comparing the coefficients, we have

$$(16) \quad b_0 = -\sqrt{2}^{l} a_l, \quad a_{l-j}(-1)^{l-j}\sqrt{2}^{l-2j} = -a_j \quad \text{for } 1\le j\le l-1 = 2k-1.$$

In particular, $a_k = (-1)^{k+1}a_k$, so $a_k = 0$ if k is even.

Applying (6) to (15) leads to

$$(17) \quad a_{2k}(-\sqrt{2})^{2k} + \cdots + a_1(-\sqrt{2})(\sqrt{2}z-\sqrt{2})^{1-2k} + b_0(\sqrt{2}z-\sqrt{2})^{-2k} + a_{2k}(z-1)^{-2k} + \cdots + a_1(z-1)^{-1}(2z-1)^{1-2k} + b_0(2z-1)^{-2k} + a_{2k}\left(\sqrt{2}z - \frac{1}{\sqrt{2}}\right)^{-2k} + \cdots + a_1\left(\sqrt{2}z-\frac{1}{\sqrt{2}}\right)^{-1}(\sqrt{2}z)^{1-2k} + b_0(\sqrt{2}z)^{-2k} + a_{2k}z^{-2k} + \cdots + a_1 z^{-1} + b_0 = 0.$$

If $k=1$, then $q(z) = a_2 z^{-2} + a_1 z^{-1} + b_0$. It follows from (16) that $b_0 = -\sqrt{2}^2 a_2$. Hence

$$q(z) = b_0(1-(\sqrt{2}z)^{-2}) + a_1 z^{-1}.$$

Suppose that $k\ge 2$. Note that from the partial fraction expansion, we have

$$\frac{1}{z^N(z-\frac{1}{2})^M} = \frac{A_1}{z} + \cdots + \frac{A_N}{z^N} + \frac{B_1}{z-\frac{1}{2}} + \cdots + \frac{B_M}{(z-\frac{1}{2})^M},$$

where $A_{N-j} = 2^{M+j}(-1)^M\binom{M+j-1}{M-1}$ $(0\le j\le N-1)$ and $B_{M-j} = 2^{N+j}(-1)^j\binom{N+j-1}{N-1}$ $(0\le j\le M-1)$. By applying the partial fraction expansion to $a_2(\sqrt{2}z - \frac{1}{\sqrt{2}})^{-2}(\sqrt{2}z)^{2-2k}$ and $a_1(\sqrt{2}z-\frac{1}{\sqrt{2}})^{-1}(\sqrt{2}z)^{1-2k}$, and the term $a_{2k-2}z^{-2k+2}$ on the left hand side of (17), the coefficient of z^{-2k+2} is $2^{-k+2}a_2 - 2^{2-k}a_1 + a_{2k-2}$, which becomes 0. So by (16), we have $a_1 = 0$.

If $k=2$, then $a_2 = 0, a_3 = 0$ and $b_0 = -\sqrt{2}^4 a_4$ by (16). Therefore

$$q(z) = b_0(1-(\sqrt{2}z)^{-4}).$$

Suppose that $k\ge 3$. We now assume $a_1 = a_2 = \cdots = a_{j-1} = 0$ for $2\le j\le k-1$. By applying, again, the partial fraction expansion on the left hand side of (17), the coefficient of $z^{-2k+j+1}$ is $\sqrt{2}^{-2k+2j+2}(-1)^{j+1}a_{j+1} + j2^{-k+j+1}(-1)^j a_j + a_{2k-j-1}$, which becomes 0. So by (16), $a_{2k-j-1} + a_{j+1}(-1)^{j+1}\sqrt{2}^{-2k+2(j+1)} = 0$, which gives $a_j = 0$. Consequently, we have $a_1 = a_2 = \cdots = a_{k-1} = 0$. Note that $a_k = 0$ if k is even.

Therefore, for $k \geq 3$, $q(z)$ has the form

$$q(z) = \begin{cases} b_0(1-(\sqrt{2}z)^{-2k}), & \text{if } k \text{ is even,} \\ b_0(1-(\sqrt{2}z)^{-2k}) + a_k z^{-k}, & \text{if } k \text{ is odd.} \end{cases}$$

Since $b_0(1-(\sqrt{2}z)^{-2k})$ satisfies (5) and (6), $q(z) = b_0(1-(\sqrt{2}z)^{-2k}) + a_k z^{-k}$ satisfies (5) and (6) if and only if $a_k z^{-k}$ satisfies (5) and (6). Note that z^{-k} satisfies (5) only when k is odd. We show that for odd k, z^{-k} satisfies (5) and (6) if and only if $k = 1$. For $q(z) = z^{-k}$, the functional equation (6) says

$$\begin{aligned} &-(z-1)^{-k} + (2z-1)^{-k}(z-1)^{-k} \\ &\quad + z^{-k}(2z-1)^{-k} + z^{-k} = 0 \end{aligned}$$

and this is 0 only when $k = 1$.

(II). The case $k \leq 0$. By applying Bol's identity [1] to (5) and (6), we have

$$\begin{aligned} 0 &= (D^{-2k+1}q)|_{2-2k}W_2(z) + D^{-2k+1}q(z), \\ 0 &= (D^{-2k+1}q)|_{2-2k}U^3(z) + (D^{-2k+1}q)|_{2-2k}U^2(z) \\ &\quad + (D^{-2k+1}q)|_{2-2k}U(z) + D^{-2k+1}q(z), \end{aligned}$$

which imply that $q^{(-2k+1)}(z)$ is a rational period function of weight $2-2k > 0$. Here, $Df(z) = \frac{1}{2\pi i}\frac{df}{dz}$. By part (I) of the proof, $q^{(-2k+1)}(z) = b_0(1-(\sqrt{2}z)^{2k-2})$, since the term $b_1 z^{-1}$ does not occur as the derivative of a rational function. Integrating $-2k+1$ times, we get

$$q(z) = b_0'(2^{k-1}z^{-1} + z^{-2k+1}) + p_k(z),$$

where b_0' is a complex number and $p_k(z)$ is a polynomial of degree $\leq -2k$.

3. Proofs of Proposition 1.3, Theorem 1.4 and Theorem 1.5.

Proof of Proposition 1.3. Suppose that F is an entire modular integral of weight $2k$ having the Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \quad y = \operatorname{Im} z > 0.$$

Since $a_n = O(n^\nu)$, for sufficiently large $\sigma = \operatorname{Re}(s)$, $\sum_{n=1}^{\infty} a_n e^{-2\pi n y} y^{s-1}$ converges uniformly on $y > 0$ and $\sum_{n=1}^{\infty}\int_0^b a_n e^{-2\pi n y} y^{s-1}dy$ converges uniformly on $b > 0$. Hence, we can integrate term by term to have

$$\begin{aligned} \Phi_F(s) &= \int_0^\infty \sum_{n=1}^{\infty} a_n e^{-2\pi n y} y^{s-1} dy \\ &= \sum_{n=1}^{\infty} a_n \int_0^\infty e^{-2\pi n y} y^{s-1} dy \\ &= (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} a_n n^{-s} = \Phi(s). \end{aligned}$$

□

Proof of Theorem 1.4. For large $\sigma = \operatorname{Re}(s)$, we may then write

$$\begin{aligned} \Phi(s) &= \int_{\frac{1}{\sqrt{2}}}^{\infty} (F(yi) - a_0) y^{s-1} dy \\ &\quad + \int_0^{\frac{1}{\sqrt{2}}} (F(iy) - a_0) y^{s-1} dy. \end{aligned}$$

Since $(\sqrt{2}z)^{-2k}F(-\frac{1}{2z}) = F(z) + q(z)$, we observe that

$$\begin{aligned} &\int_0^{\frac{1}{\sqrt{2}}} (F(yi) - a_0) y^{s-1} dy \\ &= 2^{-s}\int_{\frac{1}{\sqrt{2}}}^{\infty} \left(F\left(-\frac{1}{2\alpha i}\right) - a_0\right)\alpha^{-s-1} d\alpha \\ &= 2^{-s}\int_{\frac{1}{\sqrt{2}}}^{\infty} ((\sqrt{2}yi)^{2k}(F(yi) + q(yi)) - a_0) y^{-s-1} dy \\ &= (-1)^k 2^{k-s}\int_{\frac{1}{\sqrt{2}}}^{\infty} (F(yi) - a_0) y^{2k-s-1} dy \\ &\quad + \frac{(-1)^k 2^{-\frac{s}{2}} a_0}{s-2k} - \frac{2^{-\frac{s}{2}} a_0}{s} \\ &\quad + (-1)^k 2^{k-s}\int_{\frac{1}{\sqrt{2}}}^{\infty} q(yi) y^{2k-s-1} dy. \end{aligned}$$

Hence we get

$$\begin{aligned} \Phi(s) &= D_1(s) + (-1)^k 2^{k-s} D_2(s) \\ &\quad + 2^{-\frac{s}{2}} a_0 \left(\frac{(-1)^k}{s-2k} - \frac{1}{s}\right) + (-1)^k 2^{k-s} E(s), \end{aligned}$$

where

$$\begin{aligned} D_1(s) &= \int_{\frac{1}{\sqrt{2}}}^{\infty} (F(yi) - a_0) y^{s-1} dy, \\ D_2(s) &= \int_{\frac{1}{\sqrt{2}}}^{\infty} (F(yi) - a_0) y^{2k-s-1} dy, \\ E(s) &= \int_{\frac{1}{\sqrt{2}}}^{\infty} q(yi) y^{2k-s-1} dy. \end{aligned}$$

Since $F(z) - a_0$ vanishes exponentially as $z \to i\infty$, $D_1(s)$ and $D_2(s)$ are entire functions of s. We note that $D_1(2k-s) = D_2(s)$ and $D_2(2k-s) = D_1(s)$. We now investigate the function $E(s)$. Note that $q(z)$ has the form $q(z) = \sum_{n=-L}^{N} \alpha_n z^n$, and it follows that

$$E(s) = \int_{\frac{1}{\sqrt{2}}}^{\infty} \sum_{n=-L}^{N} \alpha_n (iy)^n y^{2k-s-1} dy$$
$$= \sum_{n=-L}^{N} \alpha_n i^n \frac{2^{\frac{s-2k-n}{2}}}{s-2k-n}$$
so long as $\mathrm{Re}(s) > 2k + N$.

We now consider $E(s)$ for the three cases $k > 1$, $k = 1$, and $k \le 0$.
(I) Let $k > 1$. Then $q(z) = b_0(1 - (\sqrt{2}z)^{-2k})$, and hence

$$E(s) = b_0 2^{-k+\frac{s}{2}} \left(\frac{1}{s-2k} - \frac{(-1)^k}{s} \right).$$

Thus,

$$\Phi(s) = D_1(s) + (-1)^k 2^{k-s} D_2(s) + 2^{-\frac{s}{2}}(a_0 + b_0)\left(\frac{(-1)^k}{s-2k} - \frac{1}{s} \right)$$

and $\Phi(s)$ has been extended analytically to the entire s-plane, with possible simple poles at $s = 0, 2k$. Moreover, $\Phi(2k-s) = (-1)^k 2^{s-k} \Phi(s)$.
(II) Suppose $k = 1$. Then $q(z) = b_0(1 - (\sqrt{2}z)^{-2}) + b_1 z^{-1}$, and hence

$$E(s) = b_0 2^{-1+\frac{s}{2}} \left(\frac{1}{s-2} + \frac{1}{s} \right) - \frac{b_1 i 2^{\frac{s-1}{2}}}{s-1}.$$

Thus,

$$\Phi(s) = D_1(s) - 2^{1-s} D_2(s) - 2^{-\frac{s}{2}}(a_0 + b_0)\left(\frac{1}{s-2} + \frac{1}{s} \right) + \frac{i 2^{\frac{1-s}{2}} b_1}{s-1}$$

and $\Phi(s)$ has been extended analytically to the entire s-plane, with possible simple poles at $s = 0$, 1 and 2. Furthermore, $\Phi(2-s) = -2^{s-1}\Phi(s)$.
(III) Let $k \le 0$. Then $q(z) = b_0(2^{k-1}z^{-1} + z^{-2k+1}) + \sum_{n=0}^{-2k} c_n z^n$ with $c_n \in \mathbf{C}$. Thus,

$$E(s) = -\frac{i 2^{\frac{s-1}{2}} b_0}{s-2k+1} + \frac{i(-1)^k 2^{\frac{s-1}{2}} b_0}{s-1} + \sum_{n=0}^{-2k} \frac{i^n 2^{\frac{s-n-2k}{2}} c_n}{s-2k-n}.$$

Hence we have

$$\Phi(s) = D_1(s) + (-1)^k 2^{k-s} D_2(s) + 2^{-\frac{s}{2}} a_0 \left(\frac{(-1)^k}{s-2k} - \frac{1}{s} \right) + 2^{k-\frac{s+1}{2}} i b_0 \left(\frac{(-1)^{k+1}}{s-2k+1} + \frac{1}{s-1} \right) + (-1)^k \sum_{n=0}^{-2k} \frac{i^n 2^{\frac{-s-n}{2}} c_n}{s-2k-n}$$

and $\Phi_F(s)$ has an analytic continuation to the entire s-plane, except for possible simple poles at $s = 2k-1, 2k, \ldots, 0, 1$. Note that the functional equation (5) implies that

$$c_{-n-2k} = (-1)^{n+1} 2^{-k-n} c_n \quad (0 \le n \le -2k). \tag{18}$$

Furthermore, it follows from (18) that $\Phi(2k-s) = (-1)^k 2^{s-k} \Phi(s)$.

Since $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ $(y = \mathrm{Im}\, z > 0)$ with $a_n = O(n^v)$, the functions $D_1(s)$ and $D_2(s)$ are bounded in every truncated strip of the form $\sigma_1 \le \mathrm{Re}(s) \le \sigma_2$, $|\mathrm{Im}(s)| \ge t_0 > 0$. Also, we note that rational functions $1/s$, $1/(s-2k)$, and $E(s)$ are also bounded in the truncated strip. Thus, $\Phi(s)$ is bounded in every truncated strip of the form above. □

Proof of Theorem 1.5. Note that it follows from $a_n = O(n^v)$ that the function

$$F(z) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is holomorphic in $\mathbf{H}$. Since $a_n = O(n^\nu)$, $\sum_{n=1}^{\infty} a_n n^{-s}$ converges in some right half-plane. It follows from the convergence of $\sum_{n=1}^{\infty} a_n n^{-s}$ and $\Phi(2k-s) = (-1)^k 2^{s-k}\Phi(s)$ that $\Phi(s)$ has at most finitely many simple poles at rational integer points. From the integral formula

$$e^{-y} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) y^{-s} ds$$

for any y, $d > 0$ and absolute convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ for $\sigma = \mathrm{Re}(s)$ sufficiently large, we see that $F(iy) - a_0$ is the inverse Mellin transform of $\Phi(s)$. Indeed,

$$F(iy) - a_0 = \sum_{n=1}^{\infty} \frac{a_n}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s)(2\pi n y)^{-s} ds \tag{19}$$

$$= \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{a_n}{n^s}y^{-s}ds$$
$$= \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\Phi(s)y^{-s}ds$$

for any large positive d. We can choose $d > 2|k|$ sufficiently large so that all of the poles of $\Phi(s)$ lie between $-d$ and d. Then the integrals

$$\int_{d+iT}^{-d+iT}\Phi(s)y^{-s}ds \quad \text{and} \quad \int_{-d-iT}^{d-iT}\Phi(s)y^{-s}ds$$

have limit 0 as $T \to \infty$ (see [6, p. 1868]). We now integrate around a rectangle with vertices $\pm d \pm iT$. By applying the residue theorem, we obtain

$$(20) \qquad F(iy) - a_0 = \frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi(s)y^{-s}ds + \sum_{n=-[d]}^{[d]}\alpha_n y^{-n}$$

where α_n is the residue of $\Phi(s)$ at $s = n$. Applying the functional equation $\Phi(2k-s) = (-1)^k 2^{s-k}\Phi(s)$, we get

$$\frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi(s)y^{-s}ds$$
$$= \frac{(-1)^k 2^k}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi(2k-s)(2y)^{-s}ds$$
$$= (i\sqrt{2}y)^{-2k}\frac{1}{2\pi i}\int_{2k+d-i\infty}^{2k+d+i\infty}\Phi(u)\left(\frac{1}{2y}\right)^{-u}du$$
$$= (\sqrt{2}iy)^{-2k}\left(F\left(\frac{i}{2y}\right) - a_0\right).$$

From (20) we have

$$(21) \qquad F(z) - a_0 = (\sqrt{2}z)^{-2k}F\left(-\frac{1}{2z}\right) - a_0(\sqrt{2}z)^{-2k} + \sum_{n=-[d]}^{[d]}\alpha_n i^n z^{-n}$$

holds for $z = iy, y > 0$. By the identity theorem, (21) holds for all $z \in \mathbf{H}$. Therefore, $F(z)$ is an entire modular integral of weight $2k$ with rational period function for $\Gamma_0^+(2)$ with poles only at 0 and ∞. $\square$

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