The second moment for counting prime geodesics

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Abstract: A brighter light has freshly been shed upon the second moment of the Prime Geodesic Theorem. We work with such moments in the two and three dimensional hyperbolic spaces. Letting $E_\Gamma(X)$ be the error term arising from counting prime geodesics associated to $\Gamma = \mathrm{PSL}_2(\mathbf{Z}[i])$, the bound $E_\Gamma(X) \ll X^{3/2+\epsilon}$ is proved in a square mean sense. Our second moment bound is the pure counterpart of the work of Balog *et al.* for $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, and the main innovation entails the delicate analysis of sums of Kloosterman sums. We also infer pointwise bounds from the standpoint of the second moment. Finally, we announce the pointwise bound $E_\Gamma(X) \ll X^{67/42+\epsilon}$ for $\Gamma = \mathrm{PSL}_2(\mathbf{Z}[i])$ by an application of the Weyl-type subconvexity.

Key words: Prime Geodesic Theorem; L-functions; subconvexity; spectral summation formulæ; Kloosterman sums; exponential sums.

1. Introduction. The purpose of this article is to elucidate applications of the second moment of the Prime Geodesic Theorem. For a brief introduction in two dimensions, we denote $\Lambda_\Gamma(P) = \log N(P_0)$ if P is a power of an underlying primitive hyperbolic element $P_0 \in \Gamma$ with $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ a cofinite Fuchsian group, and $\Lambda_\Gamma(P) = 0$ otherwise. We then define the Chebyshev-like counting function

$$\Psi_\Gamma(X) = \sum_{N(P) \leqslant X} \Lambda_\Gamma(P),$$

where $N(P)$ stands for the norm of P. As one of the early triumphs of his celebrated trace formula, Selberg [21] proved for a general cofinite Γ that

$$\Psi_\Gamma(X) = \sum_{1/2 < s_j \leqslant 1} \frac{X^{s_j}}{s_j} + E_\Gamma(X),$$

where the full main term arises from the small eigenvalues $\lambda_j = s_j(1-s_j) = 1/4 + t_j^2 < 1/4$ of the hyperbolic Laplacian acting on $L^2(\Gamma\backslash\mathcal{H})$ with $\mathcal{H}$ the upper half-plane. The error $E_\Gamma(X)$ is the principal subject in this article. It is well-known that $E_\Gamma(X) \ll X^{3/4}$ for a cofinite Γ. Given the analogue of the Riemann hypothesis for Selberg zeta functions except for a finite number of the exceptional zeros, one may expect $E_\Gamma(X) \ll X^{1/2+\epsilon}$ at least when $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ (cf. [9, p.139], [11,12]). This remains an outstanding open problem owing to the abundance of eigenvalues.

Nonetheless, when Γ is arithmetic, one can improve upon the exponent $3/4$. We record the classical achievement for $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ due to Iwaniec [9] with the exponent $35/48 + \epsilon$ by an application of the Kuznetsov formula. The constant $35/48$ was subsequently lowered to $7/10$ by Luo–Sarnak [14], $71/102$ by Cai [7], and $25/36$ by Soundararajan–Young [22]. Quite recently, the innovative rederivation of the exponent $25/36$ has been produced by Balkanova–Frolenkov [2,3] as the outcome of the effective use of Zagier's L-functions. Their exponent is the finest amongst many consequences to date. Actually, it has been understood that the exponent can further be reduced in a square mean sense. The set-up we are interested in is as follows: We consider the *second moment* of $E_\Gamma(X)$ that refers to

$$\frac{1}{\Delta}\int_V^{V+\Delta} |E_\Gamma(X)|^2 dX \tag{1.1}$$

for some V and $1 \ll \Delta \leqslant V$. We shall generalize the recent work of Cherubini–Guerreiro [8] to the following shape involving the parameters V and Δ.

Theorem 1.1. *Let* $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, *and let* $1 \ll \Delta \leqslant V$. *For any* $\epsilon > 0$ *we then have*

$$\frac{1}{\Delta}\int_V^{V+\Delta} |E_\Gamma(X)|^2 dX \ll V^{5/4+\epsilon}\left(\frac{V}{\Delta}\right)^{1/2}.$$

This bound is slightly weaker than that of Balog *et al.* [5] at the very least when $V = \Delta$. The reason for this fact will be explained later conceptually. The author thanks G. Cherubini for pointing

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out that Cherubini–Guerreiro indeed obtained the same result after their work [8]. The pointwise exponent 2/3 is known to be deduced from the Lindelöf hypothesis for Dirichlet L-functions of real primitive characters.

Analogously, one can consider the Prime Geodesic Theorem for the Picard manifold $\mathcal{M} = \mathrm{PSL}_2(\mathcal{O})\backslash\mathcal{H}^3$ with $\mathcal{O} = \mathbf{Z}[i]$ and $\mathcal{H}^3 \cong \mathrm{SL}_2(\mathbf{C})/\mathrm{SU}_2(\mathbf{C})$. For a general setting, let $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ be a cofinite Kleinian group, and let $\Psi_\Gamma(X)$ be the analogous counting function associated to Γ, which counts hyperbolic and loxodromic conjugacy classes of Γ. In our situation, the small eigenvalues $\lambda_j = s_j(2-s_j) < 1$ provide a finite number of terms that form the full main term of $\Psi_\Gamma(X)$, namely

$$E_\Gamma(X) = \Psi_\Gamma(X) - \sum_{1<s_j\leqslant 2} \frac{X^{s_j}}{s_j}.$$

In a major breakthrough, for cofinite Kleinian groups Sarnak [20] essentially established that $E_\Gamma(X) \ll X^{5/3+\epsilon}$. For $\Gamma = \mathrm{PSL}_2(\mathcal{O})$, Balkanova *et al.* [1] recently proved, by imitating the argument of Luo–Sarnak [14], that the exponent $13/8+\epsilon$ is admissible. Balog *et al.* [4] confirmed that $E_\Gamma(X) \ll X^{3/2+4\theta/7+\epsilon}$, where θ signifies a subconvex exponent for Dirichlet L-functions $L(1/2, \chi_D)$ with $\chi_D(n) = (n/D)$ the Kronecker symbol for the quadratic extension of $\mathbf{Q}(i)$. For later discussions, we mention the hybrid subconvex bound

$$L(1/2+it, \chi_D) \ll (1+|t|)^A N(D)^{\theta+\epsilon}. \tag{1.2}$$

The convexity bound renders $\theta = 1/4$, and we tackle the subconvexity problem to make θ smaller. Wu [24] succeeded in showing the subconvex exponent $\theta = 1/4 - (1-2\alpha)/16$ of Burgess quality over general number fields, where α is the sharpest result known hitherto towards the Ramanujan–Petersson conjecture. Nelson [19] recently established (in his tour de force argument) general Motohashi-type *spectral reciprocity*, followed by the Weyl-type subconvex exponent $\theta = 1/6$ (that marks the limit of current technology). His result in tandem with the estimate of Balog *et al.* [4] leads to $E_\Gamma(X) \ll X^{67/42+\epsilon}$. The original Motohashi formula [17] asserts that the fourth moment of the Riemann zeta function has a beautiful expansion as a spectral sum of third powers of central L-values of automorphic forms for $\mathrm{SL}_2(\mathbf{Z})$. We will ponder over the spectral reciprocity (in different contexts) for some families of L-functions in the forthcoming article, in order to derive subconvex bounds for them. Turning our eyes to $E_\Gamma(X)$, Koyama [13, Theorem 1.1] derived the exponent $11/7+\epsilon$ under the mean-Lindelöf hypothesis for symmetric square L-functions attached to Hecke–Maaß cusp forms on $\Gamma\backslash\mathcal{H}^3$.

Theorem 1.2 (Kaneko [10])**.** *Let* $\Gamma = \mathrm{PSL}_2(\mathcal{O})$*, and let* η *be the subconvex exponent defined by* (4.3)*. For* $1 \ll \Delta \leqslant V$*, we then have*

$$\frac{1}{\Delta}\int_V^{V+\Delta} |E_\Gamma(X)|^2 dX \ll_\epsilon V^{\frac{2(7+4\eta)}{5+2\eta}+\epsilon}\left(\frac{V}{\Delta}\right)^{\frac{4}{5+2\eta}} + V^3(\log V)^2. \tag{1.3}$$

As a corollary, Theorem 1.2 enunciates (upon taking $\eta = 1/2$ established in [1]) that the bound $E_\Gamma(X) \ll X^{3/2+\epsilon}$ is valid on average:

Corollary 1.3. *Suppose the hypotheses in Theorem* 1.2*. For any* $\epsilon > 0$*, then we have*

$$\frac{1}{\Delta}\int_V^{V+\Delta} |E_\Gamma(X)|^2 dX \ll_\epsilon \Delta^{-2/3} V^{11/3+\epsilon}.$$

The average bound on $E_\Gamma(X)$ corresponding to the estimate for the second moment is better than the best pointwise bound currently known. Our approach to prove Theorem 1.2 entails the delicate analysis of the Kuznetsov formula, and is the pure counterpart of the recent work of Balog *et al.* [5] on the modular surface. We wish that the second term in the right hand side of (1.3) would not arise. At present, if we reduce the exponent η, namely if $\eta < 1/2$, then the second term in (1.3) ends up dominating the first one. Theorem 1.2 has possibility for reaching a rather better bound if $V^3(\log V)^2$ does not appear (this second term is neglected if $\Delta \leqslant V^{(3+2\eta)/4}$). Recently for a general cofinite group Γ, Balkanova *et al.* [1, Theorem 1.2] gave the average bound $E_\Gamma(X) \ll X^{8/5+\epsilon}$. Their analysis rests on the Selberg trace formula for a suitably chosen test function, whereas we will utilize the Kuznetsov formula instead. Using a mean-to-max argument, they [1, Remark 1.4] also ascertained that a second moment bound of the type $\ll V^{\beta+\epsilon}\Delta^{-\gamma}$ in short intervals gives rise to $E_\Gamma(X) \ll X^{\alpha+\epsilon}$ with $\alpha = (\beta+\gamma)/(2+\gamma)$. Thus, Corollary 1.3 leads to $E_\Gamma(X) \ll X^{13/8+\epsilon}$. Adapting separately the first and second terms in (1.3) to $V^{\beta+\epsilon}\Delta^{-\gamma}$ conduces to $E_\Gamma(X) \ll X^{(11+4\eta)/(7+2\eta)+\epsilon}$. If we assume $\eta = 0$, this bound agrees with [13, Theorem 1.1]. Nonetheless, we announce without a proof that the above unconditional exponent 13/8 can further be superseded by the stronger pointwise bound:

Theorem 1.4. *Let* $\Gamma = \mathrm{PSL}_2(\mathcal{O})$*. Suppose* *that the subconvex bound* (1.2) *holds for some* $A>0$ *and a real number* $\theta \geqslant 0$. *For* $X \gg 1$, *we have*

$$E_\Gamma(X) \ll X^{3/2+\varpi/2+\epsilon}, \quad \varpi = \frac{24\theta - 1 + 2\eta(1+8\theta)}{23+10\eta}.$$

This is a priori the sheer counterpart of the theorem of Soundararajan–Young [22, Theorem 1.1], and the generalization of [4, Corollaries 1.4 and 1.5]. The method of proof of Theorem 1.4 is inspired by the recent works of Nelson [19] and Balog *et al.* [4]. We will employ the Weyl-type subconvexity for quadratic Dirichlet L-functions over $\mathbf{Q}(i)$. Substituting $\theta = 1/6$ and $\eta = 1/2$, Theorem 1.4 renders the state-of-the-art (unconditional) pointwise bound on $E_\Gamma(X)$, and quantitatively surpasses the quality of the result of Balog *et al.* [4, Corollary 1.2] with the exponent $13/8 - (177 - \sqrt{31049})/32 \approx 1.60023$. Their inventive idea is underpinned by the connection with the Gauß circle problem and a zero density theorem for Dirichlet L-functions $L(s,\chi_D)$ (which is in fact the three dimensional analogue of the work of Bykovskiĭ [6]). In passing, Theorem 1.4 improves upon the conditional exponent $\delta = 11/7 \approx 1.57143$ (under the assumption $\eta = 0$) due to Koyama [13].

2. Pointwise bounds in two dimensions. In this section, let $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$. For $X>1$, we define the spectral exponential sum $S(T,X) := \sum_{t_j \leqslant T} X^{it_j}$. There are advantages of introducing this function whereby one may pass between bounds on both second moments of $E_\Gamma(X)$ and $S(T,X)$ in short intervals via the explicit formula [9]. Nevertheless, one cannot simply bound $S(T,X)$ by summing up the terms with absolute values to break the barrier $S(T,X) \ll T^2$. We henceforth use the abbreviation $X \preccurlyeq Y \overset{\mathrm{def}}{\Longleftrightarrow} X \ll_\epsilon Y(NV)^\epsilon$ for some $N>1$ and free variables X and Y. By arguing similarly to [8] one sees that Theorem 1.1 follows from

$$\frac{1}{\Delta}\int_V^{V+\Delta} |S(T,X)|^2 dX \preccurlyeq \Delta^{-1/2} T^2 V^{3/4}. \tag{2.1}$$

We can actually prove a sophisticated version of this estimate with the exact classification by means of the size of T (see [5, Section 3]). One reduces the evaluation of the inequality (2.1) to that of a smoothened variant. Since the strategy for showing (2.1) is quite the same as in [8], we omit the detailed analysis. However, we have not yet found a proper generalization of [5, Theorem 1] due to technical difficulties on the estimation of some oscillatory integral. In passing, Theorem 1.1 only focuses on the full modular group $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$. For a cofinite Γ, one can mimic the argument in [8] again, inferring $\frac{1}{\Delta}\int_V^{V+\Delta} |E_\Gamma(X)|^2 dX \ll V^2 \Delta^{-2/3}$ to get back to the pointwise exponent $3/4+\epsilon$. The following claim generally connects second moment bounds with pointwise bounds.

Lemma 2.1. *Let* Γ *be a cofinite Fuchsian group and* $1 \ll \Delta \leqslant V$. *Suppose that we have the second moment bound of the shape* $\frac{1}{\Delta}\int_V^{V+\Delta} |E_\Gamma(X)|^2 dX \ll V^\beta \Delta^{-\gamma}$ *for some* $\beta, \gamma > 0$. *Then we have* $E_\Gamma(X) \ll X^{\alpha+\epsilon}$ *with* $\alpha = \beta/(2+\gamma)$.

This is the incarnation of the statement in [1, Remark 1.4]. The proof of Lemma 2.1 is back-of-the-envelope, and uses a mean-to-max argument along with reductio ad absurdum. The author thanks D. Chatzakos and G. Cherubini for letting him know the precise mechanism of that argument and a different proof of Lemma 2.1. Therefore Theorem 1.1 says $E_\Gamma(X) \ll X^{7/10+\epsilon}$ by substituting $\beta = 7/4$ and $\gamma = 1/2$, that coincides with the classical result of Luo–Sarnak [14]. There should exist an approach to further ameliorate this pointwise exponent $7/10$ from the standpoint of the second moment. On an average point of view, we believe that the $2k$-th moment of $E_\Gamma(X)$ is effective to deduce the conjectural exponent $1/2$ in some sense. However this process demands a nontrivial bound on the spectral higher moment of Rankin–Selberg L-functions. This is still open at present. To this topic we will return elsewhere.

3. Set-up in three dimensions. In the rest of this article, we consider $S(T,X)$ associated with the spectrum for $\Gamma = \mathrm{PSL}_2(\mathcal{O})$. In this case, its barrier is replaced with $S(T,X) \ll T^3$ by the Weyl law. We denote the complete set of cusp forms by $\{u_j(v) : j = 1,2,3,\dots\}$ attached to the eigenvalues $\lambda_j = 1 + t_j^2$ with the sign convention $t_j > 0$. We shall assume the u_j's to be chosen so that they are simultaneous eigenfunctions of the ring of Hecke operators and L^2-normalized. The Fourier expansion of $u_j(v)$ reads

$$u_j(v) = \sum_{n \in \mathcal{O}^*} \rho_j(n) y K_{it_j}(2\pi|n|y) e(\langle n, z\rangle), \tag{3.1}$$

where $\langle w, z\rangle$ is the standard inner product in $\mathbf{R}^2 \cong \mathbf{C}$ and $e(x) = \exp(2\pi i x)$. The Fourier coefficients $\rho_j(n)$ are proportional to the Hecke eigenvalues $\lambda_j(n)$, i.e.

$$\rho_j(n) = \rho_j(1)\lambda_j(n), \quad n \in \mathcal{O}^*. \tag{3.2}$$

For a Hecke–Maaß cusp form u_j displayed in (3.1) we introduce the Rankin–Selberg convolution

$$L(s, u_j \otimes u_j) := \sum_{n \in \mathcal{O}^*} \frac{|\rho_j(n)|^2}{N(n)^s}.$$

We then define the symmetric square L-function

$$L(s, \mathrm{sym}^2 u_j) = \sum_{n \in \mathcal{O}^*} \frac{c_j(n)}{N(n)^s} = \frac{\zeta_K(2s)}{\zeta_K(s)} \sum_{n \in \mathcal{O}^*} \frac{|\lambda_j(n)|^2}{N(n)^s},$$

where $c_j(n) = \sum_{\ell^2 k = n} \lambda_j(k^2)$. Given the convention

$$\rho_j(n) = \sqrt{\frac{\sinh \pi t_j}{t_j}} v_j(n) \quad \text{with} \quad v_j(n) = v_j(1)\lambda_j(n),$$

it is important to recast $L(s, \mathrm{sym}^2 u_j)$ as

$$L(s, \mathrm{sym}^2 u_j) = \frac{\zeta_K(2s)}{\zeta_K(s)} L(s, u_j \otimes u_j) \frac{t_j}{\sinh \pi t_j} |v_j(1)|^{-2}.$$

We now introduce the harmonic weights

$$\alpha_j := |v_j(1)|^2 = \frac{|\rho_j(1)|^2 t_j}{\sinh \pi t_j} = \frac{16\pi}{L(1, \mathrm{sym}^2 u_j)}.$$

Now, Gaussian Kloosterman sums are defined as follows: Letting $m, n, c \in \mathcal{O}$ with $c \neq 0$,

$$\mathcal{S}(m, n; c) := \sum_{a \in (\mathcal{O}/(c))^\times} e(\langle m, a/c \rangle) e(\langle n, a^*/c \rangle),$$

where $aa^* \equiv 1 \pmod{c}$. Weil's bound for the Kloosterman sums was derived in [17, (3.5)]: $|\mathcal{S}(m, n; c)| \leqslant N(c)^{1/2} |(m, n, c)| d(c)$, where $d(c)$ signifies the number of divisors of c. The Kuznetsov formula for $\mathrm{PSL}_2(\mathcal{O}) \backslash \mathcal{H}^3$ was first announced in the seminal work of Motohashi [15,16]. It is embodied in

Theorem 3.1. *Let $h(r)$ be even, holomorphic in $|\Im(r)| < 1/2 + \epsilon$ for an arbitrary fixed $\epsilon > 0$, and suppose that $h(t) \ll (1 + |t|)^{-3-\epsilon}$ in that strip. For $m, n \in \mathcal{O}^*$ we then have $D + C = U + S$, where*

$$D = \sum_{j \geqslant 1} \frac{\rho_j(n)\overline{\rho_j(m)}}{\sinh \pi t_j} t_j h(t_j),$$

$$C = 2\pi \int_{-\infty}^{\infty} \frac{\sigma_{it}(n)\sigma_{it}(m)}{|mn|^{it} |\zeta_K(1 + it)|^2} h(t) dt,$$

$$U = \pi^{-2}(\delta_{m,n} + \delta_{m,-n}) \int_{-\infty}^{\infty} t^2 h(t) dt,$$

$$S = \sum_{c \in \mathcal{O}^*} \frac{\mathcal{S}(m, n; c)}{N(c)} \psi\left(\frac{2\pi\sqrt{mn}}{c}\right).$$

Here $\delta_{m,n}$ is the Kronecker delta, and

$$\psi(z) = \int_{-\infty}^{\infty} \frac{it^2}{\sinh \pi t} h(t) \mathcal{J}_{it}(z) dt,$$

$$\mathcal{J}_\nu(z) = (|z|/2)^{2\nu} J_\nu^*(z) J_\nu^*(\overline{z}),$$

where $J_\nu^*(z) = J_\nu(z)(z/2)^{-\nu}$ with J_ν being the J-Bessel function of order ν.

4. Second moment. In order to gain a good estimate for (1.1) in three dimensions, our method relies heavily on the Kuznetsov formula and the spectral second moment bound for Rankin–Selberg L-functions. Our work is motivated by recent results on the second moment as in [5,8].

With the Kuznetsov formula in mind, for $X, T > 2$ so that $2\beta := \log X + i/T$, we deal with

$$\varphi(z) := \frac{\sinh \beta}{\pi} z \exp(iz \cosh \beta),$$

whose Bessel–Kuznetsov transform satisfies

$$\hat{\varphi}(t) := \frac{\pi i}{2 \sinh \pi t} \int_0^\infty (J_{2it}(x) - J_{-2it}(x)) \varphi(x) \frac{dx}{x}$$
$$= \frac{\sinh(\pi + 2i\beta)t}{\sinh \pi t} = X^{it} e^{-t/T} + O(e^{-\pi t}).$$

It should be stressed that the spectral weights $\hat{\varphi}(t_j)$ depend solely on the parameters X, T. One sees that D in Theorem 3.1 turns out to be

$$\sum_j \hat{\varphi}(t_j) |v_j(n)|^2 = \sum_j \alpha_j \hat{\varphi}(t_j) |\lambda_j(n)|^2$$

thanks to (3.2). This is sometimes called the spectral-arithmetic average. A routine calculation yields that the contribution of C and U is bounded by T^2. This way, our goal is reduced to analyzing the second moment of a sum of Kloosterman sums. To this end, we provide several auxiliary lemmas (without proofs). For notational convenience, we assume that $n \in \mathcal{O}$ satisfies $N(n) \sim N$, which means $N \leqslant N(n) \leqslant 2N$ for $N > 1$, and that T, X, V and Δ are real numbers satisfying $1 \ll \Delta \leqslant V \leqslant X \leqslant V + \Delta$ and $1 \ll T \leqslant V^{1/2}$. Throughout this article, we further assume that $N \ll (TX)^A$ for some fixed $A > 0$. We wish to establish the following:

Theorem 4.1. *We have*

$$\frac{1}{\Delta} \int_V^{V+\Delta} |\mathcal{S}_n(\psi)|^2 dX \preccurlyeq \Delta^{-1} N V^2 + T^3, \tag{4.1}$$

where

$$\mathcal{S}_n(\psi) = \sum_{c \in \mathcal{O}^*} \frac{\mathcal{S}(n, n; c)}{N(c)} \psi\left(\frac{2\pi\overline{n}}{c}\right).$$

A vehicle to establish Theorem 4.1 is as follows: we first need to cleverly remove some initial part of $\mathcal{S}_n(\psi)$, and then we can naturally replace ψ with a simpler function $\tilde{\psi}$. We next undertake representing $\tilde{\psi}$ in terms of the K-Bessel function of order zero, from which we infer that $\mathcal{S}_n(\psi)$ is replaceable with a certain *finite* sum of Kloosterman sums weighted by the K-Bessel function with an acceptable error term. Having reduced the estimation of the second moment of $\mathcal{S}_n(\psi)$ to that of the finite sum, we encounter an oscillatory integral involving the K-Bessel function of order zero. This integral can be easily evaluated by bounding in absolute value, and by integration by parts. Notice that we can circumvent any stationary phase analysis. We summarize the above procedure in the following (see [10] for details):

Lemma 4.2. *Let*

$$\tilde{\psi}(z) = \int_{-\infty}^{\infty} \frac{it^2\hat{\varphi}(t)}{\sinh \pi t} \left|\frac{z}{2}\right|^{2it} \Gamma(1+it)^{-2} dt.$$

We then have $\mathcal{S}_n(\psi) = \mathcal{S}_n^{\flat}(\tilde{\psi}) + O(N^{1/2+\epsilon}T^{1+\epsilon})$, *where* $\mathcal{S}_n^{\flat}(\tilde{\psi})$ *signifies the weighted sum of Gaussian Kloosterman sums, more precisely*

$$\mathcal{S}_n^{\flat}(\tilde{\psi}) = \sum_{N(c)>4\pi^2 N(n)} \frac{\mathcal{S}(n,n;c)}{N(c)} \tilde{\psi}\left(\frac{2\pi\overline{n}}{c}\right).$$

Lemma 4.3. *Let* $N \geqslant 1$ *and* $n \in \mathcal{O}$. *We then have* $\mathcal{S}_n^{\flat}(\tilde{\psi}) = \mathcal{S}_n^{\sharp}(K_0) + O((NX)^{1/2+\epsilon})$ *with* $\mathcal{S}_n^{\sharp}(K_0)$ *being a finite sum counted with Gaussian Kloosterman sums, namely*

$$\mathcal{S}_n^{\sharp}(K_0) = 2iM^2 N(n) X \times \sum_{C_1 < N(c) \leqslant C_2} \frac{\mathcal{S}(n,n;c)}{N(c)^2} K_0\left(\frac{2\pi X^{1/2} M|n|}{|c|}\right),$$

where $M = \exp(-i(\pi/2 - 1/2T))$, $C_1 = N(n)V(T\log T)^{-2}$ *and* $C_2 = N(n)V$.

Lemma 4.4. *For* $z_1, z_2 > 0$ *with* $V^{-1/2} \leqslant z_j \ll 1$ $(j=1,2)$ *one has*

$$\frac{1}{\Delta}\int_V^{V+\Delta} K_0(X^{1/2}Mz_1)\overline{K_0(X^{1/2}Mz_2)}dX \ll (z_1z_2)^{-1/2}\min(V^{-1/2}, \Delta^{-1}|z_1 - z_2|^{-1}).$$

Lemma 4.4 tells us that the weight function $K_0(X^{1/2}M|z|)$ carries some oscillation in X, when integrating over $X \in [V, V+\Delta]$. After squaring out the c-sum in the integrand in the left hand side of (4.1), we split the resulting double sum into the diagonal and off-diagonal parts. Estimating these separately along with Lemma 4.4, Hardy–Littlewood–Pólya inequality, and Weil's bound on $\mathcal{S}(n,n;c)$, we finish the proof of Theorem 4.1.

Remark 4.5. From a simple consideration, one could deduce $\mathcal{S}_n^{\sharp}(K_0) \preccurlyeq (NTX)^{1/2}$, whence we find that $\mathcal{S}_n(\psi) \preccurlyeq (NTX)^{1/2}$. By appealing to this and choosing N and T suitably, the bound $E_\Gamma(X) \ll X^{(11+4\eta)/(7+2\eta)+\epsilon}$ will follow readily (cf. Theorem 4.7).

To conclude the proof of Theorem 1.2, let $h : (0,\infty) \to \mathbf{R}$ be a smooth compactly supported function with holomorphic Mellin transform $\tilde{h} : \mathbf{C} \to \mathbf{C}$. We choose h such that it is supported in some dyadic window $[\sqrt{N}, \sqrt{2N}]$ for $N > 1$, whose derivatives satisfy $|h^{(\ell)}(\xi)| \ll N^{-\ell/2}$ for $\ell = 0, 1, 2, \ldots$, and whose mean value is $\int_{-\infty}^{\infty} h(\xi)\xi d\xi = N$. Integrating by parts ℓ-times, we find that $\tilde{h}(s) \ll_{\sigma,\ell} N^{\sigma/2}(1+|s|)^{-\ell}$, the implied constant depending continuously on σ for $s = \sigma + it$. Following [5,14], we then derive for some explicitly given constant c,

$$\sum_j X^{it_j} e^{-t_j/T} = \frac{1}{cN} \sum_{n \in \mathcal{O}^*} h(|n|)\mathcal{S}_n(\psi) - \frac{1}{cN}\int_{(1/2)} \tilde{h}(2s) M_1(s) \frac{ds}{\pi i} + O(T^2) \tag{4.2}$$

where the error emerges from the terms C and U in the Kuznetsov formula (Theorem 3.1), and we put

$$M_1(s) = \sum_j \frac{t_j}{\sinh \pi t_j} \hat{\varphi}(t_j) L(s, u_j \otimes u_j).$$

The square mean integral of the first term in the right hand side of (4.2) is bounded in Theorem 4.1. As for the second term, we argue analogously to [5]; specifically we exploit the spectral second moment bound on symmetric square L-functions due to Balkanova *et al.* [1]. After our smoothing, we state the second moment of $S(T,X)$.

Corollary 4.6. *For any* $\epsilon > 0$ *we have*

$$\frac{1}{\Delta}\int_V^{V+\Delta} |S(T,X)|^2 dX \ll_\epsilon \Delta^{-1} T^{5/2+\eta} V^{3/2+\epsilon}.$$

In view of the explicit formula of Nakasuji [18], bounding the second moment of $E_\Gamma(X)$ is concerned with that of $S(T,X)$. Hence we have completed the proof of Corollary 1.3. A finer second moment bound on $E_\Gamma(X)$ has been established by the author [10].

Denote by η the extra exponent of T for the following mean value of Rankin–Selberg L-functions:

$$\sum_{t_j \sim T} \frac{t_j}{\sinh \pi t_j} |L(w, u_j \otimes u_j)| \ll |w|^A T^{3+\eta+\epsilon}, \tag{4.3}$$

where $\Re(w) = 1/2$. The convexity bound in the spectral aspect is $O(T^{4+\epsilon})$, while the mean-Lindelöf says $\eta = 0$. The best result so far established is $\eta = 1/2$ ([1, Corollary 3.4]), to which we have already alluded. Refining the argument of Koyama [13], we have

Theorem 4.7. *For η defined in* (4.3) *we have*

$$S(T,X) \ll_\epsilon T^{(7+2\eta)/4+\epsilon} X^{1/4+\epsilon} + T^2,$$

$$E_\Gamma(X) \ll_\epsilon X^{(11+4\eta)/(7+2\eta)+\epsilon}.$$

Finally, it is worth mentioning that the spectral large sieve inequality (still conjectural) would lead to the mean-Lindelöf hypothesis. As was faithfully explained in [23, Remark 2.4], the large sieve constant of Watt falls distinctly short of being best possible.

Conjecture 4.8. Let $\Gamma = \mathrm{PSL}_2(\mathcal{O})$. For T, $N \gg 1$ and $\mathbf{a} = \{a_n\}$ a sequence of complex numbers with ℓ^2 norm $\|\mathbf{a}_N\| = \sum_{N(n)\sim N} |a_n|^2$, we have

$$\sum_{t_j \leqslant T} \left| \sum_{N(n)\sim N} a_n v_j(n) \right|^2 \ll_\epsilon (N+T^3)(NT)^\epsilon \|\mathbf{a}_N\|.$$

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