Algebraic independence of certain infinite products involving the Fibonacci numbers

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Abstract: Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence. The aim of this paper is to give explicit formulae for the infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_n}\right),\quad \prod_{n=3}^{\infty}\left(1-\frac{1}{F_n}\right)$$

in terms of the values of the Jacobi theta functions. From this we deduce the algebraic independence over $\mathbf{Q}$ of the above numbers by applying Bertrand's theorem on the algebraic independence of the values of the Jacobi theta functions.

Key words: Algebraic independence; Fibonacci numbers; Jacobi theta functions.

1. Introduction and main results. Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence defined recursively by

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0$$

with $F_0 = 0$ and $F_1 = 1$. Arithmetical properties of infinite sums and products involving the Fibonacci numbers have been investigated by several authors. In 1989, André-Jeannin [1] proved that the sum of the reciprocals of the Fibonacci numbers $\sum_{n=1}^{\infty} 1/F_n$ is irrational (see also [5,6]). Moreover, it is shown in [7] that the number $\sum_{n=1}^{\infty} 1/F_{2n-1}$ is transcendental. On the other hand, some closed forms were discovered in particular cases; for example,

$$\sum_{n=1}^{\infty}\frac{1}{F_{2^n}} = \frac{5-\sqrt{5}}{2} \tag{1}$$

(cf. [10, p. 225]), which results from the use of telescoping series. As for the infinite products, the second author [12] derived that the numbers

$$\gamma_j := \prod_{n=1}^{\infty}\left(1+\frac{j}{F_{2^n}}\right), \quad j = 1, 2, \dots \tag{2}$$

are all transcendental. The infinite products (2) can be regarded as product analogues of (1). The transcendence result on the γ's was extended in [9] to algebraic independence over $\mathbf{Q}$ of the numbers $\gamma_1, \gamma_2, \dots, \gamma_m$ for any integer $m \geq 1$. Also in the case of products, some closed forms can be obtained through the telescoping method. For example, we have

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n}}\right) = 1+\sqrt{5}, \tag{3}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n}}\right) = \frac{1+\sqrt{5}}{6}, \tag{4}$$

since

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n}}\right) = \prod_{n=1}^{\infty}\frac{(1+\alpha^{-2(n-1)})(1-\alpha^{-2(n+1)})}{(1+\alpha^{-2n})(1-\alpha^{-2n})} = \frac{2}{1-\alpha^{-2}} = 2\alpha,$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n}}\right) = \prod_{n=2}^{\infty}\frac{(1+\alpha^{-2(n+1)})(1-\alpha^{-2(n-1)})}{(1+\alpha^{-2n})(1-\alpha^{-2n})} = \frac{1-\alpha^{-2}}{1+\alpha^{-4}} = \frac{\alpha}{3},$$

where $\alpha := (1+\sqrt{5})/2$ is the golden ratio. However, it is not easy to find closed forms and to derive arithmetical properties for given infinite sums and products involving the Fibonacci numbers. For instance, it is not known whether the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\prod_{n=1}^{\infty}(1+1/F_{3n})$ are transcendental.

In this paper, we give explicit formulae for the two fundamental infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_n}\right) = 13.1509666577\dots,$$

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$$\prod_{n=3}^{\infty}\left(1-\frac{1}{F_n}\right)=0.1897891436\ldots$$

by means of the values of the Jacobi theta functions. Moreover, by using the formulae we prove that the above two numbers are algebraically independent over $\mathbf{Q}$. To state our results, we define the Jacobi theta functions

$$(5)\qquad \begin{cases} \vartheta_2(q):=2q^{1/4}\sum_{n=0}^{\infty}q^{n(n+1)}, \\ \vartheta_3(q):=1+2\sum_{n=1}^{\infty}q^{n^2}, \\ \vartheta_4(q):=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}, \end{cases}$$

which converge for all complex numbers q with $|q|<1$. Throughout this paper, let

$$(6)\qquad \beta:=\frac{1}{\alpha}=\frac{\sqrt{5}-1}{2}.$$

Our main results are the following

Theorem 1. *Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence. Then*

$$\xi_1:=\prod_{n=1}^{\infty}\left(1+\frac{1}{F_n}\right)=2\beta^{-5/4}\frac{\vartheta_2(\beta)}{\vartheta_4(\beta^4)},$$

$$\xi_2:=\prod_{n=3}^{\infty}\left(1-\frac{1}{F_n}\right)=\frac{\sqrt{5}}{6}\beta^{-5/4}\frac{\vartheta_2(\beta)\vartheta_3(\beta)\vartheta_4(\beta)}{\vartheta_4(\beta^4)}.$$

Theorem 2. *The numbers ξ_1 and ξ_2 are algebraically independent over $\mathbf{Q}$. In particular, the numbers ξ_1 and ξ_2 are both transcendental.*

2. Proofs of Theorems 1 and 2. We first prove Theorem 1. Let $\vartheta_2(q),\vartheta_3(q),\vartheta_4(q)$ be the Jacobi theta functions defined in (5). Using the triple-product identities

$$\vartheta_2(q)=q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-2})(1+q^{2n}),$$

$$\vartheta_3(q)=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,$$

$$\vartheta_4(q)=\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2$$

(cf. [4, Corollary 3.1]), we have

$$(7)\qquad q^{-1/4}\vartheta_2(q)=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-2})(1+q^{2n}),$$

$$(8)\qquad \vartheta_2(q)\vartheta_3(q)\vartheta_4(q)=2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})^3,$$

$$(9)\qquad \begin{aligned}\vartheta_4(q^4)&=\prod_{n=1}^{\infty}(1-q^{4\cdot 2n})(1-q^{4(2n-1)})^2\\ &=\prod_{n=1}^{\infty}(1-q^{4n})(1-q^{4(2n-1)})\\ &=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})\end{aligned}$$

(see [4, Proof of Proposition 3.1] for (8)). Hence, we obtain by (7), (8), and (9)

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n-1}}\right)&=\prod_{n=1}^{\infty}\frac{(1+\beta^{2n-2})(1+\beta^{2n})}{1+\beta^{4n-2}}\\ &=\beta^{-1/4}\frac{\vartheta_2(\beta)}{\vartheta_4(\beta^4)},\end{aligned}$$

$$\begin{aligned}\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n-1}}\right)&=\prod_{n=2}^{\infty}\frac{(1-\beta^{2n-2})(1-\beta^{2n})}{1+\beta^{4n-2}}\\ &=\frac{\beta^{-1/4}(1+\beta^2)}{2(1-\beta^2)}\frac{\vartheta_2(\beta)\vartheta_3(\beta)\vartheta_4(\beta)}{\vartheta_4(\beta^4)}.\end{aligned}$$

Therefore, Theorem 1 follows from the above formulae and (3), (4), (6).

Next we show Theorem 2. Using the well-known identities

$$(10)\qquad \vartheta_2^2(q)=2\vartheta_2(q^2)\vartheta_3(q^2),$$

$$(11)\qquad \vartheta_4^2(q^2)=\vartheta_3(q)\vartheta_4(q),$$

(cf. [4, Chapter 2, §2.1]), we obtain by Theorem 1

$$\xi_1^3\xi_2=\frac{16\sqrt{5}}{3}\beta^{-5}\vartheta_2^2(\beta^2),$$

$$\frac{\xi_2}{\xi_1}=\frac{\sqrt{5}}{12}\vartheta_4^2(\beta^2),$$

so that the numbers $\vartheta_2^2(\beta^2)$ and $\vartheta_4^2(\beta^2)$ belong to the field $\mathbf{Q}(\xi_1,\xi_2,\beta)$. Hence, putting $\mathbf{K}:=\mathbf{Q}(\xi_1,\xi_2)$ and noting that β is an algebraic number, we have

$$\begin{aligned}2&\geq \text{trans.deg}_{\mathbf{Q}}\mathbf{K}=\text{trans.deg}_{\mathbf{Q}}\mathbf{Q}(\xi_1,\xi_2,\beta)\\ &\geq \text{trans.deg}_{\mathbf{Q}}\mathbf{Q}(\vartheta_2^2(\beta^2),\vartheta_4^2(\beta^2))=2,\end{aligned}$$

where at the last equality we used a result of Bertrand [3, Theorem 4] (see also [8]) which is a consequence of Nesterenko's theorem [11] on the algebraic independence of the values of Ramanujan functions. This implies that $\text{trans.deg}_{\mathbf{Q}}\mathbf{K}=2$, namely, the numbers ξ_1 and ξ_2 are algebraically independent over $\mathbf{Q}$. The proof of Theorem 2 is completed.

Remark 1. *It should be noted that the transcendence of* ξ_1 *is an immediate consequence of a result of Barré-Sirieix, Diaz, Gramain and Philibert* [2] (*see also* [3, Theorem 3]). *Indeed, by Theorem* 1 *and the identities* (10) *and* (11), *we have*

$$\xi_1^2 = 8\beta^{-5/2}\frac{\vartheta_2(\beta^2)}{\vartheta_4(\beta^2)}.$$

Thus, we find by [2, Théorème (i)] *that the number* $\vartheta_2(\beta^2)/\vartheta_4(\beta^2)$ *is transcendental, and hence, so is* ξ_1.

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