On normalization of quasi-log canonical pairs

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Abstract: The normalization of an irreducible quasi-log canonical pair naturally becomes a quasi-log canonical pair.

Key words: Quasi-log canonical pairs; normalization; Du Bois singularities.

1. Introduction. In [1], Florin Ambro introduced the notion of *quasi-log varieties*, which are now called *quasi-log schemes*, in order to establish the cone and contraction theorem for *generalized log varieties*. Note that a generalized log variety is a pair (X, Δ) consisting of a normal irreducible variety X and an effective $\mathbf{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbf{R}$-Cartier. Although the main result of [1] was recovered without using the theory of quasi-log schemes in [3], it became clear that quasi-log schemes are ubiquitous in the theory of minimal models (see, for example, [4] and [7]). As Ambro said in [1], the definition of quasi-log schemes is motivated by Kawamata's X-method. Therefore, it is not surprising that quasi-log schemes often appear naturally in the theory of minimal models. In this paper, we prove that the normalization of an irreducible *quasi-log canonical pair* (*qlc pair*, for short) becomes a quasi-log canonical pair. Note that the notion of quasi-log canonical pairs is one of the useful generalizations of log canonical pairs in the framework of quasi-log schemes. In general, a quasi-log canonical pair may be reducible and may not necessarily be equidimensional. We also note that the result of this paper plays a crucial role when we show that every quasi-log canonical pair has only Du Bois singularities in [11].

Let (X, Δ) be a log canonical pair and let W be a log canonical center of (X, Δ). Then $[W, \omega]$ has a natural qlc structure, where $\omega = (K_X + \Delta)|_W$. For the details, see Example 2.11 below and [7, 6.4.1 and 6.4.2]. Let $\nu : W^\nu \to W$ be the normalization. Then we expect that there exists an effective $\mathbf{R}$-divisor Δ_{W^ν} on W^ν such that (W^ν, Δ_{W^ν}) is log canonical and that $K_{W^\nu} + \Delta_{W^\nu} \sim_{\mathbf{R}} \nu^*\omega$. However, it is still a difficult open problem to find Δ_{W^ν} with the above properties. For a related topic, see [10]. By Theorem 1.1 below, which is the main theorem of this paper, we see that $[W^\nu, \nu^*\omega]$ naturally becomes a qlc pair. Therefore, we can apply the theory of quasi-log schemes to $[W^\nu, \nu^*\omega]$.

Theorem 1.1 (Normalization of qlc pairs)**.** *Let $[X, \omega]$ be a qlc pair such that X is irreducible. Let $\nu : Z \to X$ be the normalization. Then $[Z, \nu^*\omega]$ naturally becomes a qlc pair with the following properties*:

(i) *if C is a qlc center of $[Z, \nu^*\omega]$, then $\nu(C)$ is a qlc center of $[X, \omega]$, and*

(ii) $\mathrm{Nqklt}(Z, \nu^*\omega) = \nu^{-1}(\mathrm{Nqklt}(X, \omega))$. *More precisely, the equality*

$$\nu_*\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)} = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$$

holds, where $\mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$ and $\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)}$ are the defining ideal sheaves of $\mathrm{Nqklt}(X, \omega)$ *and* $\mathrm{Nqklt}(Z, \nu^*\omega)$ *respectively.*

For the definition of qlc pairs and $\mathrm{Nqklt}(X, \omega)$, see Definitions 2.4 and 2.7, respectively. By the theory of quasi-log schemes discussed in [7, Chapter 6] and Theorem 1.1, the fundamental theorems of the minimal model program hold for $[Z, \nu^*\omega]$. More precisely, the cone and contraction theorem and the basepoint-free theorem of Reid–Fukuda type hold for $[Z, \nu^*\omega]$ by [7, Theorem 6.4.7] and [7, Theorem 6.9.1] respectively (see also [6]). We can also apply various vanishing theorems to $[Z, \nu^*\omega]$. As a special case, we have the following vanishing theorem.

Corollary 1.2 (Vanishing theorem for normalizations)**.** *We use the same notation as in Theorem* 1.1. *Let $\pi : X \to S$ be a proper morphism onto a scheme S and let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then*

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$$R^i(\pi \circ \nu)_* \mathcal{O}_Z(\nu^* L) = 0$$

for every $i > 0$.

Let us discuss some conjectures for qlc pairs. The second author poses the following conjecture on Du Bois singularities.

Conjecture 1.3. Let $[X, \omega]$ be a qlc pair. Then X has only Du Bois singularities.

The statement of Conjecture 1.3 is a complete generalization of [14, Corollary 6.32]. For the details of Du Bois singularities, see [7, Section 5.3] and [14, Chapter 6]. By Theorem 1.1, we have:

Proposition 1.4. *It is sufficient to prove Conjecture* 1.3 *under the extra assumption that* X *is normal.*

Finally, we pose the following conjecture on normal qlc pairs.

Conjecture 1.5. Let $[X, \omega]$ be a qlc pair such that X is quasi-projective and normal. Then there exists an effective $\mathbf{Q}$-divisor Δ on X such that (X, Δ) is log canonical.

We note that [5, Theorem 1.1] strongly supports Conjecture 1.5. Of course, Conjecture 1.3 follows from Conjecture 1.5 by Proposition 1.4. This is because log canonical singularities are known to be Du Bois (see [14]).

Although Theorem 1.1 may look somewhat artificial, it plays an important role in [12], [13], and [11]. Roughly speaking, in [8], we prove that X is *generalized lc* in the sense of Birkar–Zhang (see [2]) with some good properties when $[X, \omega]$ is a normal irreducible quasi-log canonical pair. We can see it as a weak solution of Conjecture 1.5. We note that [8] heavily depends on the theory of variations of mixed Hodge structure on cohomology with compact support (see [9]). Then, in [11], we completely confirm Conjecture 1.3, that is, we show that X has only Du Bois singularities if $[X, \omega]$ is a quasi-log canonical pair.

We will work over $\mathbf{C}$, the complex number field, throughout this paper. A *scheme* means a separated scheme of finite type over $\mathbf{C}$. A *variety* means a reduced scheme, that is, a reduced separated scheme of finite type over $\mathbf{C}$. We will freely use the basic notation of the minimal model program as in [3], [4], and [7]. For the details of the theory of quasi-log schemes, we recommend the reader to see [7, Chapter 6].

2. Quick review of the theory of quasi-log schemes. In this section, we quickly review the theory of quasi-log schemes because it is not so popular yet.

Before we explain the definition of quasi-log canonical pairs, we prepare some basic definitions.

Definition 2.1 ($\mathbf{R}$-divisors)**.** Let X be an equidimensional variety, which is not necessarily regular in codimension one. Let D be an $\mathbf{R}$-divisor, that is, D is a finite formal sum $\sum_i d_i D_i$, where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is a real number for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i,$$

and

$$\lceil D \rceil = \sum_i \lceil d_i \rceil D_i,$$

where $\lceil d_i \rceil$ is the integer defined by $d_i \leq \lceil d_i \rceil < d_i + 1$.

Let B_1 and B_2 be $\mathbf{R}$-Cartier divisors on X. Then $B_1 \sim_{\mathbf{R}} B_2$ means that B_1 is $\mathbf{R}$-linearly equivalent to B_2.

We note that we can define $\mathbf{Q}$-*divisors* and $\sim_{\mathbf{Q}}$ similarly.

The notion of *globally embedded simple normal crossing pairs* play a crucial role in the theory of quasi-log schemes described in [7, Chapter 6].

Definition 2.2 (Globally embedded simple normal crossing pairs)**.** Let Y be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbf{R}$-divisor on M such that Y and B have no common irreducible components and that the support of $Y + B$ is a simple normal crossing divisor on M. In this situation, (Y, B_Y), where $B_Y := B|_Y$, is called a *globally embedded simple normal crossing pair*.

Definition 2.3 (Strata of simple normal crossing divisors)**.** Let Y be a simple normal crossing divisor on a smooth variety and let $Y = \bigcup_{i \in I} Y_i$ be the irreducible decomposition of Y. A *stratum* of Y is an irreducible component of $Y_{i_1} \cap \cdots \cap Y_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$.

Let us recall the definition of *quasi-log canonical pairs*.

Definition 2.4 (Quasi-log canonical pairs)**.** Let X be a scheme and let ω be an $\mathbf{R}$-Cartier divisor (or an $\mathbf{R}$-line bundle) on X. Let $f : Y \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Y, B_Y). If B_Y is a subboundary $\mathbf{R}$-divisor, that is, $B_Y = B_Y^{\leq 1}$, $f^*\omega \sim_{\mathbf{R}} K_Y + B_Y$ holds, and the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$$

is an isomorphism, then $(X, \omega, f : (Y, B_Y) \to X)$ is called a *quasi-log canonical pair* (*qlc pair*, for short). If there is no danger of confusion, we simply say that $[X, \omega]$ is a qlc pair.

The notion of *qlc strata* and *qlc centers* is very important. It is indispensable for inductive treatments of quasi-log canonical pairs.

Definition 2.5 (Qlc strata and qlc centers)**.** Let $(X, \omega, f : (Y, B_Y) \to X)$ be a quasi-log canonical pair as in Definition 2.4. Let $\nu : Y^\nu \to Y$ be the normalization. We put

$$K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y),$$

that is, Θ is the sum of the inverse images of B_Y and the singular locus of Y. Then (Y^ν, Θ) is sub log canonical in the usual sense. Let W be a log canonical center of (Y^ν, Θ) or an irreducible component of Y^ν. Then $f \circ \nu(W)$ is called a *qlc stratum* of $(X, \omega, f : (Y, B_Y) \to X)$. If there is no danger of confusion, we simply call it a qlc stratum of $[X, \omega]$. If C is a qlc stratum of $[X, \omega]$ but it is not an irreducible component of X, then C is called a *qlc center* of $(X, \omega, f : (Y, B_Y) \to X)$ or simply of $[X, \omega]$.

One of the most important results in the theory of quasi-log schemes is *adjunction*.

Theorem 2.6 (Adjunction, see [7, Theorem 6.3.5])**.** *Let $[X, \omega]$ be a qlc pair and let X' be the union of some qlc strata of $[X, \omega]$. Then $[X', \omega|_{X'}]$ is a qlc pair such that the qlc strata of $[X', \omega|_{X'}]$ are exactly the qlc strata of $[X, \omega]$ that are contained in X'.*

We strongly recommend the reader to see [7, Theorem 6.3.5] and its proof for the details of Theorem 2.6. Theorem 2.6 is a special case of [7, Theorem 6.3.5 (i)].

Definition 2.7 (Union of all qlc centers)**.** Let $[X, \omega]$ be a qlc pair. The union of all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$. It is very important that

$$[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$$

has a quasi-log canonical structure induced from $(X, \omega, f : (Y, B_Y) \to X)$ by adjunction (see Theorem 2.6 and [7, Theorem 6.3.5 (i)]).

The vanishing theorem is also a very important result. Theorem 2.8 is a special case of [7, Theorem 6.3.5 (ii)].

Theorem 2.8 (Vanishing theorem, see [7, Theorem 6.3.5])**.** *Let $[X, \omega]$ be a qlc pair and let $\pi : X \to S$ be a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$, that is, $L - \omega$ is π-nef and $(L - \omega)|_W$ is π-big for every qlc stratum W of $[X, \omega]$. Then $R^i\pi_*\mathcal{O}_X(L) = 0$ for every $i > 0$.*

The notion of $\mathbf{Q}$-*structures* is introduced in [8].

Definition 2.9 ($\mathbf{Q}$-structures)**.** If ω is a $\mathbf{Q}$-Cartier divisor (or a $\mathbf{Q}$-line bundle) on X, B_Y is a $\mathbf{Q}$-divisor on Y, and $f^*\omega \sim_{\mathbf{Q}} K_Y + B_Y$ holds in Definition 2.4, then we say that $(X, \omega, f : (Y, B_Y) \to X)$ has a $\mathbf{Q}$-structure or simply say that $[X, \omega]$ has a $\mathbf{Q}$-structure.

Remark 2.10. If $[X, \omega]$ has a $\mathbf{Q}$-structure, then we can easily see that for any union of qlc strata X' the qlc pair $[X', \omega|_{X'}]$ naturally has a $\mathbf{Q}$-structure in Theorem 2.6. For the details, see the proof of [7, Theorem 6.3.5].

We close this section with an important example.

Example 2.11. Let (X, Δ) be a log canonical pair. We put $\omega = K_X + \Delta$. Let $f : Y \to X$ be a resolution such that $K_Y + B_Y = f^*(K_X + \Delta)$. We assume that the support of B_Y is a simple normal crossing divisor on Y. Then we can easily see that (Y, B_Y) is a globally embedded simple normal crossing pair, $B_Y = B_Y^{\leq 1}$, and the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil (-B_Y^{<1})\rceil)$$

is an isomorphism. Therefore, we can see that $(X, \omega, f : (Y, B_Y) \to X)$ is a qlc pair. In this situation, W is a qlc stratum of $[X, \omega]$ if and only if W is a log canonical center of (X, Δ) or X itself.

Anyway, we recommend the reader to see [7, Chapter 6] for the theory of quasi-log schemes.

3. Proof. Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $f : (Y, B_Y) \to X$ be a proper surjective morphism from a globally embedded simple normal crossing pair (Y, B_Y) as in Definition 2.4. By [7, Proposition 6.3.1], we may assume that the union of all strata of (Y, B_Y) mapped to $\mathrm{Nqklt}(X, \omega)$, which is denoted by Y'', is a union of some irreducible components of Y. We put $Y' = Y - Y''$ and $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} Y' & \overset{\iota}{\hookrightarrow} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \underset{p}{\longrightarrow} & X \end{array}$$

where $\iota : Y' \to Y$ is a natural closed immersion and

$$Y' \overset{f'}{\longrightarrow} V \overset{p}{\longrightarrow} X$$

is the Stein factorization of $f \circ \iota : Y' \to X$. By construction, $\iota : Y' \to Y$ is an isomorphism over the generic point of X. By construction again, the natural map $\mathcal{O}_V \to f'_*\mathcal{O}_{Y'}$ is an isomorphism and every stratum of Y' is dominant onto V. Therefore, p is birational.

Claim 1. V is normal.

Proof of Claim 1. (cf. the proof of [7, Lemma 6.3.9]). Let $\pi : V^n \to V$ be the normalization. Since every stratum of Y' is dominant onto V, there exists a closed subset Σ of Y' such that $\mathrm{codim}_{Y'}\Sigma \geq 2$ and that $\pi^{-1} \circ f' : Y' \dashrightarrow V^n$ is a morphism on $Y' \setminus \Sigma$. Let $\widetilde{Y}$ be the graph of $\pi^{-1} \circ f' : Y' \dashrightarrow V^n$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Y} & \overset{q}{\longrightarrow} & Y' \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f'} \\ V^n & \underset{\pi}{\longrightarrow} & V \end{array}$$

where q and $\widetilde{f}$ are natural projections. Note that $q : \widetilde{Y} \to Y'$ is an isomorphism over $Y \setminus \Sigma$ by construction. Since Y' is a simple normal crossing divisor on a smooth variety and $\mathrm{codim}_{Y'}\Sigma \geq 2$, the natural map $\mathcal{O}_{Y'} \to q_*\mathcal{O}_{\widetilde{Y}}$ is an isomorphism. Therefore, the composition

$$\mathcal{O}_V \to \pi_*\mathcal{O}_{V^n} \to \pi_*\widetilde{f}_*\mathcal{O}_{\widetilde{Y}} = f'_*q_*\mathcal{O}_{\widetilde{Y}} \simeq \mathcal{O}_V$$

is an isomorphism. Thus we have $\mathcal{O}_V \simeq \pi_*\mathcal{O}_{V^n}$. This implies that V is normal. □

Therefore, $p : V \to X$ is nothing but the normalization $\nu : Z \to X$. So we have the following commutative diagram.

$$\begin{array}{ccc} Y' & \overset{\iota}{\hookrightarrow} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \underset{\nu}{\longrightarrow} & X \end{array}$$

Claim 2. The natural map

$$\alpha : \mathcal{O}_Z \to f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil)$$

is an isomorphism.

Proof of Claim 2. Note that $\nu : Z \to X$ is an isomorphism over $X \setminus \mathrm{Nqklt}(X, \omega)$. Therefore, α is an isomorphism outside $\nu^{-1}(\mathrm{Nqklt}(X, \omega))$. Since Z is normal and $f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil)$ is torsion-free, it is sufficient to see that α is an isomorphism in codimension one. Let P be any prime divisor on Z such that $P \subset \nu^{-1}(\mathrm{Nqklt}(X, \omega))$. Then, by construction, there exists an irreducible component of $B_{Y'}^{=1}$ which maps onto P. We note that every fiber of f is connected by $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Therefore, the effective divisor $\lceil -(B_{Y'}^{<1})\rceil$ does not contain the whole fiber of f' over the generic point of P. Thus, α is an isomorphism at the generic point of P. This means that α is an isomorphism. □

Therefore, by Claim 2, $f' : (Y', B_{Y'}) \to Z$ defines a quasi-log structure on $[Z, \nu^*\omega]$. By construction, the property (i) automatically holds. Let us consider the following ideal sheaf:

$$\begin{aligned} \mathcal{I} &= f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) \\ &\subset f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil) = \mathcal{O}_Z. \end{aligned}$$

We note that $\mathcal{I} = \mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*\omega)}$ holds since $\mathrm{Nqklt}(Z, \nu^*\omega) = f'(Y''|_{Y'})$.

Claim 3. $\mathcal{I}_{\mathrm{Nqklt}(X,\omega)} = f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'})$ holds.

Proof of Claim 3. (cf. the proof of [7, Theorem 6.3.5 (i)]). Since $\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) \subset \mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$, we get

$$f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) \subset f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) = \mathcal{O}_X,$$

that is, $f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'})$ is an ideal sheaf on X. By construction,

$$f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$$

holds. Here, we used the fact that every fiber of f is connected. □

Claim 3 implies that

$$\begin{aligned} \nu_*\mathcal{I} &= \nu_*f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) \\ &= f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}. \end{aligned}$$

Since ν is finite, $\mathcal{I} = \nu^{-1}\mathcal{I}_{\mathrm{Nqklt}(X,\omega)} \cdot \mathcal{O}_Z$. Therefore, we have $\nu^{-1}(\mathrm{Nqklt}(X, \omega)) = \mathrm{Nqklt}(Z, \nu^*\omega)$. This means that (ii) holds. □

Proof of Corollary 1.2. This follows from Theorems 1.1 and 2.8 (see also [7, Theorem 6.3.5 (ii)]). □

Finally, we prove Proposition 1.4.

Proof of Proposition 1.4. We prove Conjecture 1.3 under the extra assumption that Conjecture 1.3 holds true for normal qlc pairs. Let $[X, \omega]$

be a qlc pair. Let X_1 be an irreducible component of X and let X_2 be the union of the irreducible components of X other than X_1. Then X_1, X_2, and $X_1 \cap X_2$ are qlc pairs by adjunction (see Theorem 2.6 and [7, Theorem 6.3.5 (i)]). In particular, they are seminormal (see [7, Remark 6.2.11]). Then we have the following short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{X_1 \cap X_2} \to 0$$

(see, for example, [14, Lemma 10.21]). By [7, Lemma 5.3.9], it is sufficient to prove Conjecture 1.3 under the extra assumption that X is irreducible by induction on $\dim X$ and the number of the irreducible components of X. Therefore, from now on, we assume that X is irreducible. Let $\nu : Z \to X$ be the normalization. Then, by Theorem 1.1, we have

$$R\nu_* \mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*\omega)} = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}. \tag{3.1}$$

By induction on dimension, $\mathrm{Nqklt}(Z, \nu^*\omega)$ and $\mathrm{Nqklt}(X, \omega)$ are Du Bois since they are qlc (see Definition 2.7). Since Z is normal and $[Z, \nu^*\omega]$ is qlc, Z is Du Bois by assumption. Therefore, by [14, Corollary 6.28] and (3.1), X is Du Bois. This is what we wanted. □

We close this section with a remark on **Q**-structures.

Remark 3.1. If $[X, \omega]$ has a **Q**-structure in Theorem 1.1, then we can easily see that $[Z, \nu^*\omega]$ also has a **Q**-structure by the proof of Theorem 1.1.

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