Infinitely many non-uniqueness examples for Cauchy problems of the two-dimensional wave and Schrödinger equations

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Abstract: In 1963, Kumano-go presented one non-uniqueness example for the two-dimensional wave equation with a time-dependent potential. We construct infinitely many non-uniqueness examples with different wave numbers at infinity for Cauchy problems of the two-dimensional wave equation and Schrödinger equation as a generalization of the construction by Kumano-go.

Key words: Non-uniqueness for Cauchy problems; unique continuation; Cauchy problems for partial differential equations.

1. Introduction and main result. Let $B_r(0) \subset \mathbb{R}^2$ be an open ball centered at 0 with radius $r > 0$. Henceforth, all functions which appear in this paper take complex values. We consider the wave equation and Schrödinger equation with time-dependent potential $V$,

\begin{equation}
Lu + V(x, t)u = 0 \text{ in } \mathbb{R}^3,
\end{equation}

where $L$ denotes $L = \Box := \partial_t^2 - \Delta$ or $L = -i\partial_t - \Delta$. We consider non-uniqueness examples for Cauchy problems with Cauchy data on a non-characteristic surface $\partial B_r(0) \times \mathbb{R}$. Due to the time dependence on the potential $V$, we have few hopes to guarantee uniqueness for Cauchy problems in general. Indeed, we construct infinitely many examples with different wave numbers at infinity which violate uniqueness based on Kumano-go [5].

There exist infinitely many smooth functions $u \in C^\infty(\mathbb{R}^3)$ and $V \in C^\infty(\mathbb{R}^3)$ which satisfy (1.1) and

\begin{align*}
\text{supp } u &= (\mathbb{R}^3 \setminus B_r(0)) \times \mathbb{R}, \\
\text{supp } V &= (B_2(0) \setminus B_1(0)) \times \mathbb{R}.
\end{align*}

Theorem 1.1 relates to the result by Kumano-go [5]. He constructed one example for non-uniqueness when $L = \Box$ in the two-dimensional case based on John’s construction [3] using Bessel functions.

We construct infinitely many examples with different wave numbers at infinity for both wave and Schrödinger equations by generalizing the result in [5]. We remark that, in our construction, the potential $V$ is not real-valued but complex-valued function, whereas the coefficients are all real-valued with a damping term in Kumano-go’s construction [5].

2. Proof of the main result.

2.1. Preliminary. We prepare several lemmas regarding an asymptotic behavior of Bessel functions. Their proofs are all presented in section 3.

Lemma 2.1. Let $\delta \in (0, 1/2)$, $p \in (0, \frac{2(1-2\eta)}{5})$, $\lambda > 0$, and $J_\lambda$ be a Bessel function of order $\lambda$. We then have the asymptotic formula uniformly for $

J_\lambda(\lambda a) = (2\eta \lambda \tanh \alpha)^{1/2} e^{\lambda(\tanh \alpha - 1)} (1 + O(\lambda^{-2}))$

as $\lambda \to \infty$, where $\alpha > 0$ is defined by $\cos \alpha = a^{-1}(> 1)$.

Let $\delta \in (0, 1/2)$ and $p \in (0, \frac{2(1-2\eta)}{5})$ be fixed. We...
consider the following assumptions on a positive sequence \( \{\lambda_m\}_{m \in \mathbb{N}} \) with \( \lambda_m > 0 \) for all \( m \in \mathbb{N} \).

\[
(2.1) \quad \forall m \in \mathbb{N}, \quad m^2 \leq \lambda_m^p.
\]

\[
(2.2) \quad \lambda_{m+1} = \lambda_m (1 + o(1)) \text{ as } m \to \infty.
\]

We can choose infinitely many positive sequences \( \{\lambda_m\}_{m \in \mathbb{N}} \) satisfying (2.1) and (2.2), for instance,

\[
\lambda_m = a_n m^n + \sum_{j=0}^{n-1} a_j m^j,
\]

where \( n \geq \frac{p}{2} \) is a positive integer, and \( a_n \geq 1 \) and \( a_j \geq 0 \) are constants for \( j = 0, \ldots, n - 1 \).

**Lemma 2.2.** Let \( \delta \in (0, \frac{1}{2}) \) and \( p \in \left(0, \frac{2(1-2\delta)}{3}\right) \) be constants. Let \( \{\lambda_m\}_{m \in \mathbb{N}} \) be a positive sequence satisfying (2.1) and (2.2). We set

\[
G_m(r) := J_{\lambda_m}(\lambda_m r), \quad r \in [0, 1 - m^{-2}].
\]

Assume (2.1). Then, for \( \ell \geq 1 \) and \( \tau := 1 - \ell m^{-2} \), we have the asymptotic formula,

\[
G_m(r) = (1 + o(1)) \frac{\sqrt{m}}{(2\pi^{\ell})^{\frac{1}{2}} \sqrt{\lambda_m}} e^{-(1+o(1)) \frac{2\sqrt{3} \lambda_m m^{-3}}{m}}
\]

as \( m \to \infty \).

**Lemma 2.3.** Let \( \delta \in (0, \frac{1}{2}) \) and \( p \in \left(0, \frac{2(1-2\delta)}{3}\right) \) be constants. Let \( \{\lambda_m\}_{m \in \mathbb{N}} \) be a positive sequence satisfying (2.1). We set

\[
F_m(r, s) := G_m(\frac{km}{\lambda_m} r), \quad m \in \mathbb{N},
\]

where \( \{k_m\}_{m \in \mathbb{N}} \) is a positive sequence satisfying

\[
(2.3) \quad \frac{k_m}{\lambda_m} = 1 - \frac{1}{m} + O(m^{-3}) \text{ as } m \to \infty,
\]

and \( r_m(s) := 1 + s - \frac{s}{m(m+1)} \) for \( s \in [0, 1] \). Then, \( F_m \) satisfies

\[
F_m(r, s) = \frac{\sqrt{me^{-(1+o(1)) \frac{2\sqrt{3} \lambda_m m^{-3}}{m}}}}{(2\pi^{\ell})^{\frac{1}{2}} \sqrt{\lambda_m}}
\]

as \( m \to \infty \). Furthermore, we define \( \gamma_{m+1} \) such that

\[
\gamma_{m+1} := \frac{F_m(r_{m+1}(2-1))}{F_{m+1}(r_{m+1}(2-1))}.
\]

If we assume (2.2), then there exists \( M \in \mathbb{N} \) such that

\[
\gamma_{m+1} \leq e^{-\lambda_m m^{-3}}
\]

holds for all \( m > M \) and there exist \( \mu > 0 \), \( C > 0 \), and \( M \in \mathbb{N} \) such that

\[
\begin{align*}
\gamma_m F_{m+1}(r_{m+1}(s)) & \leq Ce^{-\mu \lambda_m m^{-3}} F_m(r_{m+1}(s)) \\
& \text{if } s \in [0, \frac{1}{4}], \\
F_m(r_{m+1}(s)) & \leq Ce^{-\mu \lambda_m m^{-3}} \gamma_m F_{m+1}(r_{m+1}(s)) \\
& \text{if } s \in [\frac{3}{4}, 1]
\end{align*}
\]

holds for all \( m > M \).

**2.2. Proof of Theorem 1.1.**

Proof. Let \( \delta \in (0, \frac{1}{2}) \) and \( p \in \left(0, \frac{2(1-2\delta)}{3}\right) \) be constants. Let \( \{\lambda_m\}_{m \in \mathbb{N}} \) be a positive sequence satisfying (2.1) and (2.2). We remark that (2.2) implies

\[
\lambda_m \leq e^{o(m)} \text{ as } m \to \infty.
\]

Indeed, (2.2) implies there exists sufficiently large \( m_0 \in \mathbb{N} \) such that for all \( m > m_0 \),

\[
\log \lambda_m \leq \left| \log \lambda_{m_0} \right| + 1 \sum_{m=m_0}^{m-1} \left| \log(1 + r(j)) \right|
\]

holds, where \( r(j) \) satisfies \( \lim_{j \to \infty} |r(j)| = 0 \). Hence, (2.7) follows from the well-known argument. For \( r > 1 \), we set

\[
u_m(r, \theta, t) := \begin{cases} F_m(r)e^{i(\lambda_{m} \theta - k_m t)}, & L = \square, \\ F_m(r)e^{i(\lambda_{m} \theta - k_m t)}, & L = -i\partial_{\theta} - \Delta, \end{cases}
\]

where \( \{k_m\}_{m \in \mathbb{N}} \) is a positive sequence satisfying (2.3) and \((r, \theta)\) is the polar coordinate in \( \mathbb{R}^2 \). By (2.4), we obtain

\[
Lu_m = 0,
\]

because the Laplace operator \( \Delta \) is written by the polar coordinate,

\[
\Delta = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_{\theta}^2.
\]

We define closed intervals \( I_m \) and \( I_{m,j} \subset I_m \) for \( m \in \mathbb{N} \) and \( j = 1, 2, 3, 4 \) as

\[
I_m := \left[ 1 + \frac{1}{m+1}, 1 + \frac{1}{m} \right]
\]

and

\[
I_{m,j} := \left[ 1 + \frac{1}{m+1} - \frac{j}{4m(m+1)}, 1 + \frac{1}{m+1} - \frac{j-1}{4m(m+1)} \right].
\]

For sufficiently large \( M \in \mathbb{N} \) and \( m > M \), we define smooth functions

\[
A_M(r) := \begin{cases} 1, & r \geq 1 + \frac{1}{M+2} + \frac{1}{(M+1)(M+2)}, \\ 0, & 0 \leq r \leq 1 + \frac{1}{M+2}. \end{cases}
\]

and

\[
A_m(r) := \begin{cases} 1, & r \in (I_{m+1} \setminus I_{m+1,4}) \cup (I_m \setminus I_{m,1}), \\ 0, & r \in [0, 1 + \frac{1}{m+1}] \cup (1 + \frac{1}{m}, \infty). \end{cases}
\]

We define \( u = u(r, \theta, t) \) as
\[
\begin{align*}
u(r, \theta, t) := A_M(r)u_M \\
+ \sum_{m=M+1}^{\infty} \gamma_{M+1} \times \cdots \times \gamma_m A_m(r)u_m
\end{align*}
\]
and set
\[
K := [0, 1] \cup \left[ 1 + \frac{1}{M+1}, \infty \right) \cup \bigcup_{m=M+1}^{\infty} (I_{m,2} \cup I_{m,3})
\]
and
\[
V(r, \theta, t) := \left\{ \begin{array}{ll}
0, & r \in K, \\
-\frac{Lu}{u}, & r \in [0, \infty) \setminus K.
\end{array} \right.
\]
Using the chain rule, (2.5), and (2.7), we obtain
\[
\left| \frac{d^r}{dr^s} F_m(r) \right| = m^s (m+1)^f \left| \frac{d^r}{dr^s} F_m(r_m(s)) \right|
\leq C \frac{m^s (m+1)^f \lambda_m^s}{m^{3s}} (1 + o(1)) \frac{\sqrt{m}}{\sqrt{\lambda_m}} 
\times e^{-(1+o(1)) \frac{1}{2} (1+s)^2 \lambda_m m^{-3}} 
\leq C e^{o(m)} F_m(r_m(s))
\]
for \( r \in I_m, \ell \in Z_{\geq 0} \). Indeed, by (2.7), it follows that
\[
\frac{m^s (m+1)^f \lambda_m^s}{m^{3s}} \leq C e^{o(m)}.
\]
For \( r \in I_{m+1}, \) using \( r_{m+1}(s) = r_m(1+s + O(m^{-1})) \), we also have
\[
\left| \frac{d^r}{dr^s} F_m(r) \right| = (m+1)^f (m+2)^f \left| \frac{d^r}{dr^s} F_m(r_m(1+s + O(m^{-1}))) \right|
\leq C \frac{(m+1)^f (m+2)^f \lambda_m^s}{m^{3s}} (1 + o(1)) \frac{\sqrt{m}}{\sqrt{\lambda_m}} 
\times e^{-(1+o(1)) \frac{1}{2} (2+s+O(m^{-1}))^2 \lambda_m m^{-3}}
\leq C e^{o(m)} F_m(r_m(1+s)).
\]
Hence, it follows that for \( r \in I_m \cup I_{m+1} \) and \( \ell \in Z_{\geq 0} \),
\[
\left| \frac{d^r}{dr^s} F_m(r) \right| \leq C e^{o(m)} F_m(r).
\]
On \( I_{m+1} \), using (2.8), (2.3), (2.7), and (2.1), we then have
\[
(2.9) \quad |\partial^\beta u(r, \theta, t)| := \sum_{|\beta| = \ell} (\partial_\theta \partial_{\theta})^\beta u(r, \theta, t)
\leq C e^{\gamma_{M+1} \times \cdots \times \gamma_m \lambda_m^2 e^{o(m)} (F_m + \gamma_m F_{m+1})}
\leq C e^{-\lambda_m m^{-3} e^{o(m)}}
\leq C e^{-m^2 (1+o(m^{-1}))}
\leq C e^{-\frac{t}{m^2}}
\]
\leq C e^{-\frac{t}{m^2}}
\]
where we used the estimate obtained by (2.1),
\[
(2.10) \quad \lambda_m m^{-3} \geq m^{3-2} > m^2.
\]
We thus proved \( u \) is smooth in \( R^3 \).

On \( I_{m+1,1} \), since
\[
(2.11) \quad |u| \geq \gamma_{M+1} \times \cdots \times \gamma_m (|u_m| - \gamma_m |u_{m+1}|)
= \gamma_{M+1} \times \cdots \times \gamma_m (F_m(r_{m+1}(s)) - \gamma_m F_{m+1}(r_{m+1}(s)))
\geq \gamma_{M+1} \times \cdots \times \gamma_m (1 - C e^{-\mu \lambda_m m^{-3}})
\times F_m(r_{m+1}(s))
\geq \gamma_{M+1} \times \cdots \times \gamma_m (1 - C e^{-\mu m^2})
\times F_m(r_{m+1}(s)) > 0
\]
for \( s \in [0, \frac{1}{2}] \) by (2.6) and (2.10), \( |u| > 0 \) on \( I_{m+1,1} \).

Similarly, on \( I_{m+1,4} \), since
\[
(2.12) \quad |u| \geq \gamma_{M+1} \times \cdots \times \gamma_m (\gamma_m |u_m| - |u_m|)
= \gamma_{M+1} \times \cdots \times \gamma_m (\gamma_m F_{m+1}(r_{m+1}(s)) - F_m(r_{m+1}(s)))
\geq \gamma_{M+1} \times \cdots \times \gamma_m (1 - C e^{-\mu \lambda_m m^{-3}})
\times F_m(r_{m+1}(s))
\geq \gamma_{M+1} \times \cdots \times \gamma_m (1 - C e^{-\mu m^2})
\times F_m(r_{m+1}(s)) > 0
\]
for \( s \in [\frac{1}{2}, 1] \), we have \( |u| > 0 \) on \( I_{m+1,4} \). By the definition of \( u \), since \( Lu = 0 \) on \( I_{m+1,2} \cup I_{m+1,3} \), \( V \) is smooth when \( r \in (1, \infty) \).

Finally, we prove \( V \) is smooth at \( r = 1 \). On \( I_{m+1,1} \), since \( Lu = L[\gamma_{M+1} \times \cdots \times \gamma_m A_m u_m] \),
\[
(2.13) \quad |\partial^\beta L u| \leq C e^{\gamma_{M+1} \times \cdots \times \gamma_m 2^{\ell+1} e^{o(m)}}
\times \frac{F_m(r_{m+1}(s))}{F_m}
\]
holds for \( |\beta| = \ell \in Z_{\geq 0} \) by (2.8). We thus have
\[
|\partial^\beta V(r, \theta, t)| = |\partial^\beta (u^{-1} L u)|
\leq \sum_{|\beta| = \ell} \left( \frac{\beta!}{\beta_1!} \right) |\partial^\beta (u^{-1})\partial^{3-\beta_1} (L u)|
\leq C \left( \frac{\gamma_{m+1} F_{m+1}}{F_m} \right)^{\ell} \lambda_m^{\ell+1} e^{o(m)}\left(1 + \frac{\gamma_m F_{m+1}}{F_m}\right)^{\ell}
\leq C e^{-\mu \lambda_m m^{-3} + o(m)}
\leq C e^{-\frac{t}{m^2}}
\]
by (2.9), (2.11), (2.13), (2.6), (2.7), and (2.10) for \( |\beta| = \ell \in Z_{\geq 0} \). Similarly on \( I_{m+1,4} \), since \( Lu = L[\gamma_{M+1} \times \cdots \times \gamma_m A_m u_m] \),
holds for \( |\beta| = \ell \in \mathbb{Z}_{\geq 0} \) by (2.8). We thus have
\[
|\partial^3 V(r, \theta, t)| = |\partial^3 (u^{-1} L u)|
\]
\[
= \left| \sum_{|\beta| \leq \ell} \left( \frac{\beta}{\beta_1} \right) \partial^3 (u^{-1}) \partial^{3-\beta} (L u) \right|
\]
\[
\leq C \left( \frac{F_m}{\gamma_m + 1} \right) \lambda^{2(\ell+1)} e^{\alpha(m)} \left( 1 + \frac{F_m}{\gamma_m + 1} \right)^{2-\ell} \cdot \frac{x}{\alpha}.
\]
\[
\leq C e^{-\frac{\pi m^2}{\alpha^2}}
\]
by (2.9), (2.12), (2.14), (2.6), (2.7), and (2.10) for \( |\beta| = \ell \in \mathbb{Z}_{\geq 0} \).

Thus, for all \( |\beta| = \ell \in \mathbb{Z}_{\geq 0} \) on \( \Gamma_{m+1} \),
\[
|\partial^3 V(r, \theta, t)| \leq C e^{-\frac{\pi m^2}{\alpha^2}} \leq C e^{-\frac{\pi x}{\alpha}}
\]
holds.

3. Proofs of the lemmas.
Proof of Lemma 2.1. We remark that
\[
1 \geq \tanh \alpha = \sqrt{1 - \alpha^2} \geq \lambda^{\frac{p}{q}} \tag{3.1}
\]
a \in (0, 1 - \lambda^{-p}].

We use the Schl"afli’s integral formula of a Bessel function,
\[
J_\alpha(\lambda \alpha) = \frac{1}{2\pi} \int_{\Gamma_0} e^{\lambda [-i \alpha \sin z + iz]} dz
\]
where \( \Gamma_0 \) consists of three sides of rectangle with vertexes at \(-\pi + i\alpha, -\pi, \pi + i\alpha, \) and \( -\pi + i\alpha \) is oriented from \(-\pi + i\alpha \) to \( \pi + i\alpha \). We set
\[
f(z) := -i \alpha \sin z + iz
\]
\[
f(z) := a \cos z \sin y - y + i(-x - a \sin x) \cos y
\]
where \( z = x + iy \). By the Cauchy’s integral theorem, we can deform \( \Gamma_0 \) into a curve defined by \( \Gamma \) on which \( x - a \sin x \cos y = 0 \). Hence, we obtain
\[
J_\alpha(\lambda \alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda g(x)} dx
\]
where \( g \) is defined by
\[
g(x) := a \cos x \sinh y(x) - y(x)
\]
and \( y \) satisfies
\[
cosh y(x) = \frac{x}{a \sin x}
\]
\[
y(x) = \log \left( \frac{x}{a \sin x} + \frac{\sqrt{x^2 + a^2 \sin^2 x}}{1} \right)
\]
for \( x \in (-\pi, \pi) \), where \( y(0) = 0 \) is well-defined owing to \( a < 1 \).

First, we evaluate \( g \) in an interval \([-\lambda^{-q}, \lambda^{-q}]\), where \( q \) satisfying
\[
0 < q < \frac{2 - p}{4}
\]
is determined later. Since there exists a constant \( C > 0 \) such that
\[
|y'(x)| = \left| \frac{1}{\sinh y} \frac{d}{dx} \left( \frac{x}{a \sin x} \right) \right| \leq \frac{C}{\sqrt{\frac{2}{1 - \alpha^2}} - 1} \frac{|x|}{\alpha}
\]
by (3.1), we have for \( x \in [-\lambda^{-q}, \lambda^{-q}] \),
\[
|y(x) - \alpha| = |y(x) - y(0)| \leq C \lambda^{-q} \frac{|x|}{\alpha} \leq C \lambda^{-2q}.
\]
Hence, the Taylor’s theorem yields
\[
g(x) = f(x + iy(x)) = f(x + iy(y - \alpha) + i\alpha)
\]
\[
= f(x) + (x + i(y - \alpha)) (f'(x) + i f''(x) + \frac{(x + i(y - \alpha))^3}{2} f'''(x) + \frac{(x + i(y - \alpha))^3}{2} f''''(x))\]
\[
\times f''(x + iy(y - \alpha)) dx + O(\lambda^{2 - q})
\]
since \( f'(x) = 0 \), \( |f''(x + iy(y - \alpha))| \leq C \) for some \( C > 0 \), and
\[
q := \frac{1}{3} \left( 1 + \delta + \frac{p}{2} \right).
\]
We remark that (3.2) is equivalent to \( p < 2(1 - 3\delta) \). Consequently, we have
\[
\int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\lambda g(x)} dx = e^{\lambda \tanh \alpha} \int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\frac{x}{2 \sinh^2 \frac{1}{\lambda} \tanh \alpha \tanh \lambda}} dx \cdot e^{O(\lambda^{-q})}
\]
\[
eq \frac{e^{\lambda \tanh \alpha}}{\sqrt{\lambda \tanh \alpha}} \left( \int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\frac{x^2}{2 \sinh^2 \frac{1}{\lambda} \tanh \alpha \tanh \lambda}} dx \right) \times \left( 1 + O(\lambda^{-q}) \right)
\]
\[
= \frac{\sqrt{2\pi} e^{\lambda \tanh \alpha}}{\sqrt{\lambda \tanh \alpha}} \left( 1 + O(\lambda^{-q}) \right)
\]
since \( \lambda^{\frac{1}{2 + q}} \sqrt{\tanh \alpha} \geq \lambda^{\frac{1}{2 + q}} \geq \lambda^{\frac{1}{2 + q} - q} \) by (3.1) and
\[
\lambda^{\frac{1}{2 + q} - q} \frac{x^2}{2 \sinh^2 \frac{1}{\lambda} \tanh \alpha \tanh \lambda} \rightarrow 0
\]
by (3.2). Hence, we have
\[
\frac{1}{2\pi} \int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\lambda g(x)} dx = \frac{e^{\lambda \tanh \alpha}}{\sqrt{2\pi \lambda \tanh \alpha}} \left( 1 + O(\lambda^{-q}) \right)
\]
Second, we evaluate \( g \) in \(( -\pi, \pi ) \setminus [-\lambda^{-q}, \lambda^{-q}] \).
Because \( \mp g'(x) \geq 0 \) when \( 0 \leq \mp x < \pi \), it follows from (3.3) and (3.1),
\[ \frac{1}{2\pi} \left( \int_{-\pi}^{-\lambda^{-q}} + \int_{\lambda^{-q}}^{\pi} \right) e^{\lambda g(x)} dx = O(\lambda^{-\delta}) \frac{e^{\lambda(\tanh \alpha - \alpha)}}{\sqrt{2\pi \lambda \tanh \alpha}}. \]

In fact, by our assumption (3.1) and (3.2),
\[
\lambda^\delta \left| \frac{1}{2\pi} \left( \int_{-\pi}^{-\lambda^{-q}} + \int_{\lambda^{-q}}^{\pi} \right) e^{\lambda g(x)} dx \right| \leq \sqrt{\frac{\tanh \alpha}{2\pi}} \lambda^{\delta + \frac{1}{2} - \frac{1}{2} \lambda^{-q}} e^{\lambda(\tanh \alpha)} \leq \sqrt{2\pi \lambda^{\delta + \frac{1}{2}} e^{-\frac{1}{2} \lambda^{-q}}} e^{O(\lambda^{-q})} \lambda^{\lambda - \infty} = 0.
\]

holds. We complete the proof. \(\square\)

Proof of Lemma 2.2. In Lemma 2.1, taking \(a = r = 1 - \ell m^{-2}\) for \(\ell \geq 1\), which is done by our assumption (2.1), yields
\[
G_m(r) = \frac{h(r)}{\sqrt{2\pi(1 - r^2)^{\frac{1}{2}}} \sqrt{\lambda_m}} (1 + o(1)) \text{ as } m \to \infty,
\]
where
\[
h(r) := \frac{r e^{\sqrt{1 - r^2}}}{1 + \sqrt{1 - r^2}},
\]
since \(e^{-\alpha} = \frac{r}{1 + \sqrt{1 - r^2}}\) for \(\alpha > 0\). Because simple calculations yield
\[
h'(r) = \frac{1 - \sqrt{1 - r^2}}{r^2} \sqrt{1 - e^{2\sqrt{1 - r^2}}} = (1 + o(1)) \sqrt{2\sqrt{1 - r}} \text{ as } r \not\to 1,
\]
we have
\[
h(r) = h(1) - \int_r^1 h'(s) ds = 1 - (1 + o(1)) \frac{2\sqrt{2}}{3} (1 - r)^{\frac{3}{2}}\text{ as } r \not\to 1.
\]

Hence, for \(\ell \geq 1\) and \(r = 1 - \ell m^{-2}\), we obtain
\[
G_m(r) = (1 + o(1)) \frac{(1 - (1 + o(1)) \frac{2\sqrt{2}}{3} (1 - \ell m^{-2})^{\frac{3}{2}} \lambda_m}{(2\pi)^{\frac{1}{2}}} \sqrt{\lambda_m} \left( 1 - (1 - \ell m^{-2})^2 \right)^{\frac{3}{2}} = (1 + o(1)) \frac{\sqrt{m}}{(2\pi^2)^{\frac{1}{2}}} e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda_m m^{-3}} \text{ as } m \to \infty.
\]

As \(m \to \infty\). The last equality comes from
\[
\frac{(1 - (1 + o(1)) \frac{2\sqrt{2}}{3} (1 - \ell m^{-2})^{\frac{3}{2}} \lambda_m}{(2\pi)^{\frac{1}{2}}} \sqrt{\lambda_m} \left( 1 - (1 - \ell m^{-2})^2 \right)^{\frac{3}{2}} = \frac{\sqrt{m}}{(2\pi^2)^{\frac{1}{2}}} e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda_m m^{-3}} \text{ as } m \to \infty.
\]

since (2.1) implies
\[
\lambda m m^{-3} \geq m^{\frac{3}{2} - 3} > m^2 \to \infty.
\]

Proof of Lemma 2.3. (2.4) is obtained by the definition. Since
\[
k_m r_m(s) = 1 - \frac{1}{1 + s + O(m^{-1})} \to m \to \infty
\]
by our assumption (2.3), we obtain
\[
F_m r_m(s) = G_m \left( \frac{k_m r_m(s)}{\lambda m} \right) = \frac{1 + s + O(m^{-1})}{m^2} = \frac{(1 + o(1)) \sqrt{m} e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} (1 + s + O(m^{-1})) \lambda m m^{-3}}}{(2\pi^2 (1 + s + O(m^{-1}))^{\frac{1}{2}} \sqrt{\lambda m}}
\]
\[
\text{as } m \to \infty. \text{ The last equality comes from } \frac{(2\pi^2 (1 + s + O(m^{-1}))^{\frac{1}{2}} \sqrt{\lambda m}}{\sqrt{m} e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} (1 + s + O(m^{-1})) \lambda m m^{-3}} = \frac{(2\pi^2 (1 + s))^{\frac{1}{2}} \sqrt{\lambda m}}{(2\pi^2 (1 + s))^{\frac{1}{2}} \sqrt{\lambda m}} \text{ as } m \to \infty.
\]

Furthermore, since \(r_{m+1}(s) = r_m(1 + s + O(m^{-1}))\)
\[
as m \to \infty,
\]
\[
\gamma_{m+1} = \frac{F_m r_m(s)}{F_m r_m(1 + s)} = (1 + o(1)) \left( \frac{\frac{1}{2} + O(m^{-1})}{1 + \frac{1}{m}} \right)^{\frac{1}{2}} \sqrt{\lambda_{m+1} m m^{-3}} = (1 + o(1)) \left( \frac{\frac{1}{2} + O(m^{-1})}{1 + \frac{1}{m}} \right)^{\frac{1}{2}} \sqrt{\lambda_{m+1} m m^{-3}}
\]
\[
\leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}} \leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}} \leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}} \leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}} \leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}}
\]

where we use our assumption (2.2) and the mean value theorem such that \(x^2 - y^2 = \frac{\theta}{2} \sqrt{\theta(x - y)}\) for \(0 \leq y \leq \theta \leq x\), holds. Hence, we have, by the above estimate,
\[
\gamma_{m+1} \leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}} \leq e^{-\frac{1}{2}(1 + o(1)) \frac{2\sqrt{2}}{3} \lambda m m^{-3}}
\]
for sufficiently large \(m \in \mathbb{N}\).

Finally, we have by the definition of \(\gamma_{m+1}\),
as \( m \to \infty \). When \( 0 \leq s \leq \frac{1}{2} \), there exist constants \( C > 0 \) and \( \theta \) satisfying \( 2 + s + O(m^{-1}) \leq \theta \leq \frac{3}{2} + O(m^{-1}) \) such that

\[
\frac{F_m(r_m(1 + s + O(m^{-1})))}{F_m\left(r_m\left( \frac{3}{2} + O(m^{-1}) \right) \right)} \\
\geq C e^{(1+o(1))\sqrt{4+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}} \\
\geq C e^{-3(1+o(1))\sqrt{1+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}}.
\]

Furthermore, there exists \( \theta \) satisfying \( 1 + s \leq \theta \leq \frac{3}{2} \) such that

\[
\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(1))} \\
\geq C e^{-(1+o(1))\sqrt{1+O(m^{-1})}\sqrt{1-s}\lambda_m (m+1)^{-3}} \\
\geq C e^{-(1+o(1))\sqrt{1+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}}.
\]

by (2.2). There then exists \( \mu > 0 \) such that

\[
\frac{F_m(r_m(1))}{\gamma_{m+1}F_m(r_m(1))} \\
\leq C e^{-(1+o(1))\sqrt{3+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}} \\
\leq C e^{-(1+o(1))\sqrt{3+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}}.
\]

for sufficiently large \( m \in \mathbb{N} \). Moreover, when \( s \in \left[\frac{3}{2}, 1\right] \), there exist constants \( C > 0 \) and \( \theta \) satisfying \( \frac{5}{2} + O(m^{-1}) \) \( \leq \theta \leq 2 + s + O(m^{-1}) \) such that

\[
\frac{F_m(r_m(1 + s + O(m^{-1})))}{F_m\left(r_m\left( \frac{5}{2} + O(m^{-1}) \right) \right)} \\
\leq C e^{-(1+o(1))\sqrt{5+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}} \\
\leq C e^{-(1+o(1))\sqrt{5+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}}.
\]

Furthermore, there exists \( \theta \) satisfying \( \frac{3}{2} \leq \theta \leq 1 + s \) such that

\[
\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(1))} \\
\leq C e^{(1+o(1))\sqrt{5+O(m^{-1})}\sqrt{1-s}\lambda_m (m+1)^{-3}} \\
\leq C e^{(1+o(1))\sqrt{5+O(m^{-1})}\sqrt{1-s}\lambda_m m^{-3}}.
\]

\[\Box\]

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