Smooth plane curves with outer Galois points whose reduced automorphism group is A_5

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Abstract: In [8] the first author classified automorphism groups of smooth plane curves of degree not less than four into five types. If the curve has a unique outer Galois point, then the quotient group of its automorphism group by the Galois group at the point, which is called the reduced automorphism group, is a finite subgroup of one-dimensional projective linear group. This article is a sequel of [10] and [9]. In this article, we shall determine the defining equation of the curve when the reduced automorphism group is an icosahedral group and give a description of the full automorphism group.

Key words: Icosahedral group; Galois point; plane curve; automorphism group.

1. Introduction.

Notation and Conventions.

- $\mathbf{Z}_m$: a cyclic group of order m;
- ζ_m: a primitive m-th root of unity;
- D_{2m}: the dihedral group of order $2m$;
- $Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$: the quaternion group;
- $T \simeq A_4$, $O \simeq S_4$, $I \simeq A_5$: the tetrahedral, octahedral, icosahedral subgroups of $PGL(2, \mathbf{C})$;
- $\overline{I} \simeq SL(2, 5)$: the binary icosahedral subgroup of $SL(2, \mathbf{C})$.

In this paper

$$\mathrm{PBD}(2,1) := \left\{ A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \alpha \end{pmatrix} \in GL(3, \mathbf{C}) \right\} \Big/ \mathbf{C}^\times.$$

Then there exists an exact sequence

$$1 \to \mathbf{C}^\times \to \mathrm{PBD}(2,1) \xrightarrow{\rho} PGL(2, \mathbf{C}) \to 1,$$

where $\rho : \mathrm{PBD}(2,1) \to PGL(2, \mathbf{C})$ ($[A] \mapsto [(a_{ij})]$) is the natural homomorphism and $\mathbf{C}^\times$ is the subgroup of $\mathrm{PBD}(2,1)$ consisting of the elements represented by diagonal matrices of the form $\mathrm{diag}(1, 1, \alpha)$ ($\alpha \neq 0$).

The group $GL(n, \mathbf{C})$ acts on $\mathbf{C}[X_1, X_2, \ldots, X_n]$ as follows: For $A \in GL(n, \mathbf{C})$ and $f(X_1, X_2, \ldots, X_n) \in \mathbf{C}[X_1, X_2, \ldots, X_n]$,

$$A(f)(X_1, X_2, \ldots, X_n) := f((X_1, X_2, \ldots, X_n)^t A^{-1}).$$

In this article we consider smooth plane curves with a unique outer Galois point. We determine their defining equations and study the structure of their automorphism groups when the reduced automorphism group is A_5 (see Definition 1). Our method is based on the classification theorem of automorphism groups by the first author (see Theorem 1) and the theory of Galois points for smooth plane curves.

First we recall known results on automorphism groups of smooth plane curves. Let C be a smooth plane curve of degree $d \geq 4$. Then the automorphism group $\mathrm{Aut}(C)$ is naturally considered as a subgroup of $PGL(3, \mathbf{C})$. In other words, it acts on $\mathbf{P}^2$. Furthermore, we obtain a classification of automorphism groups of smooth plane curves:

Theorem 1 ([8, Theorem 2.1]). *Let C be a smooth plane curve of degree $d \geq 4$ and $G = \mathrm{Aut}(C)$. If G has a fixed point $P \in \mathbf{P}^2$, then the following hold.*

(a-i) *If $P \in C$ then G is a cyclic group whose order is at most $d(d-1)$. Furthermore, if $d \geq 5$ and $|G| = d(d-1)$, then C is projectively equivalent to the curve $YZ^{d-1} + X^d + Y^d = 0$.*

(a-ii) *If $P \notin C$ then, under a suitable coordinate system, there exists a commutative diagram*

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$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{C}^{\times} & \longrightarrow & \mathrm{PBD}(2,1) & \xrightarrow{\rho} & PGL(2,\mathbf{C}) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\rho} & G' & \longrightarrow & 1, \end{array}$$

where N is a cyclic group whose order is a factor of d and $G' = G/N$ is conjugate to $\mathbf{Z}_m$, D_{2m}, T, O or I, where m is an integer at most $d-1$. Moreover, if $G' \simeq D_{2m}$, then $m|d-2$ or N is trivial. In particular $|G| \leq \max\{2d(d-2), 60d\}$.

If G has no fixed points, then one of the following holds.

(b-i) *$G \subset \mathrm{Aut}(F_d)$, where $F_d : X^d + Y^d + Z^d = 0$ is Fermat curve of degree d.*

(b-ii) *$G \subset \mathrm{Aut}(K_d)$, where $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ is Klein curve of degree d.*

(c) *G is conjugate to one of the following subgroups of $PGL(3,\mathbf{C})$: the alternating group A_5 or A_6, the Klein group $PSL(2,7)$, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72. In particular $|G| \leq 360$.*

Definition 1. The group G' in (a-ii) of Theorem 1 is called the *reduced automorphism group* of C.

Theorem 1 is based on the classification of finite subgroups of $PGL(3,\mathbf{C})$ (see [3]) and leads us to several problems:

(1) Obtain a detailed classification of automorphism groups of smooth plane curves of fixed degree (see [2]).
(2) Study smooth plane curves with a fixed automorphism group (see [1], [12]).
(3) Study automorphism groups of smooth plane curves with special property, for example, Galois points (see the next section).

2. Smooth plane curves with Galois points. We recall basic facts on smooth plane curves with Galois points from [13]. See also [11]. Let C be a smooth plane curve of degree $d \geq 4$ and P a point of $\mathbf{P}^2$. We then have a morphism $\pi_P : C \to \mathbf{P}^1$, which is the restriction of the projection $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$ with the center P. It induces a field extension $\pi_P^* : \mathbf{C}(\mathbf{P}^1) \hookrightarrow \mathbf{C}(C)$. Put $K := \mathbf{C}(C)$ and $K_P := \pi_P^*(\mathbf{C}(\mathbf{P}^1))$.

Definition 2. The point P is called a *Galois point* for C if the field extension K/K_P is Galois. A Galois point P is said to be *inner* (resp. *outer*) if $P \in C$ (resp. $P \notin C$).

Galois points (and quasi Galois points, a wider notion (see [7])) are defined in arbitrary characteristic and deeply related to automorphism groups of smooth plane curves (see, for example, [4–6], [9], [10]).

In this article we consider smooth curves with outer Galois points.

Theorem 2 ([13, Theorem 4′, Proposition 5′])**.** *Let C be a smooth plane curve of degree $d \geq 4$. Then the number of outer Galois points for C is 0, 1 or 3. Furthermore, it is equal to 3 if and only if C is isomorphic to Fermat curve.*

Assume that C has an outer Galois point P and C is not isomorphic to Fermat curve. Then P is the unique outer Galois point. In particular $G = \mathrm{Aut}(C)$ fixes P. Hence it follows from Theorem 1 that, under a suitable coordinate system, $G \subset \mathrm{PBD}(2,1)$ and $P = (0:0:1)$. We have an exact sequence of groups

$$1 \to \mathbf{Z}_d \to G \xrightarrow{\rho} G' \to 1, \tag{1}$$

where G' is a finite subgroup of $PGL(2,\mathbf{C})$. Then G' is one of the following groups: a cyclic group $\mathbf{Z}_m$ ($m \leq d-1$), a dihedral group D_{2m} ($m|d-2$), the tetrahedral group T, the octahedral group O or the icosahedral group I. Furthermore, the following holds.

Lemma 3. *The defining equation of C is of the form $Z^d = F(X,Y)$, where F is a homogeneous polynomial of degree d without multiple factors.*

Proof. Since $P = (0:0:1) \notin C$, the defining equation of C is as follows:

$$Z^d + \sum_{k=1}^{d} a_k F_k(X,Y) Z^{d-k} = 0,$$

where $a_k \in \mathbf{C}$ and F_k is a homogeneous polynomial of degree k. Note that $\mathbf{Z}_d$ in (1) is generated by the element represented by $A = \mathrm{diag}(1,1,\zeta_d)$. Since Z^d is invariant under A, so is the left side of the equation. Then we see that $a_k = 0$ for $k = 1, 2, \ldots, d-1$. Thus C is defined by the equation $Z^d = F(X,Y)$, where $F(X,Y) = -a_d F_d(X,Y)$. Since C is smooth, $F(X,Y)$ has no multiple factors. $\square$

In what follows we assume that $G' = I$. There exists a natural commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{C}^{\times} & \longrightarrow & SL(2,\mathbf{C}) \times \mathbf{C}^{\times} & \longrightarrow & SL(2,\mathbf{C}) & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow \varpi & & \downarrow \pi & & \\ 1 & \longrightarrow & \mathbf{C}^{\times} & \longrightarrow & \mathrm{PBD}(2,1) & \underset{\rho}{\longrightarrow} & PGL(2,\mathbf{C}) & \longrightarrow & 1, \end{array}$$

where π and ϖ are natural projections. This diagram induces another commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}_d & \longrightarrow & \widetilde{G} & \longrightarrow & \overline{I} & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow\varpi & & \downarrow\pi & & \\ 1 & \longrightarrow & \mathbf{Z}_d & \longrightarrow & G & \underset{\rho}{\longrightarrow} & I & \longrightarrow & 1, \end{array}$$

where $\widetilde{G} := \varpi^{-1}(G)$. Then $\operatorname{Ker}\varpi = \{\pm E_3\}$ and $\operatorname{Ker}\pi = \{\pm E_2\}$.

Remark 1. Note that $\overline{I}$ is generated by

$$a := -\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}, b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{and} \quad c := \frac{1}{\eta^2 - \eta^3}\begin{pmatrix} \eta + \eta^{-1} & 1 \\ 1 & -(\eta + \eta^{-1}) \end{pmatrix},$$

where η is a primitive 5th root of unity. Furthermore, it is well known that the invariant ring $\mathbf{C}[X,Y]^{\overline{I}}$ is generated by

$$u = XY(X^{10} + 11X^5Y^5 - Y^{10}),$$

$$v = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20}$$

and

$$w = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30}.$$

In particular any polynomial in $\mathbf{C}[X,Y]^{\overline{I}}$ has an even degree. The generators u, v, w of $\mathbf{C}[X,Y]^{\overline{I}}$ satisfy $w^2 = 1728u^5 - v^3$.

3. Defining equations of curves. First we study the defining equation of C. Recall that it is of the form $Z^d = F(X,Y)$.

Main Result 1. *The polynomial $F(X,Y)$ belongs to the ring $\mathbf{C}[X,Y]^{\overline{I}} = \mathbf{C}[u,v,w]$, i.e., it is invariant under $\overline{I}$.*

Proof. Let A be any matrix in $\overline{I}$. Then there exists a number $\alpha \in \mathbf{C}^\times$ such that $(A,\alpha) \in \widetilde{G}$. Note that α is unique up to multiplication by a d-th root of unity. Thus we obtain a group homomorphism

$$\overline{\chi} : \overline{I} \to \mathbf{C}^\times \ (A \mapsto \alpha^d).$$

Put $\sigma = \varpi(A,\alpha)$. Then $F((X,Y)^tA^{-1}) = \alpha^{-d}F(X,Y)$ since σ acts on $C : Z^d = F(X,Y)$. Thus it suffices to show that $\overline{\chi}$ is trivial.

Put $[A] = \pi(A)$. Then A is uniquely determined by $[A]$ up to multiplication by ± 1. Hence we obtain another group homomorphism

$$\chi : I \to \mathbf{C}^\times \ ([A] \mapsto \alpha^{2d}).$$

Put $K = \operatorname{Ker}\chi$ and $\overline{K} = \operatorname{Ker}\overline{\chi}$. Note that I/K and $\overline{I}/\overline{K}$ are finite subgroups of $\mathbf{C}^\times$, hence they are cyclic. It follows that $K = I$, since I is simple and non-cyclic. Therefore χ is trivial. It follows that $\operatorname{Im}\overline{\chi} \subset \{\pm 1\}$ and $\overline{K}$ is a subgroup of index at most two in $\overline{I}$. Then $\overline{K} = \overline{I}$, since $\overline{I}$ has no subgroup of index two. Hence $\overline{\chi}$ is also trivial. □

Remark 2. For any homogeneous polynomial $F \in \mathbf{C}[u,v,w]$ without multiple factors, the smooth plane curve C defined by the equation $Z^d = F(X,Y)$ has an outer Galois point and there exists an exact sequence

$$1 \to \mathbf{Z}_d \to \operatorname{Aut}(C) \to I \to 1.$$

By Main Result 2 below, it never splits.

Remark 3. The above result may seem clear, but it is not trivial and related to the group structure of I. Indeed, if $G' \neq I$ then $F(X,Y)$ is not always invariant under $\widetilde{G}' = \pi^{-1}(G')$. For example, let C be the sextic curve defined by $Z^6 = XY(X^4 - Y^4)$. Then $G = \operatorname{Aut}(C)$ satisfies the exact sequence

$$1 \to \mathbf{Z}_6 \to G \to O \to 1.$$

Thus $G' = O$ in this case and the polynomial $XY(X^4 - Y^4)$ is not invariant under $\overline{O}$ (the binary octahedral subgroup of $SL(2,\mathbf{C})$), which is generated by

$$\begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_8^{-1} & \zeta_8^{-1} \\ \zeta_8^5 & \zeta_8 \end{pmatrix}.$$

4. The structure of automorphism groups. In this section we study the structure of G. Recall the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}_d & \longrightarrow & \widetilde{G} & \longrightarrow & \overline{I} & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow\varpi & & \downarrow\pi & & \\ 1 & \longrightarrow & \mathbf{Z}_d & \longrightarrow & G & \underset{\rho}{\longrightarrow} & I & \longrightarrow & 1. \end{array}$$

Let d_0 be the odd part of d, i.e., $d = 2^e d_0$ ($e \geqq 1$) and $2 \nmid d_0$ and $\widetilde{G}_0$ the subgroup of $\widetilde{G}$ generated by

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} b & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} c & \\ & 1 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \zeta_{2^e} \end{pmatrix},$$

where a, b and c are those in Remark 1. Then it is clear that $-E_3 \in \widetilde{G}_0$ and $\widetilde{G} = \mathbf{Z}_{d_0} \times \widetilde{G}_0$. Hence $G = \widetilde{G}/\{\pm E_3\} \simeq \mathbf{Z}_{d_0} \times G_0$, where $G_0 := \widetilde{G}_0/\{\pm E_3\}$.

Thus the structure of G is determined by the one of G_0, which is as follows:

Main Result 2. *The group $G_0 = \mathbf{Z}_{2^e}{}^\bullet I$ is a non-split extension of I by $\mathbf{Z}^{2^e}$ and it is an extension of a cyclic group $\mathbf{Z}_{2^{e-1}}$ by $\overline{I}$. Precisely*

$$G_0 = \begin{cases} \overline{I} & (\text{if } e = 1), \\ \overline{I} \rtimes \mathbf{Z}_2 & (\text{if } e = 2), \\ \overline{I}{}^\bullet \mathbf{Z}_{2^{e-1}} & (\text{if } e \geq 3). \end{cases}$$

In the rest of this section we give a proof of the above result. First we show that the exact sequence

$$1 \longrightarrow \mathbf{Z}_{2^e} \longrightarrow G_0 \longrightarrow I \longrightarrow 1 \tag{2}$$

is not split. Let $\widetilde{H}$ be the subgroup of $\widetilde{G}_0$ generated by

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} b & \\ & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & \\ & 1 \end{pmatrix}$$

and $\widetilde{G}_1$ the subgroup of $\widetilde{G}_0$ generated by

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} b & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} c & \\ & 1 \end{pmatrix}$$
$$\text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Then $-E_3 \in \widetilde{G}_1$ and $\widetilde{G}_0 = \mathbf{Z}_{2^e} \times \widetilde{H}$. Furthermore, $\widetilde{G}_1$ is the kernel of the composite of the natural projection $\widetilde{G}_0 \to \mathbf{Z}_{2^e}$ and $\mathbf{Z}_{2^e} \to \mathbf{Z}_{2^{e-1}}$ $(\alpha \mapsto \alpha^2)$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{G}_1 & \longrightarrow & \widetilde{G}_0 & \longrightarrow & \mathbf{Z}_{2^{e-1}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{I} & \longrightarrow & G_0 & \underset{\psi}{\longrightarrow} & \mathbf{Z}_{2^{e-1}} & \longrightarrow & 1, \end{array} \tag{3}$$

where ψ is defined by $\psi\left(\left[\begin{pmatrix} A & \\ & \alpha \end{pmatrix}\right]\right) = \alpha^2$.

Suppose that (2) is split. Then G_0 contains a subgroup H isomorphic to I. The map $\psi|_H$ is trivial since $\mathrm{Im}\,(\psi|_H)$ is cyclic and H is a non-cyclic simple group. It follows that $H \subset \mathrm{Ker}\,\psi = \overline{I}$. This is a contradiction, since $\overline{I}$ contains no subgroup isomorphic to I.

Next we consider the splitting of the exact sequence (3). If $e = 1$ then clearly $G_0 = \overline{I}$. If $e = 2$, then the homomorphism $s\colon \mathbf{Z}_2 \to G_0$ defined by $s(-1) = \left[\begin{pmatrix} & 1 & \\ -1 & & \\ & & i \end{pmatrix}\right]$ is a section of ψ. Thus $G_0 = \overline{I} \rtimes \mathbf{Z}_2$.

Finally suppose that $e \geq 3$ and there exists a section $\iota\colon \mathbf{Z}_{2^{e-1}} \to G_0$ of ψ. We can write $\iota(\zeta_{2^{e-1}}) = \left[\begin{pmatrix} A & \\ & \alpha \end{pmatrix}\right]$ for a matrix $A \in \overline{I}$ and a number α such that $\alpha^2 = \zeta_{2^{e-1}}$. Then $A^{2^{e-1}} = -E_2$ since $\alpha^{2^{e-1}} = -1$. Therefore the order of A in $\overline{I}$ is $2^e \geq 8$. This is a contradiction because Sylow 2-subgroups of $\overline{I}$ are conjugate to Q_8, the quaternion group.

5. Examples. We give some examples of Main Result 2. Let u, v and w be those in Remark 1, i.e.,

$$u = XY(X^{10} + 11X^5Y^5 - Y^{10}),$$
$$v = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20}$$

and

$$w = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30}.$$

Example 1. Let C be the plane curve of degree 12 defined by

$$Z^{12} = XY(X^{10} + 11X^5Y^5 - Y^{10}) \ (= u).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbf{Z}_{12} \to G \to I \to 1$$

and $G = \mathbf{Z}_3 \times (\overline{I} \rtimes \mathbf{Z}_2)$.

Example 2. Let C be the plane curve of degree 20 defined by

$$Z^{20} = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20} \ (= v).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbf{Z}_{20} \to G \to I \to 1$$

and $G = \mathbf{Z}_5 \times (\overline{I} \rtimes \mathbf{Z}_2)$.

Example 3. Let C be the plane curve of degree 30 defined by

$$Z^{30} = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30} \ (= w).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbf{Z}_{30} \to G \to I \to 1$$

and $G = \mathbf{Z}_{15} \times \overline{I}$.

Remark 4. For the above cases, we determined the structure of the automorphism group by another method in [9, Theorem 5].

Example 4. Let C be the plane curve of degree 32 defined by $Z^{32} = uv$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbf{Z}_{32} \to G \to I \to 1$$

and $G = \overline{I}^{\bullet}\mathbf{Z}_{16}$.

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