Euler and Laplace integral representations of GKZ hypergeometric functions II

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Abstract: We establish the intersection theory of the rapid decay homology group and formulate the twisted period relation in this setting. We claim that there is a standard method of constructing a basis of the rapid decay homology group which can be related to GKZ hypergeometric series. This can be carried out with the aid of a convergent regular triangulation T. When T is unimodular, we can obtain a closed formula of the homology intersection number. Finally, we obtain a Laurent series expansion formula of the cohomology intersection number in terms of the combinatorics of T.

Key words: GKZ hypergeometric systems; integral representations; twisted Gauß-Manin connections; twisted intersection numbers; quadratic relations.

1. Introduction. This is the continuation of the announcement "Euler and Laplace integral representations of GKZ hypergeometric functions I". We use the same notation. In this announcement, we claim that there is a combinatorial construction of a basis of rapid decay homology group which can be related to Γ-series solutions. Namely, to each convergent regular triangulation T we have a basis consisting of cycles Γ_T and that of series solutions Φ_T which are related to each other in terms of character matrices of finite abelian groups associated to T. As an application, when T is unimodular, we can determine the twisted intersection matrix explicitly. One can also prove an explicit expansion formula of cohomology intersection number associated to a unimodular triangulation. Details are available in [MHa, §4-].

2. Intersection theory for rapid decay homology and cohomology groups. In this section, we develop an intersection theory of rapid decay homology groups along the line of the preceding studies [CM95], [Iwa03], [KY94] and [OST03]. We consider a smooth complex affine variety U and a regular singular connection (E,∇) on U. In order to simplify the discussion and the notation, we assume that E is a trivial bundle and ∇ is given by $\nabla=d+\sum_{i=1}^k\alpha_i\frac{df_i}{f_i}\wedge$ for some regular functions $f_i\in\mathcal{O}(U)\setminus\mathbf{C}$ and complex numbers α_i. For another regular function $f\in\mathcal{O}(U)$, we set $\nabla_f=\nabla+df\wedge$. We inherit the notation of §3 of [MHb]. Let $\widetilde{V}\subset\widetilde{X}$ be an open subset. A section of the sheaf $\mathcal{P}^{<D}_{\widetilde{X}}$ (resp. $\mathcal{P}^{\mathrm{mod}D}_{\widetilde{X}}$) on $\widetilde{V}$ is a smooth function on $\widetilde{V}\setminus\widetilde{D}$ whose derivative of any order is flat (resp. of moderate growth) along D in the sense of §8.3 of [Sab13]. We set $\mathcal{A}^{?D}_{\widetilde{X}}=\mathrm{Ker}\Big(\bar\partial:\mathcal{P}^{?D}_{\widetilde{X}}\to\mathcal{P}^{?D}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathcal{O}_X}\varpi^{-1}\Omega^{(0,1)}_X\Big)$, where ? stands for one of either $<$ or mod. As in [Hie09], we set

$$DR^{?D}_{\widetilde{X}}(\nabla_f):=\mathcal{A}^{?D}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathcal{O}_{X^{an}}}\varpi^{-1}DR_{X^{an}}(\nabla_f). \tag{1}$$

Let $\nabla^\vee_f$ be the dual connection of ∇_f. With this notation, we set ${}^\vee\mathcal{S}^{<D}:=\mathcal{H}^0\Big(DR^{<D}_{\widetilde{X}}(\nabla^\vee_f)\Big)$ and $\mathcal{S}^{\mathrm{mod}D}:=\mathcal{H}^0\Big(DR^{\mathrm{mod}D}_{\widetilde{X}}(\nabla_f)\Big)$.

We first remark that $DR^{<D}_{\widetilde{X}}(\nabla^\vee_f)$ and $DR^{\mathrm{mod}D}_{\widetilde{X}}(\nabla_f)$ are resolutions of ${}^\vee\mathcal{S}^{<D}$ and $\mathcal{S}^{\mathrm{mod}D}$ respectively ([Hie09, Proposition 1]). Combining this result with the quasi-isomorphism

$$\mathcal{A}^{?D}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathcal{O}_{X^{an}}}\varpi^{-1}\Omega^r_X\xrightarrow{\sim}\Big(\mathcal{P}^{?D}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathcal{O}_X}\varpi^{-1}\Omega^{(r,\bullet)}_X,\bar\partial\Big), \tag{2}$$

we see that ${}^\vee\mathcal{S}^{<D}$ and $\mathcal{S}^{\mathrm{mod}D}$ are quasi-isomorphic to $\mathcal{P}DR^{<D}_{\widetilde{X}}(\nabla^\vee_f)\overset{def}{=}\Big(\mathcal{P}^{<D}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathcal{O}_X}\varpi^{-1}\Omega^{(\bullet,\bullet)}_X,\nabla^\vee_f,\bar\partial\Big)$ and $\mathcal{P}DR^{\mathrm{mod}D}_{\widetilde{X}}(\nabla_f)\overset{def}{=}\Big(\mathcal{P}^{\mathrm{mod}D}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathcal{O}_X}\varpi^{-1}\Omega^{(\bullet,\bullet)}_X,\nabla_f,\bar\partial\Big)$, respectively. We set

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$$\mathrm{H}^*_{r.d.}(U,\nabla_f^\vee)=\mathbf{H}^*(\tilde{X};{}^\vee\mathcal{S}^{<D}). \tag{3}$$

Let $\tilde{j}:U^{an}\to\widetilde{X}$ be the natural inclusion. Then, the canonical duality pairing $DR^{<D}_{\tilde{X}}(\nabla_f^\vee)\otimes DR^{\mathrm{mod}D}_{\tilde{X}}(\nabla_f)\to\tilde{j}_!\mathbf{C}_{U^{an}}$ ([Hie09, Theorem 3]) yields a perfect pairing

$$\begin{array}{cccc} \langle\bullet,\bullet\rangle_{ch}:\mathrm{H}^*_{dR}(U;\nabla_f)\times\mathrm{H}^{2n-*}_{r.d.}(U,\nabla_f^\vee)\to & & \mathbf{C} \\ \Psi & & \Psi \\ ([\omega],[\eta]) & \mapsto & \displaystyle\int_{U^{an}}\omega\wedge\eta, \end{array} \tag{4}$$

which we call the cohomology intersection pairing. We also give a realization of $\mathbf{H}^*(\widetilde{X},\mathcal{S}^{\mathrm{mod}D})$ in terms of a certain relative homology group. We set $\widetilde{D^{\mathrm{mod}}}=(\widetilde{D_{irr}}\setminus\tilde{f}^{-1}(S^1\infty))\cup\tilde{f}^{-1}(\{e^{\theta\sqrt{-1}}\infty\mid-\frac{\pi}{2}<\theta<\frac{\pi}{2}\})$. We consider a sequence of natural inclusions $U^{an}\xrightarrow{l}U^{an}\cup\widetilde{D^{\mathrm{mod}}}\xrightarrow{k}\tilde{X}$. By the local description of $\mathcal{S}^{\mathrm{mod}D}$ ([Hie09, p12]), we can easily confirm the equality $\mathcal{S}^{\mathrm{mod}D}=k_!l_*\mathcal{L}^\vee$. We define a sheaf $\mathcal{C}^{-p}_{U^{an}\cup\widetilde{D^{\mathrm{mod}}},\widetilde{D^{\mathrm{mod}}}}(\mathcal{L}^\vee)$ on $\tilde{X}$ as the associated sheaf of a presheaf whose section on an open subset V is given by $S_p(U^{an}\cup\widetilde{D^{\mathrm{mod}}},(U^{an}\cup\widetilde{D^{\mathrm{mod}}}\setminus V)\cup\widetilde{D^{\mathrm{mod}}};\ l_*\mathcal{L}^\vee)$. Here, S_p is the set of singular p-chains. We easily see that $\mathcal{S}^{\mathrm{mod}D}_{\tilde{X}}[2n]\simeq k_!\mathcal{C}^{-*}_{U^{an}\cup\widetilde{D^{\mathrm{mod}}},\widetilde{D^{\mathrm{mod}}}}(\mathcal{L}^\vee)$ as in the arguments after Proposition 2.1 of [MHc]. Therefore, we have a realization $\mathbf{H}^*(\tilde{X};\mathcal{S}^{\mathrm{mod}D})=\mathrm{H}^{\mathrm{mod}}_{2n-*}(U,\nabla_f)\overset{def}{=}\mathrm{H}_{2n-*}(U^{an}\cup\widetilde{D^{\mathrm{mod}}},\widetilde{D^{\mathrm{mod}}};l_*\mathcal{L}^\vee)$. Moreover, the same argument as §5 of [Hie09] proves the perfectness of the pairing

$$\begin{array}{cccc} \mathrm{H}^*_{r.d.}(U,\nabla_f^\vee)\times\mathrm{H}^{\mathrm{mod}}_{2n-*}(U,\nabla_f)\to & & \mathbf{C} \\ \Psi & & \Psi \\ ([\eta],[\delta^\vee]) & \mapsto & \displaystyle\int_{\delta^\vee}e^{-f}\prod_{i=1}^k f_i^{-\alpha_i}\eta. \end{array} \tag{5}$$

With these setups, we can naturally define the Poincaré duality isomorphism $\Phi:\mathrm{H}^{r.d.}_*(U^{an},\nabla_f^\vee)\xrightarrow{\sim}\mathrm{H}^{2n-*}_{r.d.}(U,\nabla_f^\vee)$ and $\Phi^\vee:\mathrm{H}^{\mathrm{mod}}_*(U,\nabla_f)\xrightarrow{\sim}\mathrm{H}^{2n-*}_{dR}(U,\nabla_f)$. Namely, for any element $[\gamma]\in\mathrm{H}^{r.d.}_*(U^{an},\nabla_f^\vee)$, $\Phi(\gamma)\in\mathrm{H}^{2n-*}_{r.d.}(U,\nabla_f^\vee)$ is the unique element such that the equality $\int_\gamma e^f\prod_{i=1}^k f_i^{\alpha_i}\omega=\int\Phi(\gamma)\wedge\omega$ holds for any $[\omega]\in\mathrm{H}^*_{dR}(U,\nabla_f)$. In the same way, for any element $[\delta^\vee]\in\mathrm{H}^{\mathrm{mod}}_*(U,\nabla_f)$, $\Phi^\vee(\delta^\vee)\in\mathrm{H}^{2n-*}_{dR}(U,\nabla_f)$ is the unique element such that the equality $\int_{\delta^\vee}e^{-f}\prod_{i=1}^k f_i^{-\alpha_i}\eta=\int\Phi^\vee(\delta^\vee)\wedge\eta$ holds for any $[\eta]\in\mathrm{H}^*_{r.d.}(U,\nabla_f^\vee)$. We define the homology intersection pairing $\langle\bullet,\bullet\rangle_h$ by

$$\begin{array}{cccc} \langle\bullet,\bullet\rangle_h:\mathrm{H}^{r.d.}_*(U,\nabla_f^\vee)\times\mathrm{H}^{\mathrm{mod}}_{2n-*}(U,\nabla_f)\to & & \mathbf{C} \\ \Psi & & \Psi \\ ([\gamma],[\delta^\vee]) & \mapsto & \displaystyle\int\Phi(\gamma)\wedge\Phi^\vee(\delta^\vee). \end{array} \tag{6}$$

Let us fix four bases $\{[\omega_i]\}_{i=1}^r\subset\mathrm{H}^*_{dR}(U;\nabla_f)$, $\{[\gamma_i]\}_{i=1}^r\subset\mathrm{H}^{r.d.}_*(U;\nabla_f^\vee)$, $\{[\eta_i]\}_{i=1}^r\subset\mathrm{H}^{2n-*}_{r.d.}(U;\nabla_f^\vee)$, and $\{[\delta_i^\vee]\}_{i=1}^r\subset\mathrm{H}^{\mathrm{mod}}_{2n-*}(U;\nabla_f)$. We set $I_{ch}=(\langle[\omega_i],[\eta_j]\rangle_{ch})_{i,j}$, $I_h=(\langle[\gamma_i],[\delta_j^\vee]\rangle_h)_{i,j}$, $P=(\int_{\gamma_j}e^f\prod_{l=1}^k f_l^{\alpha_l}\omega_i)_{i,j}$, and $P^\vee=(\int_{\delta_j^\vee}e^{-f}\prod_{l=1}^k f_l^{-\alpha_l}\eta_i)_{i,j}$. The twisted period relation [CM95, Theorem 2] is a transcendental analogue of Riemann-Hodge bilinear relations:

$$I_{ch}=P\,{}^tI_h^{-1}\,{}^tP^\vee. \tag{7}$$

Remark 2.1. While preparing this announcement, the author noticed that one can find a similar discussion in [FSY].

3. Γ-series solutions of GKZ systems. In this section, we briefly recall the construction of a basis of solutions of GKZ system in terms of Γ-series following the exposition of M.-C. Fernández-Fernández ([FF10]). For any commutative ring R and for any pair of finite sets I and J, the symbol $R^{I\times J}$ denotes the set of matrices with entries in R whose rows (resp. columns) are indexed by elements of I (resp. J). For any univariate function F and for any vector $w={}^t(w_1,\dots,w_d)\in\mathbf{C}^{d\times1}$, we define $F(w)$ by $F(w)=F(w_1)\cdots F(w_d)$. For any subset $\tau\subset\{1,\dots,n\}$, the symbol A_τ denotes the matrix given by the columns of A indexed by elements of τ. Let $\bar{\tau}$ be the complement of τ. In the following, we take $\sigma\subset\{1,\dots,n\}$ such that the cardinality $|\sigma|$ is equal to d and $\det A_\sigma\neq0$. Since A_σ^{-1} is naturally regarded as an element of $\mathbf{Q}^{\sigma\times d}$, we write $p_{\sigma i}(v)$ for the i-th entry $(i\in\sigma)$ of the vector $A_\sigma^{-1}v$ with $v\in\mathbf{C}^{d\times1}$. We write $\mathbf{1}_\sigma$ for the vector in $\mathbf{Z}^{\sigma\times1}$ whose entries are all unity. For any vector $\mathbf{k}\in\mathbf{Z}^{\bar{\sigma}\times1}$, we put

$$\varphi_{\sigma,\mathbf{k}}(z;\delta) \tag{8}$$

$$\overset{\mathrm{def}}{=}z_\sigma^{-A_\sigma^{-1}\delta}\sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}}\frac{(z_\sigma^{-A_\sigma^{-1}A_{\bar{\sigma}}}z_{\bar{\sigma}})^{\mathbf{k}+\mathbf{m}}}{\Gamma(\mathbf{1}_\sigma-A_\sigma^{-1}(\delta+A_{\bar{\sigma}}(\mathbf{k}+\mathbf{m})))(\mathbf{k}+\mathbf{m})!},$$

where $\Lambda_{\mathbf{k}}$ is given by

$$\Lambda_{\mathbf{k}}=\{\mathbf{k}+\mathbf{m}\in\mathbf{Z}^{\bar{\sigma}\times1}_{\geq0}\mid A_{\bar{\sigma}}\mathbf{m}\in\mathbf{Z}A_\sigma\}. \tag{9}$$

We call $\varphi_{\sigma,\mathbf{k}}(z;\delta)$ a Γ-series. It can readily be seen that $\varphi_{\sigma,\mathbf{k}}(z;\delta)$ is a solution of the GKZ hypergeometric system $M_A(\delta)$.

Now we recall the definition of a regular triangulation. In general, for any subset σ of $\{1,\dots,n\}$, $\mathrm{cone}(\sigma)$ denotes the cone generated by the set $\{\mathbf{a}(i)\}_{i\in\sigma}$. A collection T of subsets of $\{1,\dots,n\}$ is called a triangulation if $\{\mathrm{cone}(\sigma)\mid\sigma\in T\}$ is the set of cones in a simplicial fan whose support equals $\mathrm{cone}(A)$. For any generic choice of a vector $\omega\in\mathbf{R}^{1\times n}$, we can define a triangulation $T(\omega)$ as follows: a subset $\sigma\subset\{1,\dots,n\}$ belongs to $T(\omega)$ if there exists a vector $\mathbf{n}\in\mathbf{R}^{1\times d}$ such that $\mathbf{n}\cdot\mathbf{a}(i)=\omega_i$ if $i\in\sigma$ and $\mathbf{n}\cdot\mathbf{a}(j)<\omega_j$ if $j\in\overline{\sigma}$. A triangulation T is called a regular triangulation if $T=T(\omega)$ for some $\omega\in\mathbf{R}^{1\times n}$. Note that $T(\omega)$ can be empty for a specific choice of A and ω. For a fixed non-empty regular triangulation T, we say that the parameter vector δ is very generic if for any simplex $\sigma\in T$, the vector $A_\sigma^{-1}(\delta+A_{\overline{\sigma}}\mathbf{m})$ does not have any integer entry for any $\mathbf{m}\in\mathbf{Z}^{\overline{\sigma}\times 1}$. Now suppose δ is very generic. Then, it was shown in [FF10] that we have $\mathrm{rank}\,M_A(\delta)=\mathrm{vol}_{\mathbf{Z}}(\Delta_A)$. Here, we write $\mathrm{vol}_{\mathbf{R}}$ for the d-dimensional Lebesgue measure and set $\mathrm{vol}_{\mathbf{Z}}=\mathrm{vol}_{\mathbf{R}}/d!$. Let us put $H_\sigma=\{j\in\{1,\dots,n\}\mid |A_\sigma^{-1}\mathbf{a}(j)|=1\}$. Here, $|A_\sigma^{-1}\mathbf{a}(j)|$ denotes the sum of all entries of the vector $A_\sigma^{-1}\mathbf{a}(j)$. We set $U_\sigma=\{z\in(\mathbf{C}^*)^n\mid \mathrm{abs}(z_\sigma^{-A_\sigma^{-1}\mathbf{a}(j)}z_j)<R,\text{ for all } a(j)\in H_\sigma\setminus\sigma\}$, where $R>0$ is a small positive real number and abs stands for the absolute value.

Definition 3.1. A regular triangulation T is said to be convergent if it is non-empty and for any n-simplex $\sigma\in T$ and for any $j\in\overline{\sigma}$, one has the inequality $|A_\sigma^{-1}\mathbf{a}(j)|\leq 1$.

Note that a convergent regular triangulation always exists. With this terminology, the following result is a special case of [FF10, Theorem 6.7.].

Proposition 3.2. *Fix a convergent regular triangulation T. Assume δ is very generic. For each simplex $\sigma\in T$, let $\{\mathbf{k}(i)\}_{i=1}^{r_\sigma}$ be a system of complete representatives of the finite abelian group $\mathbf{Z}^{d\times1}/\mathbf{Z}A_\sigma$. Then, the set $\bigcup_{\sigma\in T}\{\varphi_{\sigma,\mathbf{k}(i)}\}_{i=1}^{r_\sigma}$ is a basis of holomorphic solutions of $M_A(\delta)$ on $U_T\overset{def}{=}\bigcap_{\sigma\in T}U_\sigma\neq\varnothing$.*

4. Combinatorial construction of integration contours via regular triangulations.

We use the notation of §2 of [MHb]. We fix a convergent regular triangulation T. After a sequence of changes of coordinates, we can construct a basis $\bigcup_{\sigma\in T}\{\Gamma_{\sigma,\tilde{\mathbf{k}}(j)}\}_{j=1}^{r_\sigma}$ of the rapid decay homology group $\mathrm{H}_{n,z}^{r.d.}$ at a generic point z. Here, the index $\{\tilde{\mathbf{k}}(j)\}_{j=1}^{r_\sigma}$ is a system of complete representatives of the finite abelian group $\mathbf{Z}^{\sigma\times1}/\mathbf{Z}^tA_\sigma$. The concrete form of the cycle $\Gamma_{\sigma,\tilde{\mathbf{k}}(j)}$ is, in general, quite complicated. For the detail, see [MHa, §6]. Let $\bigcup_{\sigma\in T}\{\check{\Gamma}_{\sigma,\tilde{\mathbf{k}}(j)}\}_{j=1}^{r_\sigma}$ be the corresponding basis of the dual homology group ([MHa, §6]).

Proposition 4.1. *If $\sigma_1\neq\sigma_2$, for any elements $\tilde{\mathbf{k}}_1\in\mathbf{Z}^{\sigma_1\times1}/\mathbf{Z}^tA_{\sigma_1}$ and $\tilde{\mathbf{k}}_2\in\mathbf{Z}^{\sigma_2\times1}/\mathbf{Z}^tA_{\sigma_2}$, one has the equality*

$$\langle\Gamma_{\sigma_1,\tilde{\mathbf{k}}_1},\check{\Gamma}_{\sigma_2,\tilde{\mathbf{k}}_2}\rangle_h=0. \tag{10}$$

Moreover, we have the following explicit relations between the basis $\bigcup_{\sigma\in T}\{\Gamma_{\sigma,\tilde{\mathbf{k}}(j)}\}_{j=1}^{r_\sigma}$ of $\mathrm{H}_{n,z}^{r.d.}$ and the Γ-series solutions.

Theorem 4.2. *Take a convergent regular triangulation T. Assume that the parameter vector δ is very generic, and $\gamma_l\notin\mathbf{Z}$ for any $l=1,\dots,k$. Then, there exists a basis $\bigcup_{\sigma\in T}\{\Gamma_{\sigma,\tilde{\mathbf{k}}(j)}\}_{j=1}^{r_\sigma}$ of the rapid decay homology group $\mathrm{H}_{n,z}^{r.d.}$ such that if one puts*

$$f_{\sigma,\tilde{\mathbf{k}}(j)}(z)=\frac{1}{(2\pi\sqrt{-1})^{n+k}}\int_{\Gamma_{\sigma,\tilde{\mathbf{k}}(j)}}e^{h_{0,z^{(0)}}(x)}\prod_{l=1}^{k}h_{l,z^{(l)}}(x)^{-\gamma_l}x^c\frac{dx}{x}, \tag{11}$$

$\bigcup_{\sigma\in T}\{f_{\sigma,\tilde{\mathbf{k}}(j)}(z)\}_{j=1}^{r_\sigma}$ is a basis of solutions of $M_A(\delta)$ on the non-empty open set U_T, where $\{\tilde{\mathbf{k}}(j)\}_{j=1}^{r_\sigma}$ is a complete system of representatives of $\mathbf{Z}^{\sigma\times1}/\mathbf{Z}^tA_\sigma$. Moreover, for each $\sigma\in T$, one has a transformation formula

$$\begin{pmatrix} f_{\sigma,\tilde{\mathbf{k}}(1)}(z)\\ \vdots\\ f_{\sigma,\tilde{\mathbf{k}}(r_\sigma)}(z)\end{pmatrix}=T_\sigma\begin{pmatrix}\varphi_{\sigma,\mathbf{k}(1)}(z;\delta)\\ \vdots\\ \varphi_{\sigma,\mathbf{k}(r_\sigma)}(z;\delta)\end{pmatrix}. \tag{12}$$

Here, T_σ is an $r_\sigma\times r_\sigma$ matrix given by

$$\begin{aligned}T_\sigma=&\frac{\mathrm{sgn}(A,\sigma)\prod_{l:|\sigma^{(l)}|>1}e^{-\pi\sqrt{-1}(1-\gamma_l)}\prod_{l:|\sigma^{(l)}|=1}e^{-\pi\sqrt{-1}\gamma_l}}{\det A_\sigma\Gamma(\gamma_1)\cdots\Gamma(\gamma_k)\prod_{l:|\sigma^{(l)}|=1}(1-e^{-2\pi\sqrt{-1}\gamma_l})}\\ &\times\mathrm{diag}(\exp\{-2\pi\sqrt{-1}{}^t\tilde{\mathbf{k}}(i)A_\sigma^{-1}\delta\})_{i=1}^{r_\sigma}\\ &\times(\exp\{-2\pi\sqrt{-1}{}^t\tilde{\mathbf{k}}(i)A_\sigma^{-1}A_{\overline{\sigma}}\mathbf{k}(j)\})_{i,j=1}^{r_\sigma}\\ &\times\mathrm{diag}\left(1-\exp\left\{-2\pi\sqrt{-1}\sum_{i\in\sigma^{(0)}}p_{\sigma i}(\delta+A_{\overline{\sigma}}\mathbf{k}(j))\right\}\right)_{j=1}^{r_\sigma},\end{aligned} \tag{13}$$

where we have put $sgn(A,\sigma)=(-1)^{k|\sigma^{(0)}|+(k-1)|\sigma^{(1)}|+\cdots+|\sigma^{(k-1)}|+\frac{k(k-1)}{2}}$.

We say that T is unimodular if for any simplex $\sigma\in T$, one has the equality $\det A_\sigma=\pm 1$. In this case, there is a single cycle $\Gamma_{\sigma,0}$ associated with each simplex $\sigma\in T$.

Theorem 4.3. *Suppose T is convergent and unimodular. We decompose σ into $\sigma=\sigma^{(0)}\cup\cdots\cup\sigma^{(k)}$ and set $\gamma_0=\sum_{i\in\sigma^{(0)}}p_{\sigma i}(\delta)$. For any simplex $\sigma\in T$, one puts*

(14)

$$\Xi=\prod_{l:|\sigma^{(l)}|>1}\left\{(1-e^{2\pi\sqrt{-1}\gamma_l})\prod_{i\in\sigma^{(l)}}(1-e^{-2\pi\sqrt{-1}p_{\sigma i}(\delta)})\right\}.$$

Then, one has

$$\langle\Gamma_{\sigma,0},\check{\Gamma}_{\sigma,0}\rangle_h=\Xi \tag{15}$$

if $\sigma^{(0)}=\varnothing$ and

$$\langle\Gamma_{\sigma,0},\check{\Gamma}_{\sigma,0}\rangle_h=(1-e^{-2\pi\sqrt{-1}\gamma_0})\Xi \tag{16}$$

if $\sigma^{(0)}\neq\varnothing$.

The proof of Theorem 4.3 is based on the explicit description of the cycle $\Gamma_{\sigma,0}$. The evaluation of the intersection number $\langle\Gamma_{\sigma,0},\check{\Gamma}_{\sigma,0}\rangle_h$ can be reduced to that of simple cycles of a simple connection. See §7 of [MHa] for the detail.

As an immediate application of the theorem above, we get an expansion formula of the cohomology intersection number. We set

(17) $\varphi^{\vee}_{\sigma,\mathbf{k}}(z;\delta)$

$$=z_\sigma^{A_\sigma^{-1}\delta}\sum_{\mathbf{k}+\mathbf{m}\in\Lambda_\mathbf{k}}\frac{(-1)^{\mathbf{k}_0+\mathbf{m}_0}e^{\pi\sqrt{-1}\sum_{i\in\sigma^{(0)}}p_{\sigma i}(A_{\bar\sigma}(\mathbf{k}+\mathbf{m}))}(z_\sigma^{-A_\sigma^{-1}A_{\bar\sigma}}z_{\bar\sigma})^{\mathbf{k}+\mathbf{m}}}{\Gamma(\mathbf{1}_\sigma+A_\sigma^{-1}(\delta-A_{\bar\sigma}(\mathbf{k}+\mathbf{m})))(\mathbf{k}+\mathbf{m})!}.$$

For any complex numbers α,β such that $\alpha+\beta\notin\mathbf{Z}_{\leq 0}$, we put $(\alpha)_\beta=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$. In general, for any vectors $\alpha=(\alpha_1,\dots,\alpha_s),\beta=(\beta_1,\dots,\beta_s)\in\mathbf{C}^s$, we put $(\alpha)_\beta=\prod_{i=1}^s(\alpha_i)_{\beta_i}$. Combining the results of §2, §3 and §4, we obtain the main result of this section.

Theorem 4.4. *Suppose that four vectors $\mathbf{a},\mathbf{a}'\in\mathbf{Z}^{n\times 1},\mathbf{b},\mathbf{b}'\in\mathbf{Z}^{k\times 1}$ and a convergent unimodular regular triangulation T are given. If the parameter δ is generic so that δ is non-resonant, $\gamma_l\notin\mathbf{Z}$ for any $l=1,\dots,k$, and $\begin{pmatrix}\gamma-\mathbf{b}\\ c+\mathbf{a}\end{pmatrix}$ and $\begin{pmatrix}\gamma+\mathbf{b}'\\ c-\mathbf{a}'\end{pmatrix}$ are very generic, then, for any $z\in U_T$, one has an identity*

$$\begin{aligned}(-1)^{|\mathbf{b}|+|\mathbf{b}'|}&\gamma_1\cdots\gamma_k(\gamma-\mathbf{b})_\mathbf{b}(-\gamma-\mathbf{b}')_{\mathbf{b}'}\\ &\times\sum_{\sigma\in T}\frac{\pi^{n+k}}{\sin\pi A_\sigma^{-1}d}\varphi_{\sigma,0}\left(z;\begin{pmatrix}\gamma-\mathbf{b}\\ c+\mathbf{a}\end{pmatrix}\right)\\ &\times\varphi^{\vee}_{\sigma,0}\left(z;\begin{pmatrix}\gamma+\mathbf{b}'\\ c-\mathbf{a}'\end{pmatrix}\right)\\ &=\frac{\langle x^\mathbf{a}h^\mathbf{b}\frac{dx}{x},x^{\mathbf{a}'}h^{\mathbf{b}'}\frac{dx}{x}\rangle_{ch}}{(2\pi\sqrt{-1})^n}.\end{aligned}\tag{18}$$

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