Infinitely many non-uniqueness examples for Cauchy problems of the two-dimensional wave and Schrödinger equations

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Abstract: In 1963, Kumano-go presented one non-uniqueness example for the two-dimensional wave equation with a time-dependent potential. We construct infinitely many non-uniqueness examples with different wave numbers at infinity for Cauchy problems of the two-dimensional wave equation and Schrödinger equation as a generalization of the construction by Kumano-go.

Key words: Non-uniqueness for Cauchy problems; unique continuation; Cauchy problems for partial differential equations.

1. Introduction and main result. Let $B_r(0) \subset \mathbf{R}^2$ be an open ball centered at 0 with radius $r > 0$. Henceforth, all functions which appear in this paper take complex values. We consider the wave equation and Schrödinger equation with time-dependent potential V,

$$Lu + V(x,t)u = 0 \text{ in } \mathbf{R}^3, \tag{1.1}$$

where L denotes $L = \Box := \partial_t^2 - \Delta$ or $L = -i\partial_t - \Delta$. We consider non-uniqueness examples for Cauchy problems with Cauchy data on a non-characteristic surface $\partial B_1(0) \times \mathbf{R}$. Due to the time dependence on the potential V, we have few hopes to guarantee uniqueness for Cauchy problems in general. Indeed, we construct infinitely many examples with different wave numbers at infinity which violate uniqueness based on Kumano-go [5]. In regard to the uniqueness theorems for the wave equation, Schrödinger equation, and more general partial differential equations with variable coefficients, readers are referred to [7] and [6]. They proved uniqueness results by assuming some analyticities on coefficients partially. Readers are also referred to [4, Chapter 2.5] and [2, Chapter 3]. Alinhac and Baouendi [1] constructed non-uniqueness examples for Cauchy problems of general partial differential equations by using geometric optics. We state our main result.

Theorem 1.1. *Let $L = \Box$ or $L = -i\partial_t - \Delta$. There exist infinitely many smooth functions $u \in C^\infty(\mathbf{R}^3)$ and $V \in C^\infty(\mathbf{R}^3)$ which satisfy* (1.1) *and*

$$\operatorname{supp} u = (\mathbf{R}^2 \setminus B_1(0)) \times \mathbf{R},$$
$$\operatorname{supp} V \subset (B_2(0) \setminus B_1(0)) \times \mathbf{R}.$$

Theorem 1.1 relates to the result by Kumano-go [5]. He constructed one example for non-uniqueness when $L = \Box$ in the two-dimensional case based on John's construction [3] using Bessel functions. We construct infinitely many examples with different wave numbers at infinity for both wave and Schrödinger equations by generalizing the result in [5]. We remark that, in our construction, the potential V is not real-valued but complex-valued function, whereas the coefficients are all real-valued with a damping term in Kumano-go's construction [5].

2. Proof of the main result.

2.1. Preliminary. We prepare several lemmas regarding an asymptotic behavior of Bessel functions. Their proofs are all presented in section 3.

Lemma 2.1. *Let $\delta \in (0, \frac{1}{2})$, $p \in (0, \frac{2(1-2\delta)}{5})$, $\lambda > 0$, and J_λ be a Bessel function of order λ. We then have the asymptotic formula uniformly for $a \in (0, 1 - \lambda^{-p}]$,*

$$J_\lambda(\lambda a) = (2\pi\lambda \tanh\alpha)^{-\frac{1}{2}} e^{\lambda(\tanh\alpha - \alpha)}(1 + O(\lambda^{-\delta}))$$

as $\lambda \to \infty$, where $\alpha > 0$ is defined by $\cosh\alpha = a^{-1} (> 1)$.

Let $\delta \in (0, \frac{1}{2})$ and $p \in (0, \frac{2(1-2\delta)}{5})$ be fixed. We

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consider the following assumptions on a positive sequence $\{\lambda_m\}_{m\in\mathbf{N}}$ with $\lambda_m > 0$ for all $m \in \mathbf{N}$.

$$\forall m \in \mathbf{N},\ m^2 \le \lambda_m^p. \tag{2.1}$$

$$\lambda_{m+1} = \lambda_m(1 + o(1)) \text{ as } m \to \infty. \tag{2.2}$$

We can choose infinitely many positive sequences $\{\lambda_m\}_{m\in\mathbf{N}}$ satisfying (2.1) and (2.2), for instance,

$$\lambda_m = a_n m^n + \sum_{j=0}^{n-1} a_j m^j,$$

where $n \ge \frac{2}{p}$ is a positive integer, and $a_n \ge 1$ and $a_j \ge 0$ are constants for $j = 0, \dots, n-1$.

Lemma 2.2. *Let* $\delta \in (0, \frac{1}{2})$ *and* $p \in (0, \frac{2(1-2\delta)}{5})$ *be constants and* $\{\lambda_m\}_{m\in\mathbf{N}}$ *be a positive sequence. Set*

$$G_m(r) := J_{\lambda_m}(\lambda_m r),\ r \in [0, 1 - m^{-2}].$$

Assume (2.1). *Then, for* $\ell \ge 1$ *and* $r := 1 - \ell m^{-2}$, *we have the asymptotic formula,*

$$G_m(r) = (1 + o(1)) \frac{\sqrt{m}}{(2\pi^2\ell)^{\frac{1}{4}}\sqrt{\lambda_m}} e^{-(1+o(1))\frac{2\sqrt{2}}{3}\ell^{\frac{3}{2}}\lambda_m m^{-3}}$$

as $m \to \infty$.

Lemma 2.3. *Let* $\delta \in (0, \frac{1}{2})$ *and* $p \in (0, \frac{2(1-2\delta)}{5})$ *be constants and let* $\{\lambda_m\}_{m\in\mathbf{N}}$ *be a positive sequence satisfying* (2.1). *We set*

$$F_m(r) := G_m\left(\frac{k_m}{\lambda_m} r\right),\ m \in \mathbf{N},$$

where $\{k_m\}_{m\in\mathbf{N}}$ *is a positive sequence satisfying*

$$\frac{k_m}{\lambda_m} = 1 - \frac{1}{m} + O(m^{-3}) \text{ as } m \to \infty, \tag{2.3}$$

and $r_m(s) := 1 + \frac{1}{m} - \frac{s}{m(m+1)}$ *for* $s \in [0, 1]$. *Then,* F_m *satisfies*

$$F_m''(r) + \frac{1}{r} F_m'(r) + \left(k_m^2 - \frac{\lambda_m^2}{r^2}\right) F_m(r) = 0 \tag{2.4}$$

and for $s \in [0, 1]$,

$$\begin{aligned} &F_m(r_m(s)) \\ &= (1 + o(1)) \frac{\sqrt{m} e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}}\sqrt{\lambda_m}} \end{aligned} \tag{2.5}$$

as $m \to \infty$. *Furthermore, we define* γ_{m+1} *such that*

$$\gamma_{m+1} := \frac{F_m(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(2^{-1}))}.$$

If we assume (2.2), *then there exists* $M \in \mathbf{N}$ *such that*

$$\gamma_{m+1} \le e^{-\lambda_m m^{-3}}$$

holds for all $m > M$ *and there exist* $\mu > 0$, $C > 0$, *and* $M \in \mathbf{N}$ *such that*

$$\begin{cases} \gamma_{m+1} F_{m+1}(r_{m+1}(s)) \le C e^{-\mu\lambda_m m^{-3}} F_m(r_{m+1}(s)) \\ \text{if } s \in [0, \frac{1}{4}], \\ F_m(r_{m+1}(s)) \le C e^{-\mu\lambda_m m^{-3}} \gamma_{m+1} F_{m+1}(r_{m+1}(s)) \\ \text{if } s \in [\frac{3}{4}, 1] \end{cases} \tag{2.6}$$

holds for all $m > M$.

2.2. Proof of Theorem 1.1.

Proof. Let $\delta \in (0, \frac{1}{2})$ and $p \in (0, \frac{2(1-2\delta)}{5})$ be constants. Let $\{\lambda_m\}_{m\in\mathbf{N}}$ be a positive sequence satisfying (2.1) and (2.2). We remark that (2.2) implies

$$\lambda_m \le e^{o(m)} \text{ as } m \to \infty. \tag{2.7}$$

Indeed, (2.2) implies there exists sufficiently large $m_0 \in \mathbf{N}$ such that for all $m > m_0$,

$$\left|\frac{\log \lambda_m}{m}\right| \le \left|\frac{\log \lambda_{m_0}}{m}\right| + \frac{1}{m}\sum_{j=m_0}^{m-1} |\log(1 + r(j))|$$

holds, where $r(j)$ satisfies $\lim_{j\to\infty} |r(j)| = 0$. Hence, (2.7) follows from the well-known argument. For $r > 1$, we set

$$u_m(r, \theta, t) := \begin{cases} F_m(r) e^{i(\lambda_m\theta + k_m t)}, & L = \Box, \\ F_m(r) e^{i(\lambda_m\theta - k_m^2 t)}, & L = -i\partial_t - \Delta, \end{cases}$$

where $\{k_m\}_{m\in\mathbf{N}}$ is a positive sequence satisfying (2.3) and (r, θ) is the polar coordinate in $\mathbf{R}^2$. By (2.4), we obtain

$$Lu_m = 0,$$

because the Laplace operator Δ is written by the polar coordinate,

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2.$$

We define closed intervals I_m and $I_{m,j} \subset I_m$ for $m \in \mathbf{N}$ and $j = 1, 2, 3, 4$ as

$$I_m := \left[1 + \frac{1}{m+1}, 1 + \frac{1}{m}\right]$$

and

$$I_{m,j} := \left[1 + \frac{1}{m} - \frac{j}{4m(m+1)}, 1 + \frac{1}{m} - \frac{j-1}{4m(m+1)}\right].$$

For sufficiently large $M \in \mathbf{N}$ and $m > M$, we define smooth functions

$$A_M(r) := \begin{cases} 1, & r \ge 1 + \frac{1}{M+2} + \frac{1}{4(M+1)(M+2)}, \\ 0, & 0 \le r \le 1 + \frac{1}{M+2}, \end{cases}$$

and

$$A_m(r) := \begin{cases} 1, & r \in (I_{m+1} \setminus I_{m+1,4}) \cup (I_m \setminus I_{m,1}), \\ 0, & r \in [0, 1 + \frac{1}{m+2}] \cup (1 + \frac{1}{m}, \infty). \end{cases}$$

We define $u = u(r, \theta, t)$ as

$$u(r,\theta,t) := A_M(r)u_M + \sum_{m=M+1}^{\infty} \gamma_{M+1}\times\cdots\times\gamma_m A_m(r)u_m$$

and set

$$K := [0,1]\cup\left[1+\frac{1}{M+1},\infty\right)\cup\bigcup_{m=M+1}^{\infty}(I_{m,2}\cup I_{m,3})$$

and

$$V(r,\theta,t) := \begin{cases} 0, & r\in K,\\ -\dfrac{Lu}{u}, & r\in[0,\infty)\setminus K.\end{cases}$$

Using the chain rule, (2.5), and (2.7), we obtain

$$\begin{aligned}\left|\frac{d^\ell}{dr^\ell}F_m(r)\right| &= m^\ell(m+1)^\ell\left|\frac{d^\ell}{ds^\ell}F_m(r_m(s))\right|\\ &\le C_\ell\frac{m^\ell(m+1)^\ell\lambda_m^\ell}{m^{3\ell}}(1+o(1))\frac{\sqrt{m}}{\sqrt{\lambda_m}}\\ &\quad\times e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}}\lambda_m m^{-3}}\\ &\le C_\ell e^{o(m)}F_m(r_m(s))\end{aligned}$$

for $r\in I_m$, $\ell\in\mathbf{Z}_{\ge0}$. Indeed, by (2.7), it follows that

$$\frac{m^\ell(m+1)^\ell\lambda_m^\ell}{m^{3\ell}}\le C_\ell e^{o(m)}.$$

For $r\in I_{m+1}$, using $r_{m+1}(s)=r_m(1+s+O(m^{-1}))$, we also have

$$\begin{aligned}\left|\frac{d^\ell}{dr^\ell}F_m(r)\right| &= (m+1)^\ell(m+2)^\ell\left|\frac{d^\ell}{ds^\ell}F_m(r_{m+1}(s))\right|\\ &= (m+1)^\ell(m+2)^\ell\left|\frac{d^\ell}{ds^\ell}F_m(r_m(1+s+O(m^{-1})))\right|\\ &\le C_\ell\frac{(m+1)^\ell(m+2)^\ell\lambda_m^\ell}{m^{3\ell}}(1+o(1))\frac{\sqrt{m}}{\sqrt{\lambda_m}}\\ &\quad\times e^{-(1+o(1))\frac{2\sqrt{2}}{3}(2+s+O(m^{-1}))^{\frac{3}{2}}\lambda_m m^{-3}}\\ &\le C_\ell e^{o(m)}F_m(r_{m+1}(s)).\end{aligned}$$

Hence, it follows that for $r\in I_m\cup I_{m+1}$ and $\ell\in\mathbf{Z}_{\ge0}$,

$$\left|\frac{d^\ell}{dr^\ell}F_m(r)\right|\le C_\ell e^{o(m)}F_m(r). \tag{2.8}$$

On I_{m+1}, using (2.8), (2.3), (2.7), and (2.1), we then have

$$\begin{aligned}|\partial^\beta u(r,\theta,t)| &:= \left|\sum_{|\beta|=\ell}(\partial_r\partial_\theta\partial_t)^\beta u(r,\theta,t)\right|\\ &\le C_\ell\gamma_{M+1}\times\cdots\times\gamma_m\lambda_m^{2\ell}e^{o(m)}(F_m+\gamma_{m+1}F_{m+1})\\ &\le C_\ell e^{-\lambda_m m^{-3}}e^{o(m)}\\ &< C_\ell e^{-m^2(1+o(m^{-1}))}\\ &\le C_\ell e^{-\frac{1}{2}m^2}\\ &\le C_\ell e^{-\frac{1}{2}((r-1)^{-1}-2)^2}\xrightarrow{r\searrow1}0,\end{aligned} \tag{2.9}$$

where we used the estimate obtained by (2.1),

$$\lambda_m m^{-3}\ge m^{\frac{2}{p}-3}>m^2. \tag{2.10}$$

We thus proved u is smooth in $\mathbf{R}^3$.

On $I_{m+1,1}$, since

$$\begin{aligned}|u| &\ge \gamma_{M+1}\times\cdots\times\gamma_m(|u_m|-\gamma_{m+1}|u_{m+1}|)\\ &= \gamma_{M+1}\times\cdots\times\gamma_m(F_m(r_{m+1}(s))\\ &\quad-\gamma_{m+1}F_{m+1}(r_{m+1}(s)))\\ &\ge \gamma_{M+1}\times\cdots\times\gamma_m(1-Ce^{-\mu\lambda_m m^{-3}})\\ &\quad\times F_m(r_{m+1}(s))\\ &\ge \gamma_{M+1}\times\cdots\times\gamma_m(1-Ce^{-\mu m^2})\\ &\quad\times F_m(r_{m+1}(s))>0\end{aligned} \tag{2.11}$$

for $s\in[0,\frac{1}{4}]$ by (2.6) and (2.10), $|u|>0$ on $I_{m+1,1}$. Similarly, on $I_{m+1,4}$, since

$$\begin{aligned}|u| &\ge \gamma_{M+1}\times\cdots\times\gamma_m(\gamma_{m+1}|u_{m+1}|-|u_m|)\\ &= \gamma_{M+1}\times\cdots\times\gamma_m(\gamma_{m+1}F_{m+1}(r_{m+1}(s))\\ &\quad-F_m(r_{m+1}(s)))\\ &\ge \gamma_{M+1}\times\cdots\times\gamma_{m+1}(1-Ce^{-\mu\lambda_m m^{-3}})\\ &\quad\times F_{m+1}(r_{m+1}(s))\\ &\ge \gamma_{M+1}\times\cdots\times\gamma_{m+1}(1-Ce^{-\mu m^2})\\ &\quad\times F_{m+1}(r_{m+1}(s))>0\end{aligned} \tag{2.12}$$

for $s\in[\frac{3}{4},1]$, we have $|u|>0$ on $I_{m+1,4}$. By the definition of u, since $Lu=0$ on $I_{m+1,2}\cup I_{m+1,3}$, V is smooth when $r\in(1,\infty)$.

Finally, we prove V is smooth at $r=1$. On $I_{m+1,1}$, since $Lu=L[\gamma_{M+1}\times\cdots\times\gamma_{m+1}A_{m+1}u_{m+1}]$,

$$|\partial^\beta Lu|\le C_\ell\gamma_{M+1}\times\cdots\times\gamma_{m+1}\lambda_m^{2\ell+2}e^{o(m)}\times F_{m+1}(r) \tag{2.13}$$

holds for $|\beta|=\ell\in\mathbf{Z}_{\ge0}$ by (2.8). We thus have

$$\begin{aligned}|\partial^\beta V(r,\theta,t)| &= |\partial^\beta(u^{-1}Lu)|\\ &= \left|\sum_{|\beta_1|\le\ell}\binom{\beta}{\beta_1}\partial^\beta(u^{-1})\partial^{\beta-\beta_1}(Lu)\right|\\ &\le C_\ell\left(\frac{\gamma_{m+1}F_{m+1}}{F_m}\right)\lambda_m^{2(\ell+1)}e^{o(m)}\left(1+\frac{\gamma_{m+1}F_{m+1}}{F_m}\right)^\ell\\ &\le C_\ell e^{-\mu\lambda_m m^{-3}+o(m)}\\ &\le C_\ell e^{-\frac{\mu}{2}m^2}\end{aligned}$$

by (2.9), (2.11), (2.13), (2.6), (2.7), and (2.10) for $|\beta|=\ell\in\mathbf{Z}_{\ge0}$. Similarly on $I_{m+1,4}$, since $Lu=L[\gamma_{M+1}\times\cdots\times\gamma_m A_m u_m]$,

$$(2.14)\qquad |\partial^\beta Lu| \le C_\ell \gamma_{M+1}\times\cdots\times\gamma_m \lambda_m^{2\ell+2} e^{o(m)} F_m(r)$$

holds for $|\beta| = \ell \in \mathbf{Z}_{\ge 0}$ by (2.8). We thus have

$$\begin{aligned}
|\partial^\beta V(r,\theta,t)| &= |\partial^\beta (u^{-1}Lu)| \\
&= \left|\sum_{|\beta_1|\le\ell} \binom{\beta}{\beta_1} \partial^\beta (u^{-1}) \partial^{\beta-\beta_1}(Lu)\right| \\
&\le C_\ell \left(\frac{F_m}{\gamma_{m+1}F_{m+1}}\right) \lambda_m^{2(\ell+1)} e^{o(m)} \left(1 + \frac{F_m}{\gamma_{m+1}F_{m+1}}\right)^\ell \\
&\le C_\ell e^{-\mu\lambda_m m^{-3} + o(m)} \\
&\le C_\ell e^{-\frac{\mu}{2}m^2}
\end{aligned}$$

by (2.9), (2.12), (2.14), (2.6), (2.7), and (2.10) for $|\beta| = \ell \in \mathbf{Z}_{\ge 0}$.

Thus, for all $|\beta| = \ell \in \mathbf{Z}_{\ge 0}$ on I_{m+1},

$$|\partial^\beta V(r,\theta,t)| \le C_\ell e^{-\frac{\mu}{2}m^2} \le C_\ell e^{-\frac{\mu}{2}((r-1)^{-1}-2)^2} \xrightarrow{r\searrow 1} 0$$

holds. □

3. Proofs of the lemmas.

Proof of Lemma 2.1. We remark that

$$(3.1)\quad 1 \ge \tanh\alpha = \sqrt{1-a^2} \ge \lambda^{-\frac{p}{2}},\ a \in (0, 1-\lambda^{-p}].$$

We use the Schläfli's integral formula of a Bessel function,

$$J_\lambda(\lambda a) = \frac{1}{2\pi}\int_{\Gamma_0} e^{\lambda(-ia\sin z + iz)} dz,$$

where Γ_0 consists of three sides of rectangle with vertexes at $-\pi + i\infty$, $-\pi$, π and $\pi + i\infty$ and is oriented from $-\pi + i\infty$ to $\pi + i\infty$. We set

$$\begin{aligned}
f(z) &:= -ia\sin z + iz \\
&= a\cos x\sinh y - y + i(x - a\sin x\cosh y),
\end{aligned}$$

where $z = x + iy$. By the Cauchy's integral theorem, we can deform Γ_0 into a curve defined by Γ on which $x - a\sin x\cosh y = 0$. Hence, we obtain

$$J_\lambda(\lambda a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda g(x)} dx,$$

where g is defined by

$$g(x) := a\cos x\sinh y(x) - y(x)$$

and y satisfies

$$\begin{aligned}
&\cosh y(x) = \frac{x}{a\sin x} \\
&\Leftrightarrow y(x) = \log\left(\frac{x}{a\sin x} + \sqrt{\frac{x^2}{a^2\sin^2 x} - 1}\right)
\end{aligned}$$

for $x \in (-\pi,\pi)$, where $y(0) = \alpha$ is well-defined owing to $a < 1$.

First, we evaluate g in an interval $[-\lambda^{-q}, \lambda^{-q}]$, where q satisfying

$$(3.2)\qquad 0 < q < \frac{2-p}{4}$$

is determined later. Since there exists a constant $C > 0$ such that

$$\begin{aligned}
|y'(x)| &= \left|\frac{1}{\sinh y}\frac{d}{dx}\left(\frac{x}{a\sin x}\right)\right| \le \frac{C}{\sqrt{\frac{x^2}{(a\sin x)^2} - 1}}\frac{|x|}{a} \\
&\le \frac{C|x|}{\sqrt{1-a^2}} \le C\lambda^{-q+\frac{p}{2}}
\end{aligned}$$

by (3.1), we have for $x \in [-\lambda^{-q}, \lambda^{-q}]$,

$$|y(x) - \alpha| = |y(x) - y(0)| \le C\lambda^{-q+\frac{p}{2}}|x| \le C\lambda^{-2q+\frac{p}{2}}.$$

Hence, the Taylor's theorem yields

$$\begin{aligned}
(3.3)\quad g(x)& \\
&= f(x + iy(x)) = f(x + i(y-\alpha) + i\alpha) \\
&= f(i\alpha) + (x + i(y-\alpha))f'(i\alpha) \\
&\quad + \frac{(x + i(y-\alpha))^2}{2} f''(i\alpha) \\
&\quad + \frac{(x + i(y-\alpha))^3}{2}\int_0^1 (1-\theta)^2 \\
&\quad \times f'''(i\alpha + \theta(x + i(y-\alpha)))d\theta \\
&= \tanh\alpha - \alpha - \frac{\tanh\alpha}{2}x^2 + O(\lambda^{-1-\delta})
\end{aligned}$$

since $f'(i\alpha) = 0$, $|f'''(i\alpha + \theta(x + i(y-\alpha)))| \le C$ for some $C > 0$, and

$$q := \frac{1}{3}\left(1 + \delta + \frac{p}{2}\right).$$

We remark that (3.2) is equivalent to $p < \frac{2(1-2\delta)}{5}$. Consequently, we have

$$\begin{aligned}
\int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\lambda g(x)}dx &= e^{\lambda(\tanh\alpha-\alpha)}\int_{-\lambda^{-q}}^{\lambda^{-q}} e^{-\frac{\lambda\tanh\alpha}{2}x^2}dx\cdot e^{O(\lambda^{-\delta})} \\
&= \frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{\lambda\tanh\alpha}}\left[\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}}d\xi - 2\int_{\lambda^{\frac{1-2q}{2}}\sqrt{\tanh\alpha}}^{\infty} e^{-\frac{\xi^2}{2}}d\xi\right] \\
&\quad \times (1 + O(\lambda^{-\delta})) \\
&= \frac{\sqrt{2\pi}e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{\lambda\tanh\alpha}}(1 + O(\lambda^{-\delta}))
\end{aligned}$$

since $\lambda^{\frac{1-2q}{2}}\sqrt{\tanh\alpha} \ge \lambda^{\frac{2-p}{4}-q}$ by (3.1) and

$$\lambda^\delta\int_{\lambda^{\frac{2-p}{4}-q}}^{\infty} e^{-\frac{\xi^2}{2}}d\xi = \lambda^\delta\sqrt{\frac{\pi}{2}}e^{-\frac{1}{4}\lambda^{\frac{2-p}{2}-2q}} \xrightarrow{\lambda\to\infty} 0$$

by (3.2). Hence, we have

$$\frac{1}{2\pi}\int_{-\lambda^{-q}}^{\lambda^{-q}} e^{\lambda g(x)}dx = \frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{2\pi\lambda\tanh\alpha}}(1 + O(\lambda^{-\delta})).$$

Second, we evaluate g in $(-\pi,\pi)\setminus[-\lambda^{-q},\lambda^{-q}]$. Because $\pm g'(x) \ge 0$ when $0 \le \mp x < \pi$, it follows from (3.3) and (3.1),

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-\lambda^{-q}}+\int_{\lambda^{-q}}^{\pi}\right)e^{\lambda g(x)}dx = O(\lambda^{-\delta})\frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{2\pi\lambda\tanh\alpha}}.$$

In fact, by our assumption (3.1) and (3.2),

$$\lambda^{\delta}\left|\frac{\frac{1}{2\pi}\left(\int_{-\pi}^{-\lambda^{-q}}+\int_{\lambda^{-q}}^{\pi}\right)e^{\lambda g(x)}dx}{\frac{e^{\lambda(\tanh\alpha-\alpha)}}{\sqrt{2\pi\lambda\tanh\alpha}}}\right|$$
$$\le\sqrt{\frac{\tanh\alpha}{2\pi}}\lambda^{\delta+\frac{1}{2}}\cdot 2(\pi-\lambda^{-q})e^{-\frac{1}{2}\lambda^{\frac{2-p}{2}-2q}}\cdot e^{O(\lambda^{-\delta})}$$
$$\le\sqrt{2\pi}\lambda^{\delta+\frac{1}{2}}e^{-\frac{1}{2}\lambda^{\frac{2-p}{2}-2q}}\cdot e^{O(\lambda^{-\delta})}\xrightarrow{\lambda\to\infty}0$$

holds. We complete the proof. □

Proof of Lemma 2.2. In Lemma 2.1, taking $a=r=1-\ell m^{-2}$ for $\ell\ge 1$, which is done by our assumption (2.1), yields

$$G_m(r)=\frac{h(r)^{\lambda_m}}{\sqrt{2\pi}(1-r^2)^{\frac{1}{4}}\sqrt{\lambda_m}}(1+o(1))\text{ as }m\to\infty,$$

where

$$h(r):=\frac{re^{\sqrt{1-r^2}}}{1+\sqrt{1-r^2}},$$

since $e^{-\alpha}=\frac{r}{1+\sqrt{1-r^2}}$ for $\alpha>0$. Because simple calculations yield

$$h'(r)=\frac{1-\sqrt{1-r^2}}{r^2}\sqrt{1-r^2}e^{\sqrt{1-r^2}}$$
$$=(1+o(1))\sqrt{2}\sqrt{1-r}\text{ as }r\nearrow 1,$$

we have

$$h(r)=h(1)-\int_r^1 h'(s)ds$$
$$=1-(1+o(1))\frac{2\sqrt{2}}{3}(1-r)^{\frac{3}{2}}$$

as $r\nearrow 1$. Hence, for $\ell\ge 1$ and $r=1-\ell m^{-2}$, we obtain

$$G_m(r)=(1+o(1))\frac{(1-(1+o(1))\frac{2\sqrt{2}}{3}\ell^{\frac{3}{2}}m^{-3})^{\lambda_m}}{(2\pi)^{\frac{1}{2}}\sqrt{\lambda_m}\Big(1-(1-\ell m^{-2})^2\Big)^{\frac{1}{4}}}$$
$$=(1+o(1))\frac{\sqrt{m}}{(2\pi^2\ell)^{\frac{1}{4}}\sqrt{\lambda_m}}e^{-(1+o(1))\frac{2\sqrt{2}}{3}\ell^{\frac{3}{2}}\lambda_m m^{-3}}$$

as $m\to\infty$. The last equality comes from

$$\frac{(1-(1+o(1))\frac{2\sqrt{2}}{3}\ell^{\frac{3}{2}}m^{-3})^{\lambda_m}}{(2\pi)^{\frac{1}{2}}\sqrt{\lambda_m}\Big(1-(1-\ell m^{-2})^2\Big)^{\frac{1}{4}}}$$
$$=\frac{\sqrt{m}}{(2\pi^2\ell)^{\frac{1}{4}}\sqrt{\lambda_m}}e^{-(1+o(1))\frac{2\sqrt{2}}{3}\ell^{\frac{3}{2}}\lambda_m m^{-3}}\text{ as }m\to\infty$$

since (2.1) implies

$$\lambda_m m^{-3}\ge m^{\frac{2}{p}-3}>m^2\xrightarrow{m\to\infty}\infty.$$

□

Proof of Lemma 2.3. (2.4) is obtained by the definition. Since

$$\frac{k_m}{\lambda_m}r_m(s)=1-\frac{1+s+O(m^{-1})}{m^2}\text{ as }m\to\infty$$

by our assumption (2.3), we obtain

$$F_m(r_m(s))=G_m\left(\frac{k_m}{\lambda_m}r_m(s)\right)$$
$$=G_m\left(1-\frac{1+s+O(m^{-1})}{m^2}\right)$$
$$=(1+o(1))\frac{\sqrt{m}e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s+O(m^{-1}))^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s+O(m^{-1})))^{\frac{1}{4}}\sqrt{\lambda_m}}$$
$$=(1+o(1))\frac{\sqrt{m}e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}}\sqrt{\lambda_m}}$$

as $m\to\infty$. The last equality comes from

$$\frac{\sqrt{m}e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s+O(m^{-1}))^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s+O(m^{-1})))^{\frac{1}{4}}\sqrt{\lambda_m}}$$
$$=\frac{\sqrt{m}e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}}\sqrt{\lambda_m}}\text{ as }m\to\infty.$$

Furthermore, since $r_{m+1}(s)=r_m(1+s+O(m^{-1}))$ as $m\to\infty$,

$$\gamma_{m+1}=\frac{F_m(r_m(\frac{3}{2}+O(m^{-1})))}{F_{m+1}(r_{m+1}(2^{-1}))}$$
$$=(1+o(1))\left(\frac{\frac{3}{2}}{\frac{5}{2}+O(m^{-1})}\right)^{\frac{1}{4}}\sqrt{\frac{\lambda_{m+1}}{\lambda_m}\frac{1}{1+\frac{1}{m}}}$$
$$\times e^{-(1+o(1))\frac{2\sqrt{2}}{3}\left\{(\frac{5}{2}+O(m^{-1}))^{\frac{3}{2}}-(\frac{3}{2})^{\frac{3}{2}}\frac{\lambda_{m+1}}{\lambda_m}(\frac{m}{m+1})^3\right\}\lambda_m m^{-3}}$$
$$\le e^{-(1+o(1))\frac{2\sqrt{2}}{3}\left\{(\frac{5}{2}+O(m^{-1}))^{\frac{3}{2}}-(\frac{3}{2}(1+o(1)))^{\frac{3}{2}}\right\}\lambda_m m^{-3}}$$
$$\le e^{-(1+o(1))\sqrt{2}\sqrt{\theta}\lambda_m m^{-3}},$$

where we use our assumption (2.2) and the mean value theorem such that $x^{\frac{3}{2}}-y^{\frac{3}{2}}=\frac{3}{2}\sqrt{\theta}(x-y)$ for $0\le y\le\theta\le x$, holds. Hence, we have, by the above estimate,

$$\gamma_{m+1}\le e^{-(1+o(1))\sqrt{3}\lambda_m m^{-3}}\le e^{-\lambda_m m^{-3}}$$

for sufficiently large $m\in\mathbf{N}$.

Finally, we have by the definition of γ_{m+1},

$$\begin{aligned}&\frac{F_m(r_{m+1}(s))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))}\\&\quad=\frac{F_m(r_{m+1}(s))}{F_m(r_{m+1}(2^{-1}))}\cdot\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))}\\&\quad=\frac{F_m(r_m(1+s+O(m^{-1})))}{F_m(r_m(\frac{3}{2}+O(m^{-1})))}\cdot\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))}\end{aligned}$$

as $m\to\infty$. When $0\le s\le\frac{1}{4}$, there exist constants $C>0$ and θ satisfying $2+s+O(m^{-1})\le\theta\le\frac{5}{2}+O(m^{-1})$ such that

$$\begin{aligned}&\frac{F_m(r_m(1+s+O(m^{-1})))}{F_m(r_m(\frac{3}{2}+O(m^{-1})))}\\&\quad\ge Ce^{(1+o(1))\sqrt{2}\sqrt{\theta}(\frac{1}{2}-s)\lambda_m m^{-3}}\\&\quad\ge Ce^{(1+o(1))\sqrt{4+O(m^{-1})}(\frac{1}{2}-s)\lambda_m m^{-3}}.\end{aligned}$$

Furthermore, there exists θ satisfying $1+s\le\theta\le\frac{3}{2}$ such that

$$\begin{aligned}&\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))}\\&\quad\ge Ce^{-(1+o(1))\sqrt{2}\sqrt{\theta}(\frac{1}{2}-s)\lambda_{m+1}(m+1)^{-3}}\\&\quad\ge Ce^{-(1+o(1))\sqrt{3}(\frac{1}{2}-s)\lambda_m m^{-3}}\end{aligned}$$

by (2.2). There then exists $\mu>0$ such that

$$\begin{aligned}&\frac{F_m(r_{m+1}(s))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))}\\&\quad\ge Ce^{(1+o(1))(\sqrt{4+O(m^{-1})}-\sqrt{3})(\frac{1}{2}-s)\lambda_m m^{-3}}\\&\quad\ge Ce^{(1+o(1))\frac{\sqrt{4+O(m^{-1})}-\sqrt{3}}{4}\lambda_m m^{-3}}\\&\quad\ge Ce^{\mu\lambda_m m^{-3}}\end{aligned}$$

for sufficiently large $m\in\mathbf{N}$. Moreover, when $s\in[\frac{3}{4},1]$, there exist constants $C>0$ and θ satisfying $\frac{5}{2}+O(m^{-1})\le\theta\le 2+s+O(m^{-1})$ such that

$$\begin{aligned}&\frac{F_m(r_m(1+s+O(m^{-1})))}{F_m(r_m(\frac{3}{2}+O(m^{-1})))}\\&\quad\le Ce^{-(1+o(1))\sqrt{2}\sqrt{\theta}(s-\frac{1}{2})\lambda_m m^{-3}}\\&\quad\le Ce^{-(1+o(1))\sqrt{5+O(m^{-1})}(s-\frac{1}{2})\lambda_m m^{-3}}.\end{aligned}$$

Furthermore, there exists θ satisfying $\frac{3}{2}\le\theta\le 1+s$ such that

$$\begin{aligned}&\frac{F_{m+1}(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(s))}\\&\quad\le Ce^{(1+o(1))\sqrt{2}\sqrt{\theta}(s-\frac{1}{2})\lambda_{m+1}(m+1)^{-3}}\\&\quad\le Ce^{(1+o(1))2(s-\frac{1}{2})\lambda_m m^{-3}}\end{aligned}$$

by (2.2). There then exists $\mu>0$ such that

$$\begin{aligned}&\frac{F_m(r_{m+1}(s))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))}\\&\quad\le Ce^{-(1+o(1))(\sqrt{5+O(m^{-1})}-2)(s-\frac{1}{2})\lambda_m m^{-3}}\\&\quad\le Ce^{-(1+o(1))\frac{\sqrt{5+O(m^{-1})}-2}{4}\lambda_m m^{-3}}\\&\quad\le Ce^{-\mu\lambda_m m^{-3}}\end{aligned}$$

for sufficiently large $m\in\mathbf{N}$. □

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References

[1] S. Alinhac and M. S. Baouendi, A nonuniqueness result for operators of principal type, Math. Z. **220** (1995), no. 4, 561–568.

[2] V. Isakov, *Inverse problems for partial differential equations*, 3rd. ed., Applied Mathematical Sciences, 127, Springer, Cham, 2017.

[3] F. John, Continuous dependence on data for solutions of partial differential equations with a prescribed bound, Comm. Pure Appl. Math. **13** (1960), 551–585.

[4] A. Katchalov, Y. Kurylev and M. Lassas, *Inverse boundary spectral problems*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 123, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[5] H. Kumano-go, On an example of non-uniqueness of solutions of the Cauchy problem for the wave equation, Proc. Japan Acad. **39** (1963), 578–582.

[6] L. Robbiano and C. Zuily, Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients, Invent. Math. **131** (1998), no. 3, 493–539.

[7] D. Tataru, Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem, Comm. Partial Differential Equations **20** (1995), no. 5–6, 855–884.