Proving dualities for qMZVs with connected sums

By Benjamin BRINDLE

University of Cologne, Department of Mathematics and Computer Sciences,
Weyertal 86-90, 50931 Cologne, Germany

(Communicated by Masaki KASHIWARA, M.J.A., April 12, 2022)

Abstract: This paper gives an application of so-called connected sums, introduced recently by Seki and Yamamoto [SY]. Special about our approach is that it proves a duality for the Schlesinger–Zudilin and the Bradley–Zhao model of qMZVs simultaneously. The latter implies the duality for MZVs and the former can be used to prove the shuffle product formula for MZVs. Furthermore, the q-Ohno relation, a generalization of Bradley–Zhao duality, is also obtained.

Key words: Multiple zeta values; q-multiple zeta values; duality; connected sums.

1. Notation and definitions. For an *admissible index* $\mathbf{k} = (k_1, \dots, k_r) \in \mathbf{N}^r$, i.e., $r \geq 0$ and $k_1 \geq 2$, its *multiple zeta value* (MZV) is defined as

$$\zeta(\mathbf{k}) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1}} \cdots \frac{1}{m_r^{k_r}}.$$

To understand the algebraic structure of MZVs better on the one hand and to get connections to holomorphic functions, in particular, modular forms (see [GKZ], [Bac]), on the other hand, it is useful to introduce q-analogs of MZVs. There are several models of q-analogs. We focus in this paper on two of them: the Bradley–Zhao model and the Schlesinger–Zudilin model. For further details on these and other models, we refer to [Zha], [Bri]. In this note q will be a formal variable or a real number with $0 < q < 1$.

The Bradley–Zhao model is defined as follows: Set $\zeta_q^{\mathrm{BZ}}(\emptyset) := 1$ and for $\mathbf{k} = (k_1, \dots, k_r)$ an admissible index define

$$\zeta_q^{\mathrm{BZ}}(\mathbf{k}) := \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1(k_1-1)}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r(k_r-1)}}{(1-q^{m_r})^{k_r}}.$$

Similarly, we define Schlesinger–Zudilin qMZVs via $\zeta_q^{\mathrm{SZ}}(\emptyset) := 1$ and for every *SZ-admissible index* $\mathbf{k}$, i.e., $\mathbf{k} \in \mathbf{N}_0^r$ for some $r \geq 0$ with $k_1 \geq 1$, we set

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) := \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}.$$

2010 Mathematics Subject Classification. Primary 11M32, 05A30.

2. Dualities. Write an admissible index $\mathbf{k}$ in the shape $\mathbf{k} = (k_1 + 1, \{1\}^{d_1 - 1}, \dots, k_r + 1, \{1\}^{d_r - 1})$ with $k_j, d_j \geq 1$ unique ($\{1\}^d$ means that 1 is repeated d-times). For the next two theorems, we need the *dual index*,

$$\mathbf{k}^\vee := (d_r + 1, \{1\}^{k_r - 1}, \dots, d_1 + 1, \{1\}^{k_1 - 1}).$$

Theorem 1 (MZV-Duality, [Zag, §9])**.** *For every admissible index* $\mathbf{k}$*, we have* $\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\vee)$*.*

The next theorem can be seen as a q-analog of MZV-duality since MZV-duality follows immediately from it (cf. the proof):

Theorem 2 (BZ-Duality, [Bra, Thm. 5])**.** *We have* $\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee)$ *for every admissible index* $\mathbf{k}$*.*

A generalization of BZ-duality is the so-called q-Ohno relation, of which BZ-duality is the special case $c = 0$:

Theorem 3 (q-Ohno relation, [Bra, Thm. 5])**.** *For any admissible index* $\mathbf{k} = (k_1, \dots, k_r)$ *and any* $c \in \mathbf{N}_0$ *we have*

$$\sum_{|\mathbf{c}| = c} \zeta_q^{\mathrm{BZ}}(\mathbf{k} + \mathbf{c}) = \sum_{|\mathbf{c}'| = c} \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee + \mathbf{c}'),$$

where we sum on the left over all $\mathbf{c} = (c_1, \dots, c_r) \in \mathbf{N}_0^r$ *with* $|\mathbf{c}| := c_1 + \cdots + c_r = c$ *and on the right we sum over all* $\mathbf{c}' = (c'_1, \dots, c'_{r'}) \in \mathbf{N}_0^{r'}$ *with* $|\mathbf{c}'| = c$ *where* r' *is the depth of* $\mathbf{k}^\vee$*. The addition of indices is to be understood componentwise.*

For the SZ-model, we write an SZ-admissible index, with $k_j, d_j \geq 0$ unique, in the shape $\mathbf{k} = (k_1 + 1, \{0\}^{d_1}, \dots, k_r + 1, \{0\}^{d_r})$ and define the *SZ-dual index*,

doi: 10.3792/pjaa.98.006

$$\mathbf{k}^\dagger := (d_r+1, \{0\}^{k_r}, \dots, d_1+1, \{0\}^{k_1}).$$

Theorem 4 (SZ-Duality, [Zha, Thm. 8.3])**.** *For all* $\mathbf{k}$ *SZ-admissible, we have* $\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = \zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger)$.

Note that BZ- and SZ-duality on algebraic level look the same, both can be obtained by the same anti-automorphism on the non-commutative free algebra in two variables (see, e.g., [Bri, Thm. 3.5, Thm. 3.16]). But they imply different things. BZ-duality gives direct duality for MZVs, while SZ-duality does not. However, SZ-duality implies another important result in the theory of MZVs, namely the shuffle product formula (cf. [EMS], [Sin], for details [Bri, Thm. 3.46]).

For some calculations in the next section we need the connection between admissible and SZ-admissible index: An index $\mathbf{k}$ is admissible if and only if $\mathbf{k}-\mathbf{1}$ is SZ-admissible ($\mathbf{k}+\mathbf{1}$ is the index, which is $\mathbf{k}$ with every entry increased by 1; similar for $\mathbf{k}-\mathbf{1}$). Furthermore, we have for $\mathbf{k}$ admissible

$$(\mathbf{k}-\mathbf{1})^\dagger = \mathbf{k}^\vee - \mathbf{1}. \tag{2.1}$$

3. Connected sums & proof of dualities. As a new tool for proving identities among (q-)multiple zeta values, Seki and Yamamoto introduced the concept of so-called connected sums (this notion is independent of connected sums in topology). With connected sums, they have proven, e.g., the duality of MZVs, Hoffman's identity, and the q-analog of Ohno's relation, cf. [Sek] or [SY].

Using connected sums, we give a proof of the duality of Schlesinger–Zudilin qMZVs, the duality of Bradley–Zhao qMZVs and the usual duality of MZVs. It turns out that the connected sum defined below has the power to prove all three statements at once. As a by-product, we also get a proof for the q-Ohno relation. The proof is inspired by the one of Seki and Yamamoto ([SY]), where the authors proved q-Ohno's relation for non-modified Bradley–Zhao qMZVs and hence, in particular, also BZ-duality. We work with modified qMZVs, which will be here the reason that we can prove all the mentioned dualities at the same time. In Remark 6 (v), we refer to Seki–Yamamotos connected sum.

Definition 5 (Connected sum)**.** Let be $r, s \geq 0$, $\mathbf{k} = (k_1, \dots, k_r) \in \mathbf{N}_0^r$, $\boldsymbol{\ell} = (\ell_1, \dots, \ell_s) \in \mathbf{N}_0^s$ and $x \in [0,1)$. Define the connected sum as

$$\begin{aligned} Z_q(\mathbf{k}; \boldsymbol{\ell}; x) & \\ := \sum_{\substack{m_1 > \cdots > m_r > m_{r+1} = 0 \\ n_1 > \cdots > n_s > n_{s+1} = 0}} & \prod_{i=1}^r \frac{q^{m_i k_i}}{(1-q^{m_i}x)(1-q^{m_i})^{k_i}} \\ & \times \prod_{j=1}^s \frac{q^{n_j \ell_j}}{(1-q^{n_j}x)(1-q^{n_j})^{\ell_j}} \\ & \times \frac{q^{m_1 n_1} f_q(m_1; x) f_q(n_1; x)}{f_q(m_1+n_1; x)}, \end{aligned}$$

where $f_q(m;x) := \prod_{h=1}^m (1-q^h x)$.

Remark 6. (i) The connected sum Z_q is symmetric in $\mathbf{k}$ and $\boldsymbol{\ell}$ by definition.
(ii) Notice that the connected sum is well-defined in the sense that it is a series over positive real numbers and hence either a positive real number (if convergent) or $+\infty$ (if not convergent).
(iii) If $k_1 \geq 1$, then $Z_q(\mathbf{k}; \emptyset; 0) = \zeta_q^{\mathrm{SZ}}(\mathbf{k})$.
(iv) If $k_1 \geq 1$, then $\lim_{x \to 1} Z_q(\mathbf{k}; \emptyset; x) = \zeta_q^{\mathrm{BZ}}(\mathbf{k}+\mathbf{1})$.
(v) In [SY], the authors define also a connected sum. Call it $Z_q^{\mathrm{SY}}(\mathbf{k}; \boldsymbol{\ell}; x)$ and assume that the indices are in reversed order than there. Then Z_q and Z_q^{SY} are connected via

$$Z_q(\mathbf{k}-\mathbf{1}; \boldsymbol{\ell}-\mathbf{1}; x) = \frac{1}{(1-q)^{|\mathbf{k}|+|\boldsymbol{\ell}|}} Z_q^{\mathrm{SY}}(\mathbf{k}; \boldsymbol{\ell}; y)$$

with $x = 1 + (1-q)y$, where $|\cdot|$ denotes the sum of entries of the corresponding index.

Proposition 7 (Boundary conditions)**.** *If* $k_1 \geq 1$, $0 < q < 1$ *and* $x \in [0,1)$, *then* $Z_q(\mathbf{k}; \emptyset; x)$ *is a well-defined real number.*

Proof. One has

$$\begin{aligned} & Z_q(\mathbf{k}; \emptyset; x) \\ &\quad = \sum_{m_1 > \cdots > m_{r+1} := 0} \prod_{i=1}^r \frac{q^{m_i k_i}}{(1-q^{m_i}x)(1-q^{m_i})^{k_i}} \\ &\quad \leq \frac{1}{(1-q)^r} \sum_{m_1 > \cdots > m_{r+1} := 0} \prod_{i=1}^r \frac{q^{m_i k_i}}{(1-q^{m_i})^{k_i}} \\ &\quad = \frac{1}{(1-q)^r} \zeta_q^{\mathrm{SZ}}(k_1, \dots, k_r), \end{aligned}$$

which is well-defined since $k_1 \geq 1$, i.e., $(k_1, \dots, k_r)$ is SZ-admissible. □

After we have checked well-definedness of Z_q, we state and prove now distinguished relations among our connected sums.

Theorem 8 (Transport relations)**.** *Let be* $r, s \geq 0$ *and* $k_1, \dots, k_r, \ell_1, \dots, \ell_s \geq 0$. *If* $s > 0$,

$$Z_q((0, k_1, \dots, k_r); (\ell_1, \dots, \ell_s); x) \tag{3.1}$$

$$= Z_q((k_1,\dots,k_r);(\ell_1+1,\ell_2,\dots,\ell_s);x)$$

and if $r>0$,

$$Z_q((k_1+1,k_2,\dots,k_r);(\ell_1,\dots,\ell_s);x) = Z_q((k_1,\dots,k_r);(0,\ell_1,\ell_2,\dots,\ell_s);x). \tag{3.2}$$

Proof. The second equality follows from the first by symmetry and the first one is obtained from

$$\begin{aligned}\sum_{a>m}\frac{1}{1-q^a x}\frac{q^{an}f_q(a;x)f_q(n;x)}{f_q(a+n;x)} &= \frac{q^n}{1-q^n}\sum_{a>m}\left(\frac{q^{(a-1)n}f_q(a-1;x)f_q(n;x)}{f_q(a+n-1;x)}\right.\\ &\qquad\left. -\frac{q^{an}f_q(a;x)f_q(n;x)}{f_q(a+n;x)}\right)\\ &= \frac{q^n}{1-q^n}\frac{q^{mn}f_q(m;x)f_q(n;x)}{f_q(m+n;x)}\end{aligned}$$

and setting $m=m_1, n=n_1, a=m_0$. □

Remark 9. Theorem 8 coincides with [SY, Thm. 2.2] under the identification of Remark 6 (v).

This theorem is the key of proving Theorems 1–4. Especially, the following corollary will be needed, together with the connection of Z_q with ζ_q^{BZ} resp. ζ_q^{SZ} (Remark 6).

Corollary 10. *For every SZ-admissible index* $\mathbf{k}$ *and* $x\in[0,1)$ *we have*

$$Z_q(\mathbf{k};\emptyset;x) = Z_q(\emptyset;\mathbf{k}^\dagger;x).$$

Proof. For all indices $\mathbf{k}$ and $\boldsymbol{\ell}$ and $k\geq 1, d\geq 0$ we obtain (by $(k,\{0\}^d,\mathbf{k})$ we mean the concatination of the indices $(k,\{0\}^d)$ and $\mathbf{k}$) by applying k-times (3.2) first and then $(d+1)$-times (3.1)

$$\begin{aligned}Z_q((k,\{0\}^d,\mathbf{k});\boldsymbol{\ell};x) &= Z_q((\{0\}^{d+1},\mathbf{k});(\{0\}^k,\boldsymbol{\ell});x)\\ &= Z_q(\mathbf{k},(d+1,\{0\}^{k-1};\boldsymbol{\ell});x).\end{aligned}$$

Now, set $\boldsymbol{\ell}=\emptyset$ and write an SZ-admissible index $\mathbf{k}$ in the form

$$\mathbf{k} = (k_1,\{0\}^{d_1},\dots,k_r,\{0\}^{d_r}).$$

Then we obtain the corollary by induction on r and using the above calculation in the induction step. □

With the connection of Z_q and ζ_q^{SZ} (Rem. 6 (iii)), SZ-duality follows directly:

Proof of Theorem 4. Take some SZ-admissible index $\mathbf{k}$. Using the symmetry of Z_q and setting $x=0$, the claim follows by Corollary 10:

$$\begin{aligned}\zeta_q^{\mathrm{SZ}}(\mathbf{k}) &= Z_q(\mathbf{k};\emptyset;0) = Z_q(\emptyset;\mathbf{k}^\dagger;0)\\ &= Z_q(\mathbf{k}^\dagger;\emptyset;0) = \zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger).\end{aligned}$$

□

Analogously, we are able to prove BZ-duality:

Proof of Theorem 2. For an admissible index $\mathbf{k}$ we have, using Remark 6 and Corollary 10,

$$\begin{aligned}\zeta_q^{\mathrm{BZ}}(\mathbf{k}) &= \lim_{x\to 1} Z_q(\mathbf{k}-\mathbf{1};\emptyset;x)\\ &= \lim_{x\to 1} Z_q(\emptyset;(\mathbf{k}-\mathbf{1})^\dagger;x)\\ &= \lim_{x\to 1} Z_q((\mathbf{k}-\mathbf{1})^\dagger;\emptyset;x)\\ &= \zeta_q^{\mathrm{BZ}}((\mathbf{k}-\mathbf{1})^\dagger+\mathbf{1}) = \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee).\end{aligned}$$

□

Example 11. We give a concrete example of applying transport relations step by step to make clear what happens:

$$\begin{aligned}Z_q((1,0);\emptyset;x) &= Z_q((0,0);(0);x)\\ &= Z_q((0);(1);x) = Z_q(\emptyset;(2);x)\end{aligned}$$

By Remark 6 (iii) respectively (iv), we obtain $\zeta_q^{SZ}(1,0)=\zeta_q^{SZ}(2)$ respectively $\zeta_q^{BZ}(2,1)=\zeta_q^{BZ}(3)$. We have $(1,0)^\dagger=(2)$ and $(2,1)^\vee=(3)$, why these results indeed correspond to SZ-duality resp. BZ-duality.

We derive in the following the proof of MZV-duality, Theorem 1, from BZ-duality:

Proof of Theorem 1. Let $\mathbf{k}$ be any admissible index. Denote by $\mathrm{wt}(\mathbf{k}) := k_1+\cdots+k_r$ the sum of all entries, the *weight* of $\mathbf{k}$. Obviously, one has $\mathrm{wt}(\mathbf{k})=\mathrm{wt}(\mathbf{k}^\vee)$. We have

$$\begin{aligned}\zeta(\mathbf{k}) &= \lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathbf{k})}\zeta_q^{\mathrm{BZ}}(\mathbf{k})\\ &= \lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathbf{k}^\vee)}\zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee) = \zeta(\mathbf{k}^\vee).\end{aligned}$$

□

We give in the following a proof of Theorem 3 via connected sums Z_q defined in this paper. The main point of the proof is a Taylor series expansion at $x=1$, which is under the correspondence of Z_q and Z_q^{SY} (Rem. 6 (v)) analogous to the one of $Z_q^{\mathrm{SY}}(\mathbf{k};\emptyset;y)$ at $y=0$ in [SY].

Consider in the following connected sums of the form $Z_q(\mathbf{k};\emptyset;x)$ and the related one of the form $Z_q(\emptyset;\boldsymbol{\ell};x)$ using transport relations. In both, we will develop all occurring terms as a Taylor series at $x=1$, mainly we use that for all $m\in\mathbf{N}$, we have

$$\begin{aligned}\frac{1}{1-q^m x} &= \frac{1}{1-q^m}\frac{1}{1-\frac{q^m}{1-q^m}(x-1)}\\ &= \frac{1}{1-q^m}\sum_{c\geq 0}\left(\frac{q^m}{1-q^m}\right)^c (x-1)^c\\ &= \sum_{c\geq 0}\frac{q^{mc}}{(1-q^m)^{c+1}}(x-1)^c.\end{aligned}$$

Proof of Theorem 3. Let $\mathbf{k}=(k_1,\ldots,k_r)$ be an admissible index. Then we have

$$\begin{aligned}&Z_q(\mathbf{k}-\mathbf{1};\emptyset;x)\\ &= \sum_{m_1>\cdots>m_r>0}\prod_{j=1}^r \frac{1}{1-q^{m_j}x}\frac{q^{m_j(k_j-1)}}{(1-q^{m_j})^{k_j-1}}\\ &= \sum_{m_1>\cdots>m_r>0}\prod_{j=1}^r\left(\sum_{c_j\geq 0}\frac{q^{m_jc_j+k_j-1}}{(1-q^{m_j})^{c_j+k_j}}(x-1)^{c_j}\right)\\ &= \sum_{\substack{c_1,\ldots,c_r\geq 0\\ m_1>\cdots>m_r>0}}\left(\prod_{j=1}^r\frac{q^{m_j(k_j+c_j-1)}}{(1-q^{m_j})^{k_j+c_j}}\right)(x-1)^{c_1+\cdots+c_r}\\ &= \sum_{c_1,\ldots,c_r\geq 0}\zeta_q^{\mathrm{BZ}}(\mathbf{k}+\mathbf{c})(x-1)^{|\mathbf{c}|}.\end{aligned}$$

Since $\mathbf{k}$ was an arbitrary admissible index and $\mathbf{k}^\vee$ is admissible too, we get

$$Z_q(\emptyset;\mathbf{k}^\vee-\mathbf{1};x)=\sum_{c_1,\ldots,c_{r'}\geq 0}\zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee+\mathbf{c})(x-1)^{|\mathbf{c}|},$$

with r' the depth of $\mathbf{k}^\vee$.

Now, since $Z_q(\mathbf{k}-\mathbf{1};\emptyset;x)=Z_q(\emptyset;\mathbf{k}^\vee-\mathbf{1};x)$ for every admissible index $\mathbf{k}$ by using the transport relations, the result follows by comparing the coefficient of $(x-1)^c$ on both sides. □

In the same way, we can consider $Z_q(\mathbf{k}-\mathbf{1};\emptyset;x)$ when developing $\frac{1}{1-q^m x}$ around some $a\in\mathbf{R}$, i.e.,

$$\begin{aligned}\frac{1}{1-q^m x} &= \frac{1}{1-aq^m-q^m(x-a)}\\ &= \frac{1}{1-aq^m}\frac{1}{1-\frac{q^m}{1-aq^m}(x-a)}\\ &= \sum_{c\geq 0}\frac{q^{mc}}{(1-aq^m)^{c+1}}(x-a)^c.\end{aligned}$$

Then it is

$$\begin{aligned}&Z_q(\mathbf{k};\emptyset;x)\\ &= \sum_{m_1>\cdots>m_r>0}\prod_{j=1}^r\frac{1}{1-q^{m_j}x}\frac{q^{m_jk_j}}{(1-q^{m_j})^{k_j}}\\ &= \sum_{m_1>\cdots>m_r>0}\prod_{j=1}^r\sum_{c_j\geq 0}\frac{q^{m_jc_j}}{(1-aq^{m_j})^{c_j+1}}\frac{q^{m_jk_j}}{(1-q^{m_j})^{k_j}}\\ &\quad\times(x-a)^{c_j}.\end{aligned}$$

Remark 12. The series

$$\sum_{m_1>\cdots>m_r>0}\prod_{j=1}^r\frac{q^{m_jc_j}}{(1-aq^{m_j})^{c_j+1}}\frac{q^{m_jk_j}}{(1-q^{m_j})^{k_j}}$$

for $c_1,\ldots,c_r\geq 0$, $k_1\geq 2$, $k_2,\ldots,k_r\geq 1$ and $a\in[0,1]$ can be seen as q-analog of MZVs: For $a=1$ we have seen already by proving the q-Ohno relation, how this works. For arbitrary a, it is not clear so far, whether we can prove more identities among qMZVs with this shape of the connected sum. This could be interesting for the future.

Acknowledgements. I would like to thank Kathrin Bringmann for her helpful comments on the paper. Furthermore, I thank Henrik Bachmann and Ulf Kühn for fruitful discussions and lots of comments while supervising my master thesis, of which this paper is part of. Also, I thank the referee for valuable comments.

The author has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 101001179).

References

[Bac] H. Bachmann, Multiple Eisenstein series and q-analogs of multiple zeta values, (2015). (Ph.D. Thesis).

[Bra] D. M. Bradley, Multiple q-zeta values, J. Algebra **283** (2005), no. 2, 752–798.

[Bri] B. Brindle, Dualities of q-analogues of multiple zeta values, (2015). (Master thesis, University of Hamburg). https://sites.google.com/view/benjamin-brindle/start

[EMS] K. Ebrahimi-Fard, D. Manchon and J. Singer, Duality and (q-)multiple zeta values, Adv. Math. **298** (2016), 254–285.

[GKZ] H. Gangl, M. Kaneko and D. Zagier, Double zeta values and modular forms, in *Automorphic forms and zeta functions*, World Sci. Publ., Hackensack, NJ, 2006, pp. 71–106.

[Sek] S. Seki, Connectors, RIMS Kôkyûroku **2160** (2020), 15–27.

[SY] S. Seki and S. Yamamoto, A new proof of the duality of multiple zeta values and its generalizations, Int. J. Number Theory **15** (2019), no. 6, 1261–1265.

[Sin] J. Singer, q-Analogues of Multiple Zeta Values and their application in renormalization, (2017). (Dissertation, Erlangen-Nürnberg University).

[Zag] D. Zagier, Values of zeta functions and their applications, in *First European Congress of Mathematics, Vol. II* (*Paris, 1992*), 497–512, Progr. Math., 120, Birkhäuser, Basel, 1994.

[Zha] J. Zhao, Uniform approach to double shuffle and duality relations of various q-analogs of multiple zeta values via Rota-Baxter algebras, in *Periods in quantum field theory and arithmetic*, Springer Proc. Math. Stat., 314, Springer, Cham, 2020, pp. 259–292.