

## Non-left-orderable surgeries on negatively twisted torus knots

By Kazuhiro ICHIHARA<sup>\*)</sup> and Yuki TEMMA<sup>\*\*)</sup>

(Communicated by Kenji FUKAYA, M.J.A., April 12, 2018)

**Abstract:** We show that certain negatively twisted torus knots admit Dehn surgeries yielding 3-manifolds with non-left-orderable fundamental groups.

**Key words:** Dehn surgery; left-orderable group; twisted torus knots.

**1. Introduction.** In [1], Boyer, Gordon and Watson raised the following conjecture, which is now called the *L-space conjecture*: An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. Here a rational homology 3-sphere is called an *L-space* if  $\text{rk}\widehat{HF}(M) = |H_1(M, \mathbf{Z})|$  holds, and a non-trivial group  $G$  is called *left-orderable* if there exists a strict total order  $<$  on  $G$  such that if  $g < h$ , then  $fg < fh$  holds for any  $f, g, h \in G$ . See [1] for more details.

One of the known approaches to the conjecture is by using Dehn surgery, for it gives a simple way to construct many L-spaces at once. For example, in [7], it is shown that a knot  $K$  in the 3-sphere  $S^3$  admits a Dehn surgery yielding an L-space, then up to replacing  $K$  by its mirror image, any Dehn surgery on  $K$  along a slope  $r$  with  $r \geq 2g(K) - 1$  yields an L-space, where  $g(K)$  denotes the genus of  $K$ .

Thus it is natural to ask if Dehn surgeries on a knot yielding 3-manifolds with non-left-orderable fundamental groups give L-spaces.

There are several known results on this line. For example, recall that finite groups cannot be left-orderable, and 3-manifolds with finite fundamental groups are known to be L-spaces. Thus a good target for such study is the class of knots admitting Dehn surgeries creating 3-manifolds with finite fundamental groups. A well-known class of the knots with such surgeries is that of twisted torus knots, originally given by Dean [4].

Here a (*positively*) *twisted torus knot* in  $S^3$  is

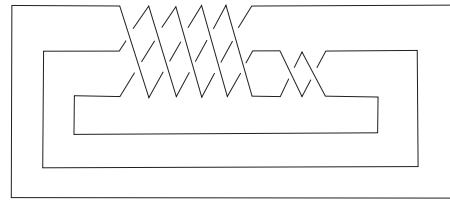


Fig. 1. 1-twisted torus knot of type (3, 5).

defined as the knot obtained from a torus knot by adding (positive) full twists along an adjacent pair of strands. See Figure 1 for example.

In fact, Clay and Watson considered Dehn surgeries on positively twisted torus knots in [3]. Precisely, they showed that for large  $r$ ,  $r$ -surgery on a positively  $s$ -twisted  $(3, 3k + 2)$ -torus knot gives a 3-manifold with non-left-orderable fundamental group if (1)  $s \geq 0$  and  $k = 1$ , or (2)  $s = 1$  and  $k \geq 0$ .

This can be compared to the result by Vafaei [8] showing that such positively twisted torus knots admit L-space surgery. Precisely, he showed that  $r$ -surgery on  $(p, kp \pm 1)$ -torus knots with  $s$  full twists on  $u$  adjacent strands gives an L-space if (1)  $u = p - 1$ , (2)  $u = p - 2$  and  $s = 1$ , or (3)  $u = 2$  and  $s = 1$ , for  $p \geq 2$ ,  $k \geq 1$ ,  $s > 0$ , and  $0 < u < p$ . In the case of  $p = 3$ , a positively 2-twisted  $(3, q)$ -torus knot ( $q > 0$ ,  $s \geq 1$ ) is of genus  $q + s - 1$ , and so, it follows that  $r$ -surgery on that knot gives a 3-manifold with non-left-orderable fundamental group if  $r \geq 2q + 2s - 3$ .

In [5], by refining the argument of Nakae [6] for pretzel knots, we have extended the result of Clay and Watson to show that  $r$ -surgery on a positively  $s$ -twisted torus knot of type  $(3, 3v + 2)$  with  $s, v \geq 0$  yields a closed 3-manifold with non-left-orderable fundamental group if  $r \geq 3(3v + 2) + 2s$ .

Soon after, Christianson, Goluboff, Hamann and Varadaraj gave further extensions in [2]. They actually show that  $r$ -surgery on  $(p, kp \pm 1)$ -torus

2010 Mathematics Subject Classification. Primary 57M50; Secondary 57M25.

<sup>\*)</sup> Department of Mathematics, College of Humanities and Sciences, Nihon University, 3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156-8550, Japan.

<sup>\*\*)</sup> MY.TECH CO., LTD., 1-40 Miyamachi, Fuchu, Tokyo 183-0023, Japan.

knots with  $s$  full twists on  $u$  adjacent strands gives a closed 3-manifold with non-left-orderable fundamental group if (1)  $u = p - 1$  and  $r \geq p(pk \pm 1) + (p - 1)^2s$ , or (2)  $u = p - 2$ ,  $s = 1$  and  $r \geq p(pk \pm 1) + (p - 2)^2$ , for  $p \geq 2$ ,  $k \geq 1$ ,  $s > 0$ ,  $0 < u < p$ . As an immediate corollary, it follows that  $r$ -surgery on a positively  $s$ -twisted torus knot of type  $(3, q)$  with  $u, q \geq 0$  yields a closed 3-manifold with non-left-orderable fundamental group if  $r \geq 3q + 4s$  for  $s > 1$  and if  $r \geq 3q + 1$  for  $s = 1$ .

We remark that all the above results concern only positively twisted torus knots. Thus it seems natural to ask what happens for *negatively* twisted torus knots. In this paper, we show the following

**Theorem 1.1.** *Let  $K$  be the  $(-1)$ -twisted  $(3, 3v + 2)$ -torus knot with  $v \geq 0$ . Then  $r$ -surgery on  $K$  yields a 3-manifold with non-left-orderable fundamental group if  $r \geq 3(3v + 2) - 2$ .*

Actually, the  $(-1)$ -twisted  $(3, 3v + 2)$ -torus knot is equivalent to the 1-twisted  $(3, 3v + 1)$ -torus knot with  $v \geq 0$ . Thus, the result in [2] guarantees that  $r$ -surgery on the knot yields a 3-manifold with non-left-orderable fundamental group if  $r \geq 3(3v + 2) + 1$ . That is, our theorem above can improve their bound slightly.

**2. Proof.** We start with recalling our technical main result in [5].

**Theorem** ([5, Theorem 1.1]). *Let  $K$  be a knot in a closed, connected 3-manifold  $M$ . Suppose that the knot group  $\pi_1(M - K)$  admits the presentation*

$$\langle a, b \mid (w_1 a^m w_1^{-1}) b^{-r} (w_2^{-1} a^n w_2) b^{r-k} \rangle.$$

Here  $w_1, w_2$  are arbitrary words with  $m, n \geq 0$ ,  $r \in \mathbf{Z}$ ,  $k \geq 0$ . Suppose further that  $a$  represents a meridian of  $K$  and  $a^{-s} w a^{-t}$  represents a longitude of  $K$  with  $s, t \in \mathbf{Z}$  and  $w$  is a word which excludes  $a^{-1}$  and  $b^{-1}$ . Then if  $q \neq 0$  and  $p/q \geq s + t$ , then Dehn surgery on  $K$  along the slope corresponding to  $p/q$  with respect to the meridian-longitude system yields a closed 3-manifold with non-left-orderable fundamental group.

Thus, to prove Theorem 1.1, we show the following

**Proposition 2.1.** *Let  $K$  be the  $u$ -twisted torus knot of type  $(3, 3v + 2)$  in  $S^3$  with  $u \in \mathbf{Z}, v \geq 0$ . Then the knot group  $\pi_1(S^3 - K)$  admits the presentation*

$$\langle a, b \mid (w_1 a^m w_1^{-1}) b^{-r} (w_2^{-1} a^n w_2) b^{r-k} \rangle$$

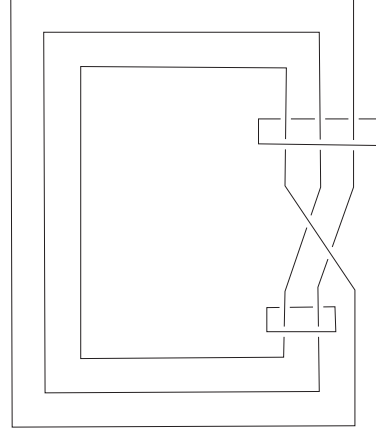


Fig. 2. Three-component link 1.

with  $m = n = 1$ ,  $r = u + 1$ ,  $k = 1$ ,  $w_1 = (ba)^{v+1}$ ,  $w_2 = (ba)^v$ . Furthermore the generator  $a$  represents a meridian of  $K$  and the preferred longitude of  $K$  is represented as  $a^{-s} w a^{-t}$  with  $s = 2u + 3(3v + 2) + 1$ ,  $t = -1$  and  $w = ((ba)^v b^{u+1})^2 (ba)^v b$ .

In the following, we will obtain a Wirtinger presentation of such a knot group. Note that, in [2,3,5], all the presentations for knot groups are obtained by using Heegaard splitting of  $S^3$ .

Let  $L$  be the three-component link, and  $D$  the diagram of  $L$  illustrated in Figure 2.

As in Figure 3, we label  $\alpha, \beta, \gamma, \xi, \psi, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7$  to the arcs in the diagram, and  $P_1, \dots, P_{12}$  to the twelve crossings in the diagram.

Next we introduce relators for the crossings of  $D$  as follows:

$$\begin{aligned} P_1 : & \xi \alpha \delta_1^{-1} \alpha^{-1}, & P_2 : & \delta_1 \beta \delta_2^{-1} \beta^{-1}, \\ P_3 : & \delta_2 \gamma \xi^{-1} \gamma^{-1}, & P_4 : & \xi^{-1} \alpha \xi \gamma^{-1}, \\ P_5 : & \xi^{-1} \beta \xi \delta_3^{-1}, & P_6 : & \xi^{-1} \gamma \xi \delta_4^{-1}, \\ P_7 : & \delta_5^{-1} \gamma \delta_3 \gamma^{-1}, & P_8 : & \delta_6^{-1} \gamma \delta_4 \gamma^{-1}, \\ P_9 : & \psi \delta_5 \delta_7^{-1} \delta_5^{-1}, & P_{10} : & \delta_7 \delta_6 \psi^{-1} \delta_6^{-1}, \\ P_{11} : & \psi^{-1} \delta_5 \psi \alpha^{-1}, & P_{12} : & \psi^{-1} \delta_6 \psi \beta^{-1}. \end{aligned}$$

Then the following are obtained from the above.

$$\begin{aligned} \delta_1 &= \alpha^{-1} \xi \alpha, & \delta_2 &= \gamma \xi \gamma^{-1}, \\ \delta_3 &= \xi^{-1} \beta \xi, & \delta_4 &= \xi^{-1} \gamma \xi, \\ \delta_5 &= \gamma \delta_3 \gamma^{-1} = \gamma \xi^{-1} \beta \xi \gamma^{-1}, \\ \delta_6 &= \gamma \delta_4 \gamma^{-1} = \gamma \xi^{-1} \gamma \xi \gamma^{-1}. \end{aligned}$$

Moreover, from the relations at  $P_{11}$  and  $P_{12}$ , we have  $\delta_5 = \psi \alpha \psi^{-1}$  and  $\delta_6 = \psi \beta \psi^{-1}$ . From these, together with the relations at  $P_9$  and  $P_{10}$ , we have  $\psi^{-1} \beta^{-1} \alpha^{-1} \psi \alpha \beta$ . From these, by erasing  $\delta_1, \dots, \delta_7$ , we have the following lemma consequently. Here let

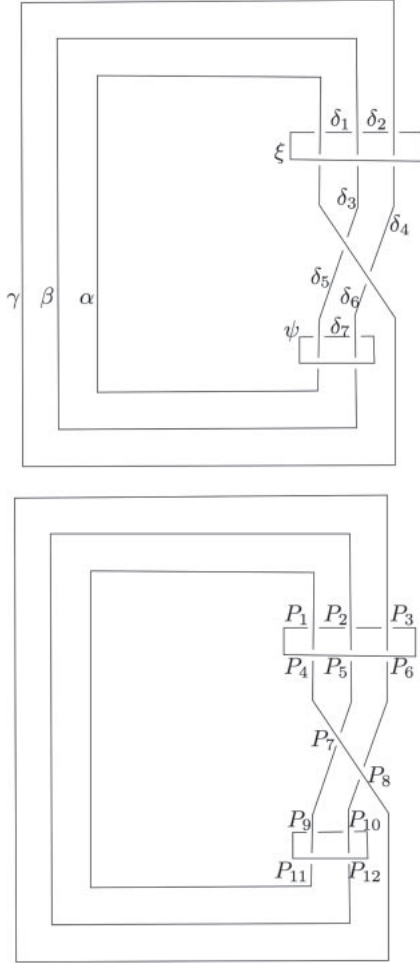


Fig. 3. Three-component link.

$l_1$  be an upper trivial component,  $l_2$  be a lower trivial component, and  $l_0$  be the other component in Figure 2.

**Lemma 2.2.** *For the three-component link  $L$  in Figure 2, the link group  $\pi_1(S^3 - L)$  admits the presentation*

$$\left\langle \alpha, \beta, \gamma, \xi, \psi \left| \begin{array}{l} \xi^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1}\xi\alpha\beta\gamma, \\ \xi^{-1}\alpha\xi\gamma^{-1}, \\ \psi^{-1}\gamma\xi^{-1}\beta\xi\gamma^{-1}\psi\alpha^{-1}, \\ \psi^{-1}\gamma\xi^{-1}\gamma\xi\gamma^{-1}\psi\beta^{-1}, \\ \psi^{-1}\beta^{-1}\alpha^{-1}\psi\alpha\beta \end{array} \right. \right\rangle.$$

Furthermore, for the component  $l_1$  of  $L$ , a meridian is represented by  $\xi$  and the preferred longitude is represented by  $\alpha\beta\gamma$ . For the component  $l_2$  of  $L$ , a meridian is represented by  $\psi$  and the preferred

longitude is represented by  $\alpha\beta$ . Also, for the component  $l_0$ , a meridian is represented by  $\alpha$  and a longitude is represented by  $\xi\xi\gamma^{-1}\psi\xi\gamma^{-1}\psi$ .

In fact,  $u$ -twisted torus knots of type  $(3, 3v+2)$  are obtained from the link  $L$  by adding  $u$  full twists along the component  $l_2$ , and  $v$  full twists along the component  $l_1$ . We note that adding  $v$  full twists along  $l_1$  is realized by the Dehn surgery on  $l_1$  along the slope  $-\frac{1}{v+1}$ . Also note that adding  $u$  full twists along  $l_2$  is realized by the Dehn surgery on  $l_2$  along the slope  $-\frac{1}{u}$ . Hence, to obtain a presentation of the knot group for a  $u$ -twisted torus knots of type  $(3, 3v+2)$ , it is sufficient to add relations  $\xi(\alpha\beta\gamma)^{-v-1} = 1$  and  $\psi(\alpha\beta)^u = 1$  to the presentation given in Lemma 2.2.

*Proof of Proposition 2.1.* Let  $K$  be a  $u$ -twisted torus knot of type  $(3, 3v+2)$ . By Lemma 2.2 and the argument above, the knot group of  $K$  admits the presentation

$$\left\langle \alpha, \beta, \gamma, \xi, \psi \left| \begin{array}{l} \xi^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1}\xi\alpha\beta\gamma, \\ \xi^{-1}\alpha\xi\gamma^{-1}, \\ \psi^{-1}\gamma\xi^{-1}\beta\xi\gamma^{-1}\psi\alpha^{-1}, \\ \psi^{-1}\gamma\xi^{-1}\gamma\xi\gamma^{-1}\psi\beta^{-1}, \\ \psi^{-1}\beta^{-1}\alpha^{-1}\psi\alpha\beta, \\ \xi(\alpha\beta\gamma)^{-v-1}, \psi(\alpha\beta)^u \end{array} \right. \right\rangle.$$

We only have to prove that this presentation can be transformed to the presentation in [5, Theorem 1.1]. Now we set  $g := \xi\gamma^{-1}$ ,  $h := \alpha\beta\gamma$ . Then the generators  $\alpha, \beta, \gamma, \xi, \psi$  of the presentation above are described by using  $g$  and  $h$  as follows:

$$\begin{aligned} \xi &= (\alpha\beta\gamma)^{v+1} = h^{v+1}, \\ \gamma &= g^{-1}\xi = g^{-1}h^{v+1}, \\ \psi &= (\alpha\beta)^u = (h^{-v}g)^u, \\ \alpha &= \xi\gamma\xi^{-1} = h^{v+1}g^{-1}, \\ \beta &= gh^{-v-1}h^{-v}g = gh^{-2v-1}g. \end{aligned}$$

Thus the relators of the presentation are represented by using  $g$  and  $h$  as follows:

$$\begin{aligned} &\xi^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1}\xi\alpha\beta\gamma \\ &= h^{-v-1}h^{-1}h^{v+1}h = 1, \\ &\xi^{-1}\alpha\xi\gamma^{-1} \\ &= h^{-v-1}h^{v+1}g^{-1}g = 1, \\ &\gamma\xi^{-1}\beta\xi\gamma^{-1}\psi\alpha^{-1}\psi^{-1} \\ (1) \quad &= g^{-1}gh^{-2v-1}gg(h^{-v}g)^u gh^{-v-1}(h^{-v}g)^{-u}, \end{aligned}$$

$$\begin{aligned}
& \gamma\xi^{-1}\gamma\xi\gamma^{-1}\psi\beta^{-1}\psi^{-1} \\
(2) \quad & = g^{-1}g^{-1}h^{v+1}g(h^{-v}g)^u g^{-1}h^{2v-1}g^{-1}(h^{-v}g)^{-u}, \\
& \psi^{-1}\beta^{-1}\alpha^{-1}\psi\alpha\beta \\
& = (g^{-1}h^v)^u g^{-1}h^v(h^{-v}g)^u h^{-v}g = 1.
\end{aligned}$$

We note that Eq. (1) is equivalent to

$$h^{-v-1}g^2(h^{-v}g)^{u-1}h^{-v}g^2h^{-v-1}(g^{-1}h^v)^{u-1}g^{-1}$$

and Eq. (2) is equivalent to

$$g(h^{-v}g)^{u-1}h^{v+1}g^{-2}h^v(g^{-1}h^v)^{u-1}g^{-2}h^{v+1}.$$

It then follows that Eqs. (1) and (2) are equivalent by considering the inverses. Also note that the last equation corresponds to the commutativity of the meridian and longitude for the component  $l_2$ , and so, can be obtained from the other relations.

Therefore, we have the following presentation of the knot group of  $K$ ,

$$\langle g, h \mid g^2(h^{-v}g)^u gh^{-v-1}(h^{-v}g)^{-u}h^{-2v-1} \rangle$$

which is equivalent to the following

$$\langle g, h \mid h^{v+1}g^{-1}(h^{-v}g)^{-u}g^{-2}h^{2v+1}(h^{-v}g)^u \rangle.$$

Furthermore, by setting  $a = g^{-1}h^{v+1}$  and  $b = ha^{-1}$ , i.e.,  $h = ba$  and  $g = h^{v+1}a^{-1} = (ba)^{v+1}a^{-1} = (ba)^v b$ , we have the following

$$\langle a, b \mid (ba)^{v+1}a(ba)^{-v-1}b^{-u-1}(ba)^{-v}a(ba)^vb^u \rangle.$$

Consequently the knot group  $\pi_1(S^3 - K)$  admits the presentation

$$\langle a, b \mid (w_1 a^m w_1^{-1}) b^{-r} (w_2^{-1} a^n w_2) b^{r-k} \rangle$$

with  $m = n = 1$ ,  $r = u + 1$ ,  $k = 1$ ,  $w_1 = (ba)^{v+1}$ ,  $w_2 = (ba)^v$ .

By Lemma 2.2, a meridian of  $K$  is represented by  $\alpha$ . This  $\alpha$  is represented by  $h^{v+1}g^{-1}$  which equals to  $a$ . Also by Lemma 2.2, a longitude of  $K$  is represented by  $\xi\xi\gamma^{-1}\psi\xi\gamma^{-1}\psi$ . This element is represented by  $h^{v+1}(g(h^{-v}g)^u)^2$  which equals to  $(ba)^{v+1}(ba)^vb^{u+1}(ba)^vb^{u+1}$ . Adding  $a^{-1}$ , we have  $a^{-1}(ba)^vb^{u+1}(ba)^vb^{u+1}(ba)^{v+1}$ . Finally, as noted in the paragraph just below [3, Proposition 3.2], the

presentation of the preferred longitude of  $K$  is obtained from the above by adding  $a^{-3(3v+2)-2u}$  as follows:

$$a^{-3(3v+2)-2u-1}(((ba)^vb^{u+1})^2(ba)^vb)a.$$

Therefore the preferred longitude of  $K$  is represented as  $a^{-s}wa^{-t}$  with  $s = 2u + 3(3v + 2) + 1$ ,  $t = -1$  and  $w = ((ba)^vb^{u+1})^2(ba)^vb$ .

This completes the proof of Proposition 2.1 since  $u$  is any integer.  $\square$

We remark that the condition for the presentation of the preferred longitude such as  $w$  is a positive word in [5, Theorem 1.1] holds in the case of  $u = -1$ , although it does not hold in the case of  $u \leq -2$ .

**Acknowledgement.** The first author was partially supported by JSPS KAKENHI Grant Number 26400100.

## References

- [ 1 ] S. Boyer, C. McA. Gordon and L. Watson, On L-spaces and left-orderable fundamental groups, *Math. Ann.* **356** (2013), no. 4, 1213–1245.
- [ 2 ] K. Christianson, J. Goluboff, L. Hamann and S. Varadaraj, Non-left-orderable surgeries on twisted torus knots, *Proc. Amer. Math. Soc.* **144** (2016), no. 6, 2683–2696.
- [ 3 ] A. Clay and L. Watson, Left-orderable fundamental groups and Dehn surgery, *Int. Math. Res. Not. IMRN* **2013**, no. 12, 2862–2890.
- [ 4 ] J. C. Dean, Small Seifert-fibered Dehn surgery on hyperbolic knots, *Algebr. Geom. Topol.* **3** (2003), 435–472.
- [ 5 ] K. Ichihara and Y. Temma, Non-left-orderable surgeries and generalized Baumslag-Solitar relators, *J. Knot Theory Ramifications* **24** (2015), no. 1, 1550003, 8 pp.
- [ 6 ] Y. Nakae, A good presentation of  $(-2, 3, 2s + 1)$ -type pretzel knot group and  $\mathbf{R}$ -covered foliation, *J. Knot Theory Ramifications* **22** (2013), no. 1, 1250143, 23 pp.
- [ 7 ] P. S. Ozsváth and Z. Szabó, Knot Floer homology and rational surgeries, *Algebr. Geom. Topol.* **11** (2011), no. 1, 1–68.
- [ 8 ] F. Vafaee, On the knot Floer homology of twisted torus knots, *Int. Math. Res. Not. IMRN* **2015**, no. 15, 6516–6537.