Euler tangent numbers modulo 720 and Genocchi numbers modulo 45

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Abstract: We establish congruences for higher order Euler polynomials modulo 720. We apply this result for constructing analogues of Stern congruences for Euler secant numbers $E_{4n} \equiv 5 (\mathrm{mod}\ 60), E_{4n+2} \equiv -1 (\mathrm{mod}\ 60)$ to Euler tangent numbers and Genocchi numbers. We prove that Euler tangent numbers satisfy the following congruences $E_{4n+1} \equiv 16 (\mathrm{mod}\ 720)$, and $E_{4n+3} \equiv -272 (\mathrm{mod}\ 720)$. We establish 12-periodic property of Genocchi numbers modulo 45.

Key words: Higher-order Euler numbers; secant numbers; tangent numbers; Genocchi numbers; Ramanujan congruences.

1. Introduction. Euler numbers are defined as coefficients of Taylor series of the function

$$\mathrm{sech}(x) + \tanh(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!}.$$

Euler numbers and higher-order Euler numbers were studied in [4], [2]. Their congruences were studied in [3], [5], [6]. Higher-order Euler numbers are defined as coefficients of secant power

$$\mathrm{sech}^q x = \sum_{n\geq 0} (-1)^n E_{2n}^{(q)} \frac{x^{2n}}{(2n)!}.$$

Here q might be any number, positive or negative. One can understand q as a formal parameter and consider $E_{2n}^{(q)}$ as a polynomial of q. We set

$$L_{2n}(q) = (-1)^n E_{2n}^{(-q)}.$$

For Euler secant numbers Stern established the following congruences ([1, p. 124], [4])

$$E_{4n} \equiv 5 (\mathrm{mod}\ 60),\ n > 0,$$
$$E_{4n+2} \equiv -1 (\mathrm{mod}\ 60),\ n \geq 0.$$

Later these congruences were re-discovered by Ramanujan.

The main result of our paper is the following

Theorem 1.1. *For any $n > 0$ and for any integer q the following congruences are valid*

$$L_{4n}(24q - 2) \equiv L_4(24q - 2) (\mathrm{mod}\ 720),$$
$$L_{4n+2}(24q - 2) \equiv L_6(24q - 2) (\mathrm{mod}\ 720).$$

Since Euler tangent numbers are particular cases of higher-order Euler numbers, $E_{2n+1} = L_{2n}(-2)$, we obtain the following consequence of Theorem 1.1.

Theorem 1.2. *For any $n > 0$ the following congruences hold*

$$E_{4n+1} \equiv 16 (\mathrm{mod}\ 720), \quad E_{4n+3} \equiv -272 (\mathrm{mod}\ 720).$$

If

$$E_{4n+1} \equiv E_5 (\mathrm{mod}\ N_1), \quad E_{4n+3} \equiv E_7 (\mathrm{mod}\ N_3),$$

for any $n > 0$ with $N_1, N_3 \geq 720$, then $N_1 = N_3 =$ 720.

The Genocchi numbers G_n are a sequence of integers defined by the generating function

$$\sum_{n\geq 1} G_n \frac{x^n}{n!} = \frac{2x}{e^x + 1}.$$

In particular, $G_1 = 1$ and $G_n = 0$, if $n > 1$ is odd. All Genocchi numbers G_{2n} are odd integers.

Combinatorial meaning of Genocchi numbers: $|G_{2n}|$ counts the number of permutations $\sigma \in S_{2n-1}$ with descends after even numbers and ascends after odd numbers. For example, if $n = 3$ then $\{42135, 21435, 34215\}$ is a list of Genocchi permutations, and $|G_3| = 3$.

Euler tangent numbers are closely related to Bernoulli numbers B_n and Genocchi numbers G_n,

$$B_{2n} = \frac{2n}{4^{2n} - 2^{2n}} E_{2n-1},$$
$$G_n = 2(1 - 2^n) B_n,$$
$$G_{2n} = -2^{2-2n}\, n\, E_{2n-1}.$$

Application of Theorem 1.2 for Genocchi numbers gives us the following result.

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Theorem 1.3. *For any $n \geq 0$, Genocchi numbers satisfy the following congruences*

$$G_{12n} \equiv 3n (\mathrm{mod}\ 45),$$
$$G_{12n+2} \equiv -6n - 16 (\mathrm{mod}\ 45), \quad n > 0,$$
$$G_{12n+4} \equiv 3n + 1 (\mathrm{mod}\ 45),$$
$$G_{12n+6} \equiv -6n - 3 (\mathrm{mod}\ 45),$$
$$G_{12n+8} \equiv 3n + 17 (\mathrm{mod}\ 45),$$
$$G_{12n+10} \equiv -6n + 25 (\mathrm{mod}\ 45).$$

Proof of theorem 1.1 is based on the following property of polynomial $L_{2n}(q)$.

Theorem 1.4. *The polynomials $L_{2n}(q)$ can be constructed by recurrence relation*

(1) $L_{2n}(q) =$
$$\sum_{i=1}^{n} \left(\binom{2n-1}{2i-1}(q+1) - \binom{2n}{2i} \right) L_{2(n-i)}(q), n > 0,$$
$$L_0(q) = 1.$$

Proof of Theorem 1.4. A function $f : A \to Q$ where $A = \{1, 2, \ldots, 2n\}$, $Q = \{1, 2, \ldots, q\}$ is called an *even pre-image*, if for any $j \in Q$ the pre-image $f^{-1}(j) \subseteq A$ has an even number of elements. Call for any $j \in Q$ pre-image set

$$B_j = f^{-1}(j) = \{i | f(i) = j\}$$

as j-th block. Then $|B_j| = p_j$ is even. In particular, p_j might be 0.

Let us prove that $L_{2n}(q)$ is a number of even pre-image functions. Let λ be partiton of n with length $k = l(\lambda)$ and $\lambda = 1^{r_1} 2^{r_2} \cdots n^{r_n}$ is a multiplicity form of this partition, i.e., r_i is a number of components of λ equal to i. Then

$$r_1 + \cdots + r_n = k,$$
$$1r_1 + 2r_2 + \cdots + nr_n = n.$$

Say that even pre-image function $f : A \to Q$ has partition type 2λ if f has k non-empty blocks $B_{j_1}, \ldots, B_{j_k}$, and their lengths generate partition 2λ. Let us calculate the number of even pre-image functions with partition type $2^{r_1} 4^{r_2} \cdots (2n)^{r_n}$. Blocks B_{j_i} with $|B_{j_i}| = 2\lambda_i$ can be selected in

$$\frac{(2n)!}{\prod_{i=1}^{k} (2\lambda_i)!} = \frac{(2n)!}{\prod_{i=1}^{n} (2i)!^{r_i}}$$

ways. Since blocks with equal size can be permuted, number of block selections is equal to

$$\frac{1}{r_1! r_2! \cdots r_n!} \, \frac{(2n)!}{\prod_{i=1}^{n} (2i)!^{r_i}}.$$

First block might be a pre-image of q elements, second block might be a pre-image of $q - 1$ elements, etc. So, the number of pre-image possibilities is equal to falling factorial

$$q_k = q(q-1) \cdots (q-k+1).$$

Therefore, the number of even pre-image functions is

$$L_{2n}(q) = \sum_{\lambda \vdash n} \frac{(2n)!}{2!^{r_1} 4!^{r_2} \cdots (2n)!^{r_n} \, r_1! r_2! \cdots r_n!} \, q_{l(\lambda)}.$$

Taylor series of hyperbolic cosine is

$$\cosh x = \sum_{k \geq 0} \frac{x^{2k}}{(2k)!}.$$

Therefore,

$$\cosh^q x = \sum_{n \geq 0} \sum_{k_1 + \cdots + k_q = n} \frac{(2n)!}{(2k_1)! \cdots (2k_q)!} \, \frac{x^{2n}}{(2n)!}.$$

Hence the coefficient of $\cosh^q x$ at $\frac{x^n}{n!}$ is $L_{2n}(q)$. This means that one can interpret higher-order Euler polynomial $L_n(q)$ for integer q as a number of even pre-image functions $f : A \to Q$.

Now we will show how to obtain formula (1). To do that, we will calculate the number of even pre-image functions in two ways.

First way. Suppose that $f : A \to Q$ is an even pre-image function, $1 \in A$ belongs to block B_{j_1}, and $|B_{j_1}| = 2s$. Then, j_1 can be selected in q ways, and other elements of block B_{j_1}, except 1, can be selected in $\binom{2n-1}{2s-1}$ ways. The number of even pre-image functions $g : A \setminus B_{j_1} \to Q \setminus \{j_1\}$ is $L_{2n-2s}(q-1)$. Therefore,

$$L_{2n}(q) = \sum_{s=1}^{n} q \binom{2n-1}{2s-1} L_{2(n-s)}(q-1). \tag{2}$$

Second way. We study pre-image possibilities for $1 \in Q$. If $f^{-1}(1) = \emptyset$, then the number of such even pre-image functions is $L_{2n}(q-1)$. If $|f^{-1}(1)| = 2s \neq 0$, then the elements of block B_1 can be selected in $\binom{2n}{2s}$ ways. The number of even pre-image functions $h : A \setminus B_1 \to Q \setminus \{1\}$ is $L_{2(n-s)}(q-1)$. Therefore,

$$L_{2n}(q) = \sum_{s=0}^{n} \binom{2n}{2s} L_{2(n-s)}(q-1). \tag{3}$$

By (2) and (3) we obtain

$$L_{2n}(q-1)=\sum_{s=1}^{n}\left(q\binom{2n-1}{2s-1}-\binom{2n}{2s}\right)L_{2(n-s)}(q-1).$$

All that remains is to change $q-1$ to q to obtain (1).

Sketch of the Proof of Theorem 1.1. We have

$$L_4(q)=3q^2-2q,\quad L_6(q)=15q^3-30q^2+16q.$$

Therefore,

$$L_4(24q-2)=16-336q+1728q^2,$$
$$L_6(24q-2)=-272+7584q-69120q^2+207360q^3.$$

Hence Theorem 1.1 can be formulated as follows:

$$L_{4n}(24q-2)\equiv 16+384q+288q^2(\mathrm{mod}\ 720),$$
$$L_{4n+2}(24q-2)\equiv 448+384q(\mathrm{mod}\ 720).$$

Since $720=2^4\cdot 3^2\cdot 5$, by the Chinese remainder theorem these congruences are equivalent to the following congruences

(4) $$L_{2n}(24q-2)\equiv 0(\mathrm{mod}\ 16),\quad n>1,$$

(5) $$L_{2n}(24q-2)\equiv 6q-2(\mathrm{mod}\ 9),\quad n>0,$$

(6) $$L_{4n}(24q-2)\equiv 1+4q+3q^2(\mathrm{mod}\ 5),\quad n>0,$$

(7) $$L_{4n+2}(24q-2)\equiv 4q-2(\mathrm{mod}\ 5),\quad n\geq 0.$$

Proof of congruence (4). We proceed by induction on $n\geq 0$. If $n=2$, then

$$L_4(24q-2)=3(24q-2)^2-2(24q-2)=$$
$$=16(-1+9q)(-1+12q).$$

Therefore, relation (4) is true for $n=2$. Suppose that this relation is valid for $n-1\geq 2$. Then by Theorem 1.4 we have

$$L_{2n}(24q-2)=X_1+X_2+X_3,$$

where

$$X_1=((24q-1)-1)L_0(24q-2)=24q-2,$$
$$X_2=\left(\binom{2n-1}{2(n-1)-1}(24q-1)-\binom{2n}{2(n-1)}\right)\cdot L_2(24q-2),$$
$$X_3=\sum_{i=1}^{n-2}\left(\binom{2n-1}{2i-1}(24q-1)-\binom{2n}{2i}\right)\cdot L_{2(n-i)}(24q-2).$$

By the inductive hypothesis we have

$$L_{2(n-i)}(24q-2)\equiv 0(\mathrm{mod}\ 16),\quad 1\leq i\leq n-2.$$

Hence

$$X_3=\sum_{i=1}^{n-2}\left(\binom{2n-1}{2i-1}(24q-1)-\binom{2n}{2i}\right)\cdot L_{2(n-i)}(24q-2)\equiv 0(\mathrm{mod}\ 16).$$

Therefore,

$$L_{2n}(24q-2)\equiv X_1+X_2$$
$$\equiv 24q-2+(2n-1)((n-1)(24q-1)-n)(24q-2)$$
$$\equiv 8(-1+12q)X(\mathrm{mod}\ 16),$$

where

$$X=(n-1)(-n-6q+12nq).$$

Note that X is even for any $n:$ if n is odd, then $(n-1)$ is even, and if n is even, $(-n-6q+12nq)$ is even. Hence, we have

$$L_{2n}(24q-2)\equiv 0(\mathrm{mod}\ 16),$$

and the congruence (4) is proved.

Proof of the congruence (5). For $n=0$ our statement is evident. Suppose that it is true for $n-1$. Then by Theorem 1.4

$$L_{2n}(24q-2)=((24q-1)-1)L_0(24q-2)+Y,$$

where

$$Y=\sum_{i=1}^{n-1}\left(\binom{2n-1}{2i-1}(24q-1)-\binom{2n}{2i}\right)\cdot L_{2(n-i)}(24q-2).$$

By induction hypothesis

$$L_{2(n-i)}(24q-2)\equiv 6\,q-2(\mathrm{mod}\ 9),\quad 1\leq i\leq n-1.$$

Further,

$$\sum_{i=1}^{n-1}\binom{2n-1}{2i-1}=4^{n-1}-1,\ \text{if}\ n>1,$$
$$\sum_{i=1}^{n-1}\binom{2n}{2i}=2^{2n-1}-2,\ \text{if}\ n>0.$$

Therefore, we have the following modulo 9

$$Y\equiv\sum_{i=1}^{n-1}\left(\binom{2n-1}{2i-1}(24q-1)-\binom{2n}{2i}\right)(6q-2)$$
$$\equiv(6q-2)(24q-1)\sum_{i=1}^{n-1}\binom{2n-1}{2i-1}$$
$$-(6q-2)\sum_{i=1}^{n-1}\binom{2n}{2i}$$

$$\equiv (6q-2)\{(24q-1)(4^{n-1}-1)-(2^{2n-1}-2)\}$$
$$\equiv 3(4^{n-1}-1)(6q-2)(8q-1).$$

Since $3(4^{n-1}-1) \equiv 0 (\bmod 9)$, we obtain

$$Y \equiv 0 (\bmod 9).$$

Therefore,

$$L_{2n}(24q-2) = ((24q-1)-1)L_0(24q-2)+Y$$
$$\equiv 24q-2 \equiv 6q-2 (\bmod 9),$$

and the congruence (5) is proved completely.

Similar arguments show that (6) and (7) are valid as well.

All that remains is to use Chinese remainder theorem to get that

$$E_{4n+1} \equiv E_5 (\bmod 720), \quad E_{4n+3} \equiv E_7 (\bmod 720),$$

for any $n > 0$. Suppose that for some integers $N_1 \geq 720$ and $N_3 \geq 720$ the following congruences are valid

$$E_{4n+1} \equiv E_5 (\bmod N_1), \quad E_{4n+3} \equiv E_7 (\bmod N_3),$$

for any $n > 0$. In particular, they are valid for $n = 2, 3$. We have

$$E_5 = 16, E_7 = -272, E_9 = 7936,$$
$$E_{11} = -353792, E_{13} = 22368256,$$
$$E_{15} = -1903757312,$$

and,

$$GCD(E_9 - E_5, E_{13} - E_5) = 720,$$
$$GCD(E_7 - E_{11}, E_7 - E_{15}) = 720.$$

Therefore,

$$N_1 = 720, \quad N_3 = 720.$$

So, the number 720 as a base of congruence in Theorem 1.1 is optimal.

Sketch of the proof of Theorem 1.3. The proof is based on the following four facts. First, by Theorem 1.2 $E_{4n+1} \equiv 16 (\bmod 45)$, $E_{4n+3} \equiv -272 (\bmod 45)$. Second, we use the following connection between Genocchi numbers and tangent numbers $G_{2n} = -2^{2-2n}\, n\, E_{2n-1}$. Third, Genocchi numbers are odd. Fourth, $16^n (\bmod 45) \equiv 1, 16, 31$, if $n \equiv 0, 1, 2 (\bmod 3)$ respectively.

Let us give the proof of congruences for $G_{12n}, G_{12n+2}, G_{12n+4}, G_{12n+6}$. Other congruences mentioned in Theorem 1.3 can be established by similar arguments.

We have

$$G_{12n} = -2^{2-12n} \cdot 6n \cdot E_{12n-1} = -16^{-3n} \cdot 3n \cdot 2^3 E_{12n-1},$$
$$2^3 E_{12n-1} \equiv -2^3 \cdot 272 (\bmod 45) \equiv -16 (\bmod 45).$$

Therefore,

$$G_{12n} \equiv 1 \cdot 48n (\bmod 45) \equiv 3n (\bmod 45).$$

Further,

$$G_{12n+2} = -2^{-12n} \cdot (6n+1) \cdot E_{12n+1}$$
$$= -16^{-3n}(6n+1)E_{12n+1},$$
$$E_{12n+1} \equiv 16 (\bmod 45),$$

and,

$$G_{12n+2} \equiv -16^{-3n} \cdot 16(6n+1) (\bmod 45)$$
$$\equiv -6n - 16 (\bmod 45).$$

We have

$$G_{12n+4} = -2^{2-12n-4}(6n+2)E_{12n+3}$$
$$= -16^{-3n}2^{-1}(3n+1)E_{12n+3},$$
$$2^{-1}E_{12n+3} \equiv -1 (\bmod 45).$$

Hence,

$$G_{12n+4} \equiv 3n + 1 (\bmod 45).$$

Further,

$$G_{12n+6} = -2^{2-12n-6}(6n+3)E_{12n+5}$$
$$= -16^{-3n}(6n+3)16^{-1}E_{12n+5},$$
$$16^{-1}E_{12n+5} \equiv 1 (\bmod 45).$$

Therefore,

$$G_{12n+6} \equiv -16^{-3n}(6n+3) \equiv -6n - 3 (\bmod 45).$$

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