Erdősian functions and an identity of Gauss

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Abstract: A famous identity of Gauss gives a closed form expression for the values of the digamma function $\psi(x)$ at rational arguments x in terms of elementary functions. Linear combinations of such values are intimately connected with a conjecture of Erdős which asserts non vanishing of an infinite series associated to a certain class of periodic arithmetic functions. In this note we give a different proof for the identity of Gauss using an orthogonality like relation satisfied by these functions. As a by product we are able to give a new interpretation for nth Catalan number in terms of these functions.

Key words: Dirichlet series; Erdős conjecture; Gauss identity; digamma function.

1. Introduction. The search for an analytic function which generalizes the notion of the well defined factorial function for natural numbers led to the definition of the complex valued Gamma function $\Gamma(z)$ which is defined as

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x}\, dx$$

when $Re(z) > 0$ and for the rest of the complex plane with the help of the functional equation

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

For more details on Gamma function and its properties reader may consult the book [2]. A well known function related to Gamma function is the digamma function $\psi(z)$ which is defined as the logarithmic derivative of the $\Gamma(z)$ as follows:

$$\psi(z) := \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Special values of the digamma function are important because of its frequent occurrence in the diverse areas of mathematics and physics. Unfortunately at present, very less is known about the nature of special values of the digamma function [15]. In fact it is not even known whether the value $\psi(1)$ is a rational or an irrational number which follows from the fact that $\psi(1) = -\gamma$ and the famous conjecture that γ is an irrational number [12]. However in 1812, in the celebrated memoir [9] on hypergeometric series by the famous mathematician Johann Carl Friedrich Gauss, the following closed form formula for the digamma function at rational arguments was given.

Theorem 1.1 (Gauss' Identity)**.** *For positive integers m, q such that $m < q$, the digamma function can be expressed in terms of Euler's constant γ and a finite number of elementary functions in the following way*

$$\psi(m/q) + \gamma = -\log 2q - \frac{\pi}{2}\cot\frac{\pi m}{q} + 2\sum_{a=1}^{\lfloor (q-1)/2\rfloor} \cos\left(\frac{2\pi a m}{q}\right)\log\left(\sin\frac{\pi a}{q}\right). \tag{2}$$

Contrary to the above identity (2), most of the known formulas for $\psi(z)$ rather follow easily from the definition of digamma function and some standard trigonometrical identities. On the other hand the Gauss' proof is still not straightforward even after it was simplified by Jensen [10] using Abel's theorem on continuity of convergent power series. As of now several different proofs exist in literature [3,7,11,13,16] which involve variety of tools such as functional equation of Hurwitz zeta function, Simpson's dissection method, Discrete Fourier analysis and integral representation of digamma function.

Here we shall give another proof of the identity (2) which uses a combinatorial identity exhibited by a certain class of periodic arithmetic functions.

Definition 1.2 (Erdősian function modulo q)**.** A function f is said to be Erdősian modulo q if it is a periodic arithmetic function with period q which

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takes values in the set $\{-1,+1\}$ everywhere except at q where it is 0.

For example, it is easy to see that we have the following 8 Erdősian functions modulo 4.

n	1	2	3	4
f_1	1	1	1	0
f_2	1	1	−1	0
f_3	1	−1	1	0
f_4	−1	1	1	0
f_5	1	−1	−1	0
f_6	−1	1	−1	0
f_7	−1	−1	1	0
f_8	−1	−1	−1	0

The functions are called Erdősian [5] mainly because of the following conjecture due to Erdős which was written to A. Livingston [14] in 1965.

Conjecture 1.3 (P. Erdős). For any Erdősian function f, the infinite series $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ can never be equal to zero.

For more details on the progress made on this conjecture and similar non-vanishing results see [4–6,16]. We recall that our goal here is to prove identity (2) and for this let us fix some notations. Let

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

denote the Dirichlet series associated to an arithmetic function f and

$$\zeta(s,x) := \frac{1}{(n+x)^s}$$

be the Hurwitz zeta function initially defined for real part of s greater than 1 and then over the whole complex plane except at $s=1$ using the principle of analytic continuation. Now for an Erdősian function f with period q it is easy to see that

$$L(s,f) = q^{-s} \sum_{a=1}^{q} f(a)\zeta\left(s, \frac{a}{q}\right), \tag{3}$$

where $\zeta(s,x)$ is the Hurwitz zeta function. It is well known that the Hurwitz zeta function has a simple pole at $s=1$ and admits the following expansion at its unique simple pole [1]

$$\zeta(s,x) = \frac{1}{s-1} - \psi(x) + O(s-1). \tag{4}$$

From the Eqs. (3) and (4) it is straightforward to deduce that $L(s,f)$ has a simple pole at $s=1$ with residue R_f where

$$R_f := \frac{1}{q}\sum_{a=1}^{q} f(a).$$

Therefore for Erdősian functions f, $L(1,f)$ converges if and only if R_f is equal to zero. Indeed, this can only happen when the period q is odd and hence the Erdős conjecture holds trivially for even q. Further in the case when q is odd one needs to only consider the case when R_f is zero. For such functions we state an orthogonality like relation below. The standard binomial coefficient $\binom{n}{m}$ is equal to the quantity $\frac{n!}{m!(n-m)!}$ and for a finite set S we write $|S|$ for the cardinality of S.

Proposition 1.4 (Orthogonality like relation). *Let $f_1, f_2, \ldots, f_{N(q)}$ denote the $N := N(q)$ Erdősian functions modulo q such that $R_{f_i} = 0$ for $1 \le i \le N(q)$. Let m and n be two positive integers. Then we have*

$$\sum_{r=1}^{N} f_r(m)f_r(n) = \begin{cases} N & \text{if } m \equiv n \ (mod\, q) \text{ and } q \nmid m, \\ 0 & \text{if } q \mid n \text{ or } q \mid m, \\ h(q) & \text{if } m \not\equiv n \ (mod\, q) \end{cases}$$

where $N(q) = \binom{q-1}{(q-1)/2}$, $h(q) = -2C_{\frac{q-3}{2}}$ *and* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the n^{th} Catalan number.*

Proof. First we observe that N is equal to the number of ways of assigning values $+1$ in half of the available $q-1$ positions which is nothing but $\binom{q-1}{(q-1)/2}$. Notice that all these f_r functions exhaust the possible ways of arranging 1 and -1 in $q-1$ positions provided $R_{f_r} = 0$. If $m \equiv n \ (mod\, q)$ then for any f_r, the product $f_r(m)f_r(n) = 1$ provided $q \nmid m$. Further if $q \mid m$ or $q \mid n$ then $f_r(m)f_r(n) = 0$. The last case where $m \not\equiv n \ (mod\, q)$ the value $h(q) := \sum_{r=1}^{N} f_r(m)f_r(n)$ will satisfy

$$h(q) = |\{r \mid f_r(m)f_r(n) = 1\}| - |\{r \mid f_r(m)f_r(n) = -1\}|. \tag{5}$$

The value of each term in the sum $\sum_{r=1}^{N} f_r(m)f_r(n)$ depends on whether the value of $f_r(m)$ and $f_r(n)$ is 1 or -1. If both are of same sign then the product $f_r(m)f_r(n)$ gives 1 otherwise -1. Same sign occurs when both of them are $+1$ or -1. Corresponding to each such case we have $q-3$ positions left to be filled which will add upto either $+2$ or -2. So number of ways that both $f_r(m)$ and $f_r(n)$ are of same sign are the possible ways of filling $\frac{q-1}{2}$ places in $q-3$ places with either $+1$ or -1. Thus we have

$$|\{r \mid f_r(m)f_r(n) = 1\}| = 2\binom{q-3}{(q-1)/2}.$$

As there are N terms in the sum $\sum_{r=1}^{N} f_r(m)f_r(n)$, we have

$$N = |\{r \mid f_r(m)f_r(n) = 1\}| + |\{r \mid f_r(m)f_r(n) = -1\}|. \tag{6}$$

Now from the above Eqs. (5) and (6) we get

$$\begin{aligned} h(q) &= 4\binom{q-3}{(q-1)/2} - \binom{q-1}{(q-1)/2} \\ &= 4\frac{(q-3)!}{\left(\frac{q-1}{2}\right)!\left(\frac{q-5}{2}\right)!} - \frac{(q-1)!}{\left(\frac{q-1}{2}\right)!\left(\frac{q-1}{2}\right)!} \\ &= -\frac{(q-1)!}{\left(\frac{q-1}{2}\right)!\left(\frac{q-1}{2}\right)!}\left[1 - 4\frac{\frac{q-3}{2}.\frac{q-1}{2}}{(q-2)(q-1)}\right] \\ &= -\frac{(q-1)!}{\left(\frac{q-1}{2}\right)!\left(\frac{q-1}{2}\right)!}.\frac{q-1}{(q-2)(q-1)} \\ &= \frac{-2}{\frac{q-1}{2}}\frac{(q-3)!}{\left(\left(\frac{q-3}{2}\right)!\right)^2} \\ &= -2C_{\frac{q-3}{2}}. \end{aligned}$$

This completes the proof. □

As a corollary we give a different way (see [17] for various other related counting problems) of producing Catalan numbers using the Erdősian functions.

Corollary 1.5. *The n^{th} Catalan number C_n is equal to $-\frac{1}{2}\sum_{r=1}^{N} f_r(1)f_r(2)$, where $N = \binom{2n+2}{n+1}$ and f_r's are all the Erdősian functions modulo $2n+3$.*

Now we are ready to give the proof of the Gauss' identity (2), but before that we need some preliminary lemmas.

2. Some Lemmas. In the rest of the section the symbol ζ_m shall denote the complex m^{th} root of unity $e^{\frac{2\pi i}{m}}$. It is easy to verify that

$$\sum_{a=1}^{q} \zeta_q^{-a(b-n)} = \begin{cases} q, & \text{if } b \equiv n \bmod q \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The following result follows from the identities (3) and (4) and is given as Theorem 16 in [15].

Lemma 2.1. *Let f be any function defined on the integers and with period q. Then, $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ converges if and only if $\sum_{a=1}^{q} f(a) = 0$, and in the case of convergence, the value of the series is*

$$-\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q).$$

Using the idea of the proof of Theorem 5 given in [16] one can derive the following expression for the value $L(1,f)$.

Lemma 2.2. *For a q periodic arithmetic function f satisfying $R_f = 0$, the value of $L(1,f)$ is given by*

$$L(1,f) = -\pi i\left(\frac{f(q)}{2q} - \frac{1}{q}\sum_{b=1}^{q-1}\frac{f(b)}{\zeta_q^b - 1}\right) - \frac{1}{q}\sum_{b=1}^{q} f(b)\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right). \tag{8}$$

Proof. As $L(1,f)$ is convergent because of Lemma 2.1, we can write

$$L(1,f) = \lim_{x\to\infty}\left(\sum_{b=1}^{q} f(b)\sum_{\substack{n\le x\\ n\equiv b \bmod q}}\frac{1}{n}\right).$$

Now with the help of Eq. (7) we get

$$\begin{aligned} L(1,f) &= \lim_{x\to\infty}\left(\sum_{b=1}^{q} f(b)\sum_{n\le x}\frac{1}{n}\left(\frac{1}{q}\sum_{a=1}^{q}\zeta_q^{-a(n-b)}\right)\right) \\ &= \frac{1}{q}\sum_{a=1}^{q-1}\sum_{b=1}^{q} f(b)\zeta_q^{ab}(-\log(1-\zeta_q^{-a})). \end{aligned}$$

Since

$$1-\zeta^{-a}=\zeta^{-a}(\zeta^{a/2}-\zeta^{-a/2})$$

we have on taking the principal logarithm

$$\log(1-\zeta_q^{-a})=\left(\frac{1}{2}-\frac{a}{q}\right)\pi i+\log\left(2\sin\frac{\pi a}{q}\right).$$

Substituting the above value in the expression for $L(1,f)$ we get

$$\begin{aligned}L(1,f)&=-\pi i\left(\sum_{a=1}^{q-1}\frac{1}{q}\sum_{b=1}^{q}f(b)\zeta_q^{ab}\left(\frac{1}{2}-\frac{a}{q}\right)\right)\\&\quad-\sum_{a=1}^{q-1}\frac{1}{q}\sum_{b=1}^{q}f(b)\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right)\\&=-\pi i\left(\frac{1}{2q}\sum_{b=1}^{q-1}f(b)\sum_{a=1}^{q-1}\zeta_q^{ab}-\frac{1}{q^2}\sum_{b=1}^{q}f(b)\sum_{a=1}^{q-1}a\zeta_q^{ab}\right)\\&\quad-\frac{1}{q}\sum_{b=1}^{q}f(b)\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right)\\&=-\pi i\left(\frac{f(q)}{2q}-\frac{1}{q}\sum_{b=1}^{q-1}\frac{f(b)}{\zeta_q^b-1}\right)\\&\quad-\frac{1}{q}\sum_{b=1}^{q}f(b)\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right).\end{aligned}$$

□

We also recall an important multiplication formula for the Gamma function which is attributed to Legendre.

$$\prod_{a=1}^{q}\Gamma\left(z+\frac{a-1}{q}\right)=q^{1/2-qz}(2\pi)^{(q-1)/2}\Gamma(qz).$$

Logarithmically differentiating the above identity and substituting $z=1/q$, we get

$$\sum_{b=1}^{q}\psi(b/q)=-q\log q-\gamma q. \tag{9}$$

Finally we evaluate the digamma function at rational arguments.

3. Proof of Theorem 1.1. In order to use the orthogonality like relations we let f_r denote an Erdősian function modulo an odd integer q and consider $m\in\mathbf{N}$ such that $1\le m<q$. Then by Lemma 2.2 we have

$$\begin{aligned}&\sum_{r=1}^{N}f_r(m)L(1,f_r)\\&=\sum_{r=1}^{N}f_r(m)\Bigg[-\pi i\left(\frac{f_r(q)}{2q}+\frac{1}{q}\sum_{b=1}^{q-1}\frac{f_r(b)}{1-\zeta_q^b}\right)\\&\quad-\frac{1}{q}\sum_{b=1}^{q}f_r(b)\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right)\Bigg]\\&=-\frac{\pi i}{q}\sum_{r=1}^{N}\sum_{b=1}^{q-1}\frac{f_r(m)f_r(b)}{1-\zeta_q^b}\\&\quad-\frac{1}{q}\sum_{r=1}^{N}f_r(m)\sum_{b=1}^{q-1}f_r(b)\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right).\end{aligned}$$

Further from Lemma 2.1 we get

$$\begin{aligned}&-\frac{1}{q}\sum_{b=1}^{q-1}\sum_{r=1}^{N}f_r(m)f_r(b)\psi(b/q)\\&=-\frac{\pi i}{q}\sum_{r=1}^{N}\sum_{b=1}^{q-1}\frac{f_r(m)f_r(b)}{1-\zeta_q^b}\\&\quad-\frac{1}{q}\sum_{r=1}^{N}f_r(m)\sum_{b=1}^{q-1}f_r(b)\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right).\end{aligned}$$

Now when $b=m$ then $f_r(m)f_r(b)=1$ otherwise $b\not\equiv m(\operatorname{mod} q)$. Therefore in the equation

$$\begin{aligned}&-\frac{1}{q}\sum_{b=1}^{q-1}\psi(b/q)\sum_{r=1}^{N}f_r(m)f_r(b)\\&=-\frac{\pi i}{q}\sum_{b=1}^{q-1}\frac{1}{1-\zeta_q^b}\sum_{r=1}^{N}f_r(m)f_r(b)\\&\quad-\frac{1}{q}\sum_{b=1}^{q-1}\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right)\sum_{r=1}^{N}f_r(m)f_r(b)\end{aligned}$$

we separate the terms when $b=m$ and $b\neq m$, which gives us

$$\begin{aligned}&-\frac{h(q)}{q}\sum_{\substack{b=1\\b\neq m}}^{q-1}\psi(b/q)-\frac{N}{q}\psi(m/q)\\&=-\frac{\pi i h(q)}{q}\sum_{\substack{b=1\\b\neq m}}^{q-1}\frac{1}{1-\zeta_q^b}-\frac{\pi i N}{q}\frac{1}{1-\zeta_q^m}\\&\quad-\frac{h(q)}{q}\sum_{\substack{b=1\\b\neq m}}^{q-1}\sum_{a=1}^{q-1}\zeta_q^{ab}\log\left(2\sin\frac{\pi a}{q}\right)\\&\quad-\sum_{a=1}^{q-1}\frac{N\zeta_q^{am}}{q}\log\left(2\sin\frac{\pi a}{q}\right).\end{aligned}$$

Further, with the help of Eq. (9) we get

$$\begin{aligned}&-h(q)\psi(m/q)+h(q)(\gamma-\gamma q)-qh(q)\log q\\&\quad+N\psi(m/q)=\pi i h(q)\sum_{b=1}^{q-1}\frac{1}{1-\zeta_q^m}-\pi i h(q)\frac{1}{1-\zeta_q^b}\end{aligned}$$

$$+ \pi i N \frac{1}{1-\zeta_q^m} + h(q)\sum_{a=1}^{q-1}\log\left(2\sin\frac{\pi a}{q}\right)\sum_{b=1}^{q-1}\zeta_q^{ab}$$
$$- h(q)\sum_{a=1}^{q-1}\log\left(2\sin\frac{\pi a}{q}\right)\zeta_q^{am}$$
$$+ N\sum_{a=1}^{q-1}\log\left(2\sin\frac{\pi a}{q}\right)\zeta_q^{am}.$$

Some more simplification and dividing the above equation with $h(q)$ gives

$$\begin{aligned}\frac{N-h(q)}{h(q)}\psi(m/q) + (\gamma-\gamma q) - q\log q \\ = \pi i\left(\frac{q-1}{2}\right) + \pi i\frac{N-h(q)}{h(q)}\frac{1}{1-\zeta_q^m} \\ - \sum_{a=1}^{q-1}\log\left(2\sin\frac{\pi a}{q}\right) \\ + \frac{N-h(q)}{h(q)}\sum_{a=1}^{q-1}\log\left(2\sin\frac{\pi a}{q}\right)\zeta_q^{am}.\end{aligned}$$

Now observe that $\dfrac{N-h(q)}{h(q)} = -(q-1)$. Further by using the expression

$$\sum_{a=1}^{q-1}\log\left(\sin\frac{\pi a}{q}\right) = \log q - (q-1)\log 2,$$

and

$$\frac{1}{1-\zeta_q^m} = \frac{1}{2} - \frac{1}{i}\cot\frac{\pi m}{q}$$

we obtain

$$\begin{aligned}\psi(m/q) = -\gamma - \log q - \frac{\pi i}{2}\left(\frac{1}{i}\cot\frac{\pi m}{q}\right) - \log 2 \\ + \sum_{a=1}^{q-1}\log\left(\sin\frac{\pi a}{q}\right)\zeta_q^{am}.\end{aligned}$$

Now write $\zeta_q^{am} = \cos\frac{2\pi am}{q} + i\sin\frac{2\pi am}{q}$ in the above equation and use the fact that cosine is an even function and sine is an odd function to obtain the desired result. To complete the proof we only need to consider the case when q is even. For this we give an outline of the proof as there is not much difference from the odd q case. Indeed, if we are given an even q we can alter the definition of an Erdősian function just by removing the 0 element from the range. In this case $L(1,f)$ will converge if and only if q is even and $R_f = 0$. The orthogonality like relation for these modified Erdősian functions can be similarly deduced as done in the proof of Proposition 1.4. The rest of the arguments are similar to the proof for the odd q case. This completes the proof. □

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