Categories over quantum affine algebras and monoidal categorification

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Abstract: Let $U_q'(\mathfrak{g})$ be a quantum affine algebra of untwisted affine ADE type, and $\mathscr{C}_{\mathfrak{g}}^0$ the Hernandez-Leclerc category of finite-dimensional $U_q'(\mathfrak{g})$-modules. For a suitable infinite sequence $\widehat{w}_0 = \cdots s_{i_{-1}} s_{i_0} s_{i_1} \cdots$ of simple reflections, we introduce subcategories $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ of $\mathscr{C}_{\mathfrak{g}}^0$ for all $a \leqslant b \in \mathbf{Z} \sqcup \{\pm\infty\}$. Associated with a certain chain $\mathfrak{C}$ of intervals in $[a,b]$, we construct a real simple commuting family $M(\mathfrak{C})$ in $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$, which consists of Kirillov-Reshetikhin modules. The category $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ provides a monoidal categorification of the cluster algebra $K(\mathscr{C}_{\mathfrak{g}}^{[a,b]})$, whose set of initial cluster variables is $[M(\mathfrak{C})]$. In particular, this result gives an affirmative answer to the monoidal categorification conjecture on $\mathscr{C}_{\mathfrak{g}}^-$ by Hernandez-Leclerc since it is $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0]}$, and is also applicable to $\mathscr{C}_{\mathfrak{g}}^0$ since it is $\mathscr{C}_{\mathfrak{g}}^{[-\infty,\infty]}$.

Key words: Monoidal categorification; quantum affine algebra; cluster algebra; Kirillov-Reshetikhin module; T-system.

1. Introduction. Let $U_q'(\mathfrak{g})$ be a quantum affine algebra. The category $\mathscr{C}_{\mathfrak{g}}$ of finite-dimensional integrable modules over $U_q'(\mathfrak{g})$ has been intensively studied due to its rich structure. For instances, every object M in $\mathscr{C}_{\mathfrak{g}}$ has its left M^* and right dual ${}^*M$, and the q-characters of Kirillov-Reshetikhin modules in $\mathscr{C}_{\mathfrak{g}}$ provide a solution of the T-system, a system of differential equations appearing in solvable lattice models ([3,5,18,19]).

On the other hand, the cluster algebras were introduced by Fomin and Zelivinsky in [2] to investigate upper global bases and total positivity in an aspect of combinatorics.

Interestingly, it is proved in [6,7,9] that the Grothendieck rings $K(\mathcal{C})$ of monoidal subcategories $\mathcal{C} = \mathscr{C}_N$ $(N \in \mathbf{Z}_{\geqslant 1})$, $\mathscr{C}_{\mathcal{Q}}$, $\mathscr{C}_{\mathfrak{g}}^-$ of $\mathscr{C}_{\mathfrak{g}}$ have cluster algebra structures $\mathscr{A}$, and conjectured that every cluster monomial corresponds to the isomorphism class of a *real* simple module in $\mathcal{C}$; that is, $\mathcal{C}$ is expected to be a *monoidal categorification* of $\mathscr{A}$. The conjectures for $\mathscr{C}_N$ $(N \in \mathbf{Z}_{\geqslant 1})$ of untwisted affine ADE types are proved in [6,8,20] and [23]. Also, the conjecture for the subcategory $\mathscr{C}_{\mathcal{Q}} \subset \mathscr{C}_{\mathfrak{g}}$, determined by a Q-data $\mathcal{Q} = (Q, \phi_Q)$ ([4,16]), is proved in [11] via the quantum affine Weyl-Schur duality functor $\mathcal{F}_{\mathcal{Q}}$ ([10,12,17,21]) from the category $\mathcal{C}_{QH}$ of finite-dimensional graded modules over the symmetric quiver Hecke algebra to $\mathscr{C}_{\mathcal{Q}}$. More precisely, the category $\mathcal{C}_{QH}$ provides a monoidal categorification of the quantum cluster algebra $A_q(\mathfrak{n})$, the quantum unipotent coordinate algebra of finite simply-laced type ([1]). Since $\mathcal{F}_{\mathcal{Q}}$ is an exact monoidal functor preserving simplicity, we can prove the conjecture for $\mathscr{C}_{\mathcal{Q}}$ in an indirect way. However, this method could not be applicable to other $\mathcal{C}$ directly.

Recently, in [13], the authors of the present paper (KKOP) developed $\mathbf{Z}$-valued invariants $\Lambda, \Lambda^\infty, \widetilde{\Lambda}, \mathfrak{d}$ for pairs of modules in $\mathscr{C}_{\mathfrak{g}}$, which is extracted from distinguished $U_q'(\mathfrak{g})$-module homomorphisms, called *R-matrices*. Furthermore, KKOP provided a criterion for a monoidal subcategory $\mathscr{C} \subset \mathscr{C}_{\mathfrak{g}}$ to become a monoidal categorification of a cluster algebra by using those invariants. This paper can be understood as a continuation of [13], since we will apply the above criterion to various subcategories $\mathscr{C}$ of $\mathscr{C}_{\mathfrak{g}}$, including $\mathscr{C}_{\mathfrak{g}}^0$, $\mathscr{C}_{\mathfrak{g}}^-$

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and $\mathscr{C}_N$. We also give their initial monoidal seeds in a uniform manner.

Let $\mathfrak{g}_0$ be a finite-dimensional simple Lie algebra of *ADE* type with a Cartan matrix $\mathsf{A} = (a_{ij})_{i,j\in I_0}$, W the Weyl group generated by simple reflections s_i $(i \in I_0)$, $\mathfrak{g}$ the untwisted affine Kac-Moody algebra associated with $\mathfrak{g}_0$, and $U_q'(\mathfrak{g})$ the quantum affine algebra associated with $\mathfrak{g}$. In [6], Hernandez-Leclerc defined the full subcategory $\mathscr{C}_{\mathfrak{g}}^0$ of $\mathscr{C}_{\mathfrak{g}}$. Since every simple modules in $\mathscr{C}_{\mathfrak{g}}$ is a tensor product of suitable parameter shifts of simple modules in $\mathscr{C}_{\mathfrak{g}}^0$, it is enough to consider subcategories of $\mathscr{C}_{\mathfrak{g}}^0$.

By extending a reduced expression $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ of the longest element w_0 of the Weyl group W, we obtain an infinite sequence

$$\widehat{w}_0 = \cdots s_{i_{-2}}s_{i_{-1}}s_{i_0}s_{i_1}s_{i_2}\cdots \tag{1.1}$$

of simple reflections satisfying properties (a) and (b) in Section 2, and then we define fundamental modules $V[k]^{\widehat{w}_0}$ $(k \in \mathbf{Z})$. For each *interval* $[a,b] = \{k \in \mathbf{Z} \mid a \leqslant k \leqslant b\}$ with $a \leqslant b \in \mathbf{Z} \sqcup \{\pm\infty\}$, we define the subcategory $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ of $\mathscr{C}_{\mathfrak{g}}^0$ which is the smallest full monoidal subcategory containing $V[k]^{\widehat{w}_0}$ for all $k \in [a,b]$. Then $\mathscr{C}_{\mathfrak{g}}^0$ is nothing but $\mathscr{C}_{\mathfrak{g}}^{[-\infty,+\infty]}$ and the subcategory $\mathscr{C}_{\mathfrak{g}}^-$ introduced by Hernandez-Leclerc ([9]) can be identified with $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0]}$ (Remark 2.3).

We say that an interval $[a,b]$ is an *i-box* if $i_a = i_b$. For each i-box $[a,b]$, we define a simple module $M[a,b]$, which can be understood as a *quantum affine analogue of the determinantial module* (see Remark 3.2). In Theorem 3.1, we show that $M[a,b]$ is a Kirillov-Reshetikhin module and give a sufficient condition for the simplicity of the tensor product $M[a,b] \otimes M[a',b']$ for i-boxes $[a,b]$ and $[a',b']$. Then we define the notion of an *admissible chain* $\mathfrak{C} = \{[a_k,b_k] \subset [a,b] \mid 1 \leqslant k \leqslant b-a+1,\ i_{a_k} = i_{b_k}\}$ *of i-boxes* for an interval $[a,b]$ satisfying certain properties (Definition 3.3). For each admissible chain $\mathfrak{C}$, the family of Kirillov-Reshetikhin modules $M(\mathfrak{C}) = \{M[a_k,b_k]\}_{1\leqslant k\leqslant b-a+1}$ in $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ forms a commuting family of real simple modules (Theorem 3.4).

The next step is to show that $K(\mathscr{C}_{\mathfrak{g}}^{[a,b]})$ has a cluster algebra structure, $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ provides a monoidal categorification of $K(\mathscr{C}_{\mathfrak{g}}^{[a,b]})$, and any admissible chain $\mathfrak{C}$ gives a monoidal seed $M(\mathfrak{C})$ (Theorem 4.3). Based on the criterion in [13], we shall prove this by showing the assertion for a special chain $\mathfrak{C}$, and then by extending it to a general $\mathfrak{C}$. Namely, we proceed by proving

(i) the existence of a Λ-admissible monoidal seed $\mathscr{S}$ of $K(\mathscr{C}_{\mathfrak{g}}^{[a,b]})$ whose set of initial cluster variable modules is $M(\mathfrak{C})$ for some admissible chain $\mathfrak{C}$,

(ii) the existence of sequences of mutations among the $M(\mathfrak{C})$'s *only employing* T-systems,

which implies that any admissible chain $\mathfrak{C}$ gives a Λ-admissible monoidal seed for all $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$. In particular, we prove that $\mathscr{C}$ is a monoidal categorification of the cluster algebra $K(\mathscr{C})$ for $\mathscr{C} = \mathscr{C}_{\mathfrak{g}}^0$ and $\mathscr{C} = \mathscr{C}_{\mathfrak{g}}^-$. Note that we need in step (i) above the existence of the cluster algebra structure on $K(\mathscr{C}_{\mathfrak{g}}^-)$ proved in [9].

This paper is an announcement whose details will appear elsewhere.

2. Subcategories. We take the algebraic closure $\mathbf{k}$ of $\mathbf{C}(q)$ inside $\bigcup_{m>0}\mathbf{C}((q^{1/m}))$ as the base field for $U_q'(\mathfrak{g})$. Recall that $\mathscr{C}_{\mathfrak{g}}$ is the category of finite-dimensional integrable modules over $U_q'(\mathfrak{g})$. There is a family $\{V(\varpi_i)_c \mid i \in I_0, c \in \mathbf{k}^\times\}$ in $\mathscr{C}_{\mathfrak{g}}$ of simple modules, called the *fundamental modules*.

For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we say that M and N *strongly commute* if $M \otimes N$ is simple, and M is *real* if $M^{\otimes k}$ is simple for all $k \in \mathbf{Z}_{\geqslant 1}$.

Let us denote by Ψ the quiver whose set of vertices is

$$\widehat{I}_0 := \{(i,k) \in I_0 \times \mathbf{Z} \mid k \equiv d(1,i) \bmod 2\}.$$

and the arrows of Ψ consist of two types:

$$\begin{cases} \text{(A)} & (i,t) \to (j,s) \text{ with } d(i,j) = 1 \text{ and } \\ & s - t = 1, \\ \text{(B)} & (i,s+2) \to (i,s). \end{cases} \tag{2.1}$$

Here $d(i,j)$ denotes the distance between the vertices i and j in the Dynkin diagram of $\mathfrak{g}_0$ and $1 \in I_0$ is an arbitrary chosen element.

We say that an infinite sequence

$$\widehat{w}_0 = \cdots s_{i_{-1}}s_{i_0}s_{i_1}\cdots$$

of simple reflections in the Braid group $B(\mathfrak{g}_0)$ ([15]) of type $\mathfrak{g}_0$ is *admissible* if

(a) there exists a sequence $\{t_k\}_{k\in\mathbf{Z}}$ of integers such that
   (1) $(i_k,t_k) \in \widehat{I}_0$,
   (2) $t_{k^+} = t_k + 2$, and
   (3) $t_k > t_{k'}$ if $k > k'$ and $d(i_k,i_{k'}) = 1$.

(b) $s_{i_k}\cdots s_{i_{k+\ell-1}} = w_0$ for all $k \in \mathbf{Z}$, where ℓ denotes the length of longest element $w_0 \in \mathsf{W}$.

Here, for $k \in \mathbf{Z}$ and $j \in I_0$, we set

$$\begin{aligned} k^+ &:= \min\{p \mid k < p,\ i_k = i_p\},\\ k^- &:= \max\{p \mid p < k,\ i_k = i_p\},\\ k(j)^+ &:= \min\{p \mid k \leqslant p,\ i_p = j\},\\ k(j)^- &:= \max\{p \mid p \leqslant k,\ i_p = j\}. \end{aligned}$$

Remark 2.1.

(i) We have $i_{k+\ell} = i_k^*$, where * denotes the involution on I_0 induced by w_0.

(ii) $\widehat{w}_0$ completely determines $\{(i_k, t_k)\}_{k\in\mathbf{Z}}$ up to an even translation.

(iii) For every $k \in \mathbf{Z}$, the reduced expression $s_{i_k}\cdots s_{i_{k+\ell-1}}$ in (b) is *adapted* to some Dynkin quiver Q of type $\mathfrak{g}_0$. Conversely, for any Dynkin quiver Q of type $\mathfrak{g}_0$, there exists a sequence $\widehat{w}_0$ satisfying (a) and (b) such that $s_{i_1}\cdots s_{i_\ell}$ is adapted to Q.

For each $k \in \mathbf{Z}$, we define the fundamental module

$$V[k]^{\widehat{w}_0} := V(\varpi_{i_k})_{(-q)^{t_k}}.$$

Then we have

$$V[k^\pm]^{\widehat{w}_0} \simeq V[k]^{\widehat{w}_0}_{(-q)^{\pm 2}}, \quad V[k+\ell]^{\widehat{w}_0} = \mathscr{D}(V[k]^{\widehat{w}_0}),$$

where $\mathscr{D}$ denotes the right dual functor.

Definition 2.2. For each interval $[a,b]$, we denote by $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ satisfying the following conditions:

(i) it is stable under taking subquotients, extensions, tensor products and

(ii) it contains $V[k]^{\widehat{w}_0}$ for all $a \leqslant k \leqslant b$ and the trivial module $\mathbf{1}$.

Remark 2.3. Many of known subcategories $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}$ can be identified with $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ by taking suitable $\widehat{w}_0$ and $[a,b]$:

(1) $\mathscr{C}_{\mathfrak{g}}^{[-\infty,\infty]}$ coincides with the subcategory $\mathscr{C}_{\mathfrak{g}}^0$.

(2) The subcategory $\mathscr{C}_{\mathcal{Q}}$ associated to a Q-data $\mathcal{Q}$ coincides with $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ for some interval $[a,b]$ with $|[a,b]| := b-a+1 = \ell$.

(3) By taking $s_{i_1}\cdots s_{i_\ell}$ in (b) as adapted to the Dynkin quiver Q with $\{1,2\} \ni \phi_Q(k) \equiv d(1,i_k) \pmod 2$ and $t_k = \phi(i_k)$ for $1 \leqslant k \leqslant |I_0|$, $\mathscr{C}_N$ can be identified with $\mathscr{C}_{\mathfrak{g}}^{[a,0]}$ where $a = 1-(N\times|I_0|)$, and $\mathscr{C}_{\mathfrak{g}}^-$ can be identified with $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0]}$. Those subcategories $\mathscr{C}_N$, $\mathscr{C}_{\mathfrak{g}}^-$ of $\mathscr{C}_{\mathfrak{g}}^0$ are introduced in [6,9].

3. Real simple commuting family associated to an admissible chain of i-boxes. Let us fix an admissible sequence $\widehat{w}_0$ and $\{t_k\}_{k\in\mathbf{Z}}$. We write $V[k]$ for $V[k]^{\widehat{w}_0}$. We say that an interval $c = [a,b]$ is an *i-box* if $i_a = i_b$. For each i-box $[a,b]$, the module $M[a,b]$ in $\mathscr{C}_{\mathfrak{g}}^0$ is defined as follows:

$$M[a,b] := \mathrm{hd}(V[b]\otimes V[b^-]\otimes\cdots\otimes V[a^+]\otimes V[a]),$$

where $\mathrm{hd}(M)$ for $M \in \mathscr{C}_{\mathfrak{g}}$ denotes the head of M. In particular, $V[a] = M[a,a]$.

Theorem 3.1.

(i) *$M[a,b]$ is a Kirillov-Reshetikhin module with a dominant extremal weight $s\varpi_{i_a}$ where $s = |\{k \mid a \leqslant k \leqslant b,\ i_k = i_a\}|$.*

(ii) *For i-boxes $[a,b]$ and $[c,d]$, $M[a,b]$ and $M[c,d]$ commutes if either*

$$a^- < c \leqslant d < b^+ \quad \text{or} \quad c^- < a \leqslant b < d^+.$$

(iii) *For any i-box $[a,b]$, there exists an exact sequence in terms of $M[a,b]$'s as follows*:

$$(3.1)\ 0 \to \bigotimes_{d(i_a,j)=1} M[a(j)^+,\ b(j)^-] \to M[a^+,b]\otimes M[a,b^-] \to M[a,b]\otimes M[a^+,\ b^-] \to 0,$$

We call it a T-system.

Remark 3.2. For any reduced expression $\underline{w_0} = s_{j_1}\cdots s_{j_\ell}$ of w_0 (*not necessarily adapted*) and $[a,b]$ with $j_a = j_b$ and $b-a+1 \leqslant \ell$, there exists a real simple module $\mathrm{D}[a,b]^{\underline{w_0}}$ in $\mathcal{C}_{QH}$ of type $\mathfrak{g}_0$, called the *determinantial module*, and there exists an exact sequence (called the T-system)

$$0 \to \bigotimes_{d(i_a,j)=1} \mathrm{D}[a(j)^+, b(j)^-] \to \mathrm{D}[a^+,b]\otimes\mathrm{D}[a,b^-] \to \mathrm{D}[a,b]\otimes\mathrm{D}[a^+,b^-] \to 0$$

in $\mathcal{C}_{QH}$, which is analogous to (3.1). More precisely, when $\underline{w_0}$ is adapted to some Dynkin quiver Q of type $\mathfrak{g}_0$, quantum affine Weyl-Schur duality functor $\mathcal{F}_{\mathcal{Q}}$ associated with some Q-data $\mathcal{Q} = (Q,\phi_Q)$ transforms the above exact sequence in $\mathcal{C}_{QH}$ to the T-system (3.1). Thus $M[a,b]$ can be understood as a *quantum affine analogue of the determinantial module.* (See [7, Proposition 4.1] and [11] for more detail.)

For an interval $c := [a,b]$, we introduce i-boxes

$$[a,b\} := [a, b(i_a)^-], \quad \{a,b] := [a(i_b)^+, b],$$
$$\mathcal{L}(c) := [a-1,b\}, \quad \mathcal{R}(c) := \{a,b+1].$$

Definition 3.3. A chain $\mathfrak{C}$ of i-boxes

$$(c_k = [a_k,b_k])_{1\leqslant k\leqslant l} \quad (l \in \mathbf{Z}_{\geqslant 1}\sqcup\{\infty\})$$

is called *admissible* if $\widetilde{c}_k = [\widetilde{a}_k,\widetilde{b}_k] := \bigcup_{1\leqslant j\leqslant k}[a_j,b_j]$ satisfies $|\widetilde{c}_k| = k$ and one of the following two statements.

(1) $c_k = \mathcal{L}(\widetilde{c}_{k-1})$,
(2) $c_k = \mathcal{R}(\widetilde{c}_{k-1})$.
(Please do not confuse l and ℓ.) The sequence of intervals $\{\widetilde{c}_k\}_{1\leqslant k\leqslant l}$ is called the *envelope* of the chain $\mathfrak{C}$ and $\widetilde{c}_l$ is called the *range* of $\mathfrak{C}$.

Thus, for an admissible chain $\mathfrak{C}$ of i-boxes, we can associate a pair $(a, \mathfrak{T})$ consisting of an integer a and a sequence $\mathfrak{T} = (T_1, T_2, \ldots, T_{l-1})$ such that

(i) $T_i \in \{\mathcal{L}, \mathcal{R}\}$ $(1 \leqslant i \leqslant l-1)$,

(ii) $a = a_1 = b_1$, and

(iii) $[\widetilde{a}_k, \widetilde{b}_k] = \begin{cases} [\widetilde{a}_{k-1} - 1, \widetilde{b}_{k-1}] & \text{if } T_{k-1} = \mathcal{L}, \\ [\widetilde{a}_{k-1}, \widetilde{b}_{k-1} + 1] & \text{if } T_{k-1} = \mathcal{R}. \end{cases}$

Hence we have $c_k = [a_k, b_k] = T_{k-1}[\widetilde{a}_{k-1}, \widetilde{b}_{k-1}]$ $(k \geqslant 2)$, and the interval $\widetilde{c}_k$ is obtained from $\widetilde{c}_{k-1}$ by adding an element from the left or from the right according that $T_{k-1} = \mathcal{L}$ or $T_{k-1} = \mathcal{R}$.

For an admissible chain $\mathfrak{C} = (c_k)_{1\leqslant k\leqslant l}$ with the associated pair $(a, \mathfrak{T})$ and for $1 \leqslant s < l$, we say that an i-box c_s is *movable* if $s = 1$ or $T_{s-1} \neq T_s$ $(s \geqslant 2)$. For a movable c_s in $\mathfrak{C}$, we define a new admissible chain $B_s(\mathfrak{C})$ whose associated pair $(a', \mathfrak{T}')$ is given

(i) $\begin{cases} a' = a \pm 1 & \text{if } s = 1 \text{ and } T_1 = \mathcal{R} \text{ (resp. } \mathcal{L}), \\ a' = a & \text{if } s > 1, \end{cases}$

(ii) $T'_k = T_k$ for $k \notin \{s-1, s\}$, and

(iii) $T'_k \neq T_k$ for $k \in \{s-1, s\}$.

That is, $B_s(\mathfrak{C})$ is the admissible chain obtained from $\mathfrak{C}$ by moving $\widetilde{c}_s$ by 1 to the right or to the left inside $\widetilde{c}_{s+1}$.

Theorem 3.4. *Let $\mathfrak{C} = (c_k)_{1\leqslant k\leqslant l}$ be an admissible chain and set*

$$M(\mathfrak{C}) := \{M[a_k, b_k] \mid 1 \leqslant k \leqslant l\}.$$

Then we have the followings
(a) *$M(\mathfrak{C})$ is a real simple commuting family.*
(b) *If $M[c, d]$ commutes with all $M[a_k, b_k]$ and $[c, d] \subset [\widetilde{a}_l, \widetilde{b}_l]$, then $[c, d] \in \mathfrak{C}$.*
(c) *For another admissible chain $\mathfrak{C}' = (c'_k)_{1\leqslant k\leqslant l}$ with the same range, there exists a finite sequence $(t_1, t_2, \ldots, t_r) \in \{1, 2, \ldots, l\}^r$ such that*

$$B_{t_r}(\cdots(B_{t_2}(B_{t_1}(\mathfrak{C}))\cdots) = \mathfrak{C}'.$$

4. Monoidal categorification. Let $K = K_{\mathrm{ex}} \sqcup K_{\mathrm{fr}}$ be a countable index set. Let $\mathscr{C}$ be a full subcategory of $\mathscr{C}^0_{\mathfrak{g}}$ stable under taking subquotients, extensions and tensor products.

A *monoidal seed in $\mathscr{C}$* is a pair $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ consisting of a commuting family $\{M_i\}_{i\in K}$ of real simple objects in $\mathscr{C}$ and a $\mathbf{Z}$-valued $K \times K_{\mathrm{ex}}$-matrix $\widetilde{B} = (b_{ij})_{(i,j)\in K\times K_{\mathrm{ex}}}$ such that (i) for each $j \in K_{\mathrm{ex}}$, there exist finitely many $i \in K$ such that $b_{ij} \neq 0$, (ii) the *principal part* $B := (b_{ij})_{i,j\in K_{\mathrm{ex}}}$ is skew-symmetric. For $i \in K$, we call M_i the i-th *cluster variable module* of $\mathscr{S}$.

For a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$, let $\Lambda^{\mathscr{S}} = (\Lambda^{\mathscr{S}}_{ij})_{i,j\in K}$ be the skew-symmetric matrix given by $\Lambda^{\mathscr{S}}_{ij} = \Lambda(M_i, M_j)$ (see [14]).

A monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ is called Λ-*admissible* if
(i) $(\Lambda^{\mathscr{S}} \widetilde{B})_{jk} = -2\delta_{jk}$ for $(j, k) \in K \times K_{\mathrm{ex}}$, and
(ii) for each $k \in K_{\mathrm{ex}}$, there exist a simple object M'_k of $\mathcal{C}$ commuting with M_i for any $i \neq k$ and an exact sequence in $\mathscr{C}$

$$(4.1)\qquad 0 \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to M_k \otimes M'_k \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to 0.$$

Under the following two assumptions

(4.2) (a) there exists a Λ-admissible monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ in $\mathscr{C}$,
(b) $K(\mathscr{C})$ is isomorphic to the cluster algebra $\mathscr{A}([\mathscr{S}])$,

KKOP ([13, Theorem 6.10]) proved that $\mathscr{C}$ provides a monoidal categorification of $\mathscr{A}([\mathscr{S}])$. Here $[\mathscr{S}] := (\{[M_i]\}_{i\in K}, \widetilde{B})$ is a seed in $K(\mathscr{C})$, and $\mathscr{A}([\mathscr{S}])$ denotes the cluster algebra with the initial seed $[\mathscr{S}]$.

Set $\widehat{I}_0^- := \widehat{I}_0 \cap (I_0 \times \mathbf{Z}_{\leqslant 0})$ and let Ψ^- be the full subquiver of Ψ whose set of vertices is $\widehat{I}_0^-$. In [9], Hernandez-Leclerc proved that $\mathscr{A}^- := K(\mathscr{C}^-_{\mathfrak{g}})$ has a cluster algebra structure whose initial cluster variable modules $\{\mathcal{M}_{(i,t)}\}_{(i,t)\in\widehat{I}_0^-}$ consist of certain KR-modules. For a suitable choice of $\widehat{w}_0$ (Remark 2.3), we have $\mathscr{C}^-_{\mathfrak{g}} = \mathscr{C}^{[-\infty,0]}_{\mathfrak{g}}$ and $\{\mathcal{M}_{(i,t)}\}_{(i,t)\in\widehat{I}_0^-}$ can be described as $M(\mathfrak{C}^-)$ for the following admissible chain $\mathfrak{C}^-$ of i-boxes:

$$\mathfrak{C}^- = (0, \mathfrak{T} = (\mathcal{L}, \mathcal{L}, \mathcal{L}, \ldots)).$$

More precisely, for $(i, t) = (i_a, t_a)$ $(a \leqslant 0)$, we have

$$\mathcal{M}_{(i,t)} = M[a, 0\}.$$

The following theorem gives an affirmative answer for the conjecture on $\mathscr{C}^-_{\mathfrak{g}}$:

Theorem 4.1. *The monoidal seed*

$$\mathscr{S}^- := (M(\mathfrak{C}^-), \widetilde{B}^-) \text{ is } \Lambda\text{-admissible},$$

where $\widetilde{B}^-$ is the matrix associated to Ψ^-. Hence $\mathscr{C}_{\mathfrak{g}}^-$ provides a monoidal categorification of $\mathscr{A}^-$.

Now we shall generalize the above theorem to an arbitrary $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$.

Proposition 4.2. *Let $\mathfrak{C} = (c_k)_{1 \leqslant k \leqslant l}$ be an admissible chain of i-boxes with the range $[a, b]$ and the envelope $\{\widetilde{c}_k\}_{1 \leqslant k \leqslant l}$. Assume that $\mathscr{C} := \mathscr{C}_{\mathfrak{g}}^{[a,b]}$ provides a monoidal categorification of $K(\mathscr{C})$ with a Λ-admissible monoidal seed $(M(\mathfrak{C}), \widetilde{B})$. Let c_s be a movable i-box of $\mathfrak{C}$ and set $\mathfrak{C}' = B_s(\mathfrak{C})$. If $\widetilde{c}_{s+1} \neq c_{s+1}$, then $M(\mathfrak{C}')$ is equal to $M(\mathfrak{C})$ up to a permutation. If $\widetilde{c}_{s+1} = c_{s+1}$, then $M(\mathfrak{C}')$ is the monoidal mutation of $M(\mathfrak{C})$ at s. Moreover the corresponding exact sequence* (4.1) *is given by the T-system* (3.1).

The above proposition and Theorem 3.4 show that all $M(\mathfrak{C})$ with the same range are mutation equivalent.

Now we state our main theorem:

Theorem 4.3. *For any admissible chain $\mathfrak{C} = (c_k)_{1 \leqslant k \leqslant l}$ for $l \in \mathbf{Z}_{\geqslant 1} \sqcup \{\infty\}$ with the range*

$$\widetilde{c}_l = [a, b] \quad \text{for } a \leqslant b \in \mathbf{Z} \sqcup \{\pm\infty\},$$

there exists a Λ-admissible monoidal seed $\mathscr{S}$ of $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ such that

(i) *its set of cluster variable modules is $M(\mathfrak{C})$,*

(ii) *its set of frozen variable modules is $\{M[a(i)^+, b(i)^-] \mid i \in I_0, -\infty < a(i)^+ \leqslant b(i)^- < +\infty\}$, and*

(iii) *$K(\mathscr{C}_{\mathfrak{g}}^{[a,b]})$ has a cluster algebra structure with the initial seed $[\mathscr{S}]$, and $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ provides a monoidal categorification of $\mathscr{A}([\mathscr{S}]) \simeq K(\mathscr{C}_{\mathfrak{g}}^{[a,b]})$.*

By Remark 2.3, we have the following

Corollary 4.4. *The Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}}^0)$ has a cluster algebra structure, and $\mathscr{C}_{\mathfrak{g}}^0$ provides a monoidal categorification of $K(\mathscr{C}_{\mathfrak{g}}^0)$.*

Remark 4.5. We can generalize the above results to an arbitrary quantum affine algebra $U_q'(\mathfrak{g})$ by applying a similar framework with the results in [12,17,21,22].

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