Inequalities for free multi-braid arrangements

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(Communicated by Masaki Kashiwara, M.J.A., March 12, 2018)

Abstract: Abe, Nuida, and Numata (2009) describe a large class of free multiplicities on the braid arrangement arising from signed-eliminable graphs. On a large cone in the multiplicity lattice, we prove that these are the only free multiplicities on the braid arrangement. We also give a conjecture on the structure of all free multiplicities on the braid arrangement.

Key words: Freeness of multi-arrangements; braid arrangement; multi-derivations.

1. Introduction. Let $V \cong \mathbf{K}^{\ell+1}$ be a vector space over a field $\mathbf{K}$ of characteristic zero, V^* its dual space and $S = \mathrm{Sym}(V^*) \cong \mathbf{K}[x_0, \ldots, x_\ell]$. For $0 \le i < j \le \ell$, set $\alpha_{ij} = x_i - x_j$ and write H_{ij} for the corresponding hyperplane. The *braid arrangement of type* $A_\ell \subset V$ is $\cup_{0 \le i<j \le \ell} H_{ij}$. A *multiplicity* on A_ℓ is a map $\mathbf{m} : \{H_{ij}\} \to \mathbf{N}$; we represent $\mathbf{m}$ as the tuple $(m_{ij} = \mathbf{m}(H_{ij}))$ in the *multiplicity lattice* $\Lambda_\ell = \mathbf{N}^{\binom{\ell+1}{2}}$. The order of the subscripts is not important, i.e., $m_{ij} = m_{ji}$. The pair $(A_\ell, \mathbf{m})$ is called a multi-braid arrangement; it is *free* if the module

$$D(A_\ell, \mathbf{m}) := \{\theta \in \mathrm{Der}(S) : \theta(\alpha_{ij}) \in \alpha_{ij}^{m_{ij}} S, 0 \le i < j \le \ell\}$$

of multi-derivations is a free module over S. In this case we call $\mathbf{m}$ a *free multiplicity.*

Write Λ_ℓ^b for the cone of multiplicities in Λ_ℓ satisfying the inequalities $m_{ij} \le m_{ik} + m_{jk} + 1$ for every triple i, j, k. We call Λ_ℓ^b the *balanced cone* of multiplicities since the exponents of every sub-A_2 multi-arrangement differ by at most one by a result of Wakamiko [9]. In [2], Abe, Nuida, and Numata completely characterize free multiplicities $\mathbf{m} \in \Lambda_\ell^b$ of the form $m_{ij} = n_i + n_j + \epsilon_{ij}$, where $n_0, \ldots, n_\ell \in \mathbf{Z}_{\ge 0}$ and $\epsilon_{ij} \in \{-1, 0, 1\}$ for $0 \le i < j \le \ell$. We call a multiplicity of this kind an *ANN multiplicity.* In this note we prove that the free ANN multiplicities are the *only* free multiplicities in Λ_ℓ^b (partially generalizing the result of [5]).

To state our result precisely, we need some notation. It is natural to associate A_ℓ to the complete graph $K_{\ell+1}$ on the $\ell+1$ vertices $\{v_0, \ldots, v_\ell\}$, where the hyperplane $H_{ij} \in A_\ell$ corresponds to the edge $\{v_i, v_j\} \in K_{\ell+1}$. Then a multiplicity $\mathbf{m}$ on A_ℓ yields a labeling of the edges of $K_{\ell+1}$ by positive integers; the edge $\{v_i, v_j\}$ is labeled by m_{ij}. We call a three-cycle in $K_{\ell+1}$ with edges $\{v_i, v_j\}, \{v_j, v_k\}, \{v_k, v_i\}$ an *odd three-cycle* of $\mathbf{m}$ if $m_{ij} + m_{jk} + m_{ki}$ is an odd integer. Given a four-cycle C with edges $\{v_i, v_j\}, \{v_j, v_s\}, \{v_s, v_t\}, \{v_t, v_i\}$ in $K_{\ell+1}$, we put $\mathbf{m}(C) = |m_{ij} - m_{js} + m_{st} - m_{ti}|$. Notice that $\mathbf{m}(C)$ does not depend on a particular ordering of the edges of C, so it is well-defined.

Given a subset $U \subset \{v_0, \ldots, v_\ell\}$ of size at least four, write $\mathbf{m}_U$ for the restriction of $\mathbf{m}$ to the subset $\{H_{ij} : \{v_i, v_j\} \subset U\}$. We define the *deviation of* $\mathbf{m}$ *over* U *as* $\mathrm{DV}(\mathbf{m}_U) = \sum_{C \subset U} \mathbf{m}(C)^2$, where the index runs over all four-cycles of $K_{\ell+1}$ which are contained in U. Moreover, we define q_U to be the number of odd three-cycles of $\mathbf{m}$ contained in U. Our main result is:

Theorem 1.1. *Suppose* $(A_\ell, \mathbf{m})$ *is a multi-braid arrangement with* $\mathbf{m} \in \Lambda_\ell^b$. *The following are equivalent.*

(1) $(A_\ell, \mathbf{m})$ *is free.*
(2) $\mathrm{DV}(\mathbf{m}_U) \le q_U(|U| - 1)$ *for every subset* $U \subset \{v_0, \ldots, v_\ell\}$ *where* $|U| \ge 4$.
(3) $\mathbf{m}$ *is a free ANN multiplicity.*

The proof of Theorem 1.1 is at the end of §5. In §6 we also introduce the notion of a *free vertex* and present a conjecture about the structure of all free multiplicities on braid arrangements.

2. Examples. We illustrate Theorem 1.1 for the A_3 arrangement, which corresponds to the complete graph K_4. Write $\mathbf{m}$ for a multiplicity and v_0, v_1, v_2, v_3 for the vertices of K_4. Since there are only four vertices, the criterion (2) in Theorem

2010 Mathematics Subject Classification. Primary 13N15; Secondary 05E40, 14N20.

doi: 10.3792/pjaa.94.36

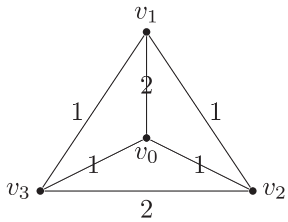

Fig. 1. The multiplicity $\mathbf{m}_1$ in Example 2.1.

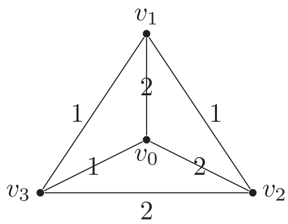

Fig. 2. The multiplicity $\mathbf{m}_2$ in Example 2.2.

1.1 only needs to be considered for $U = \{v_0, v_1, v_2, v_3\}$. There are three four-cycles with edges:

- $C_1 : \{v_0, v_2\}, \{v_2, v_1\}, \{v_1, v_3\}, \{v_3, v_0\}$
- $C_2 : \{v_0, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}, \{v_1, v_0\}$
- $C_3 : \{v_0, v_3\}, \{v_3, v_2\}, \{v_2, v_1\}, \{v_1, v_0\}$,

so $\mathrm{DV}(\mathbf{m}) = |\mathbf{m}(C_1)|^2 + |\mathbf{m}(C_2)|^2 + |\mathbf{m}(C_3)|^2$. Hence

$$\begin{aligned}\mathrm{DV}(\mathbf{m}) = (m_{02} - m_{12} + m_{13} - m_{03})^2 \\ + (m_{02} - m_{23} + m_{13} - m_{01})^2 \\ + (m_{03} - m_{23} + m_{12} - m_{01})^2.\end{aligned}$$

Moreover, there are four three-cycles of K_4.

Example 2.1. Consider the multiplicity $\mathbf{m}_1$ defined by $m_{01} = m_{23} = 2$ and $m_{02} = m_{03} = m_{12} = m_{13} = 1$, shown in Figure 1. We can check that $\mathbf{m}_1 \in \Lambda^b_\ell$. We compute $\mathbf{m}(C_1) = 0$ and $\mathbf{m}(C_2) = \mathbf{m}(C_3) = 2$, so $\mathrm{DV}(\mathbf{m}_1) = 8$. Also, there are no odd three-cycles, so $q_U = 0$. Since $\mathrm{DV}(\mathbf{m}_1) > 0 = q_U \cdot \cdot (|U| - 1)$, $(A_3, \mathbf{m}_1)$ is not free by Theorem 1.1.

Example 2.2. Consider the multiplicity $\mathbf{m}_2$ defined by $m_{01} = m_{02} = m_{23} = 2$ and $m_{03} = m_{12} = m_{13} = 1$, shown in Figure 2. We can check that $\mathbf{m}_2 \in \Lambda^b_\ell$. We compute $\mathbf{m}(C_1) = \mathbf{m}(C_2) = 1$ and $\mathbf{m}(C_3) = 2$, so $\mathrm{DV}(\mathbf{m}_2) = 6$. Also, there are two odd three-cycles, so $q_U = 2$. Since $\mathrm{DV}(\mathbf{m}_2) = 6 \le 2 \cdot 3 = q_U \cdot (|U| - 1)$, $(A_3, \mathbf{m}_2)$ is free by Theorem 1.1.

3. Mixed products in the balanced cone. We will prove the implication (1) $\Longrightarrow$ (2) in Theorem 1.1 using the notion of local and global mixed products from [3], which we now explain. If $D(A_\ell, \mathbf{m})$ is free we list its (non-zero) exponents as a non-increasing sequence $(d_1, \ldots, d_\ell)$. Put $|\mathbf{m}| = \sum_{0 \le i < j \le \ell} m_{ij}$. Then $\sum_{i=1}^{\ell} d_i = |\mathbf{m}|$ by Saito's criterion. For a free multi-arrangement, the kth *global mixed product* is $\mathrm{GMP}(k) = \sum d_{i_1} d_{i_2} \cdots d_{i_k}$, where the sum runs across all k-tuples satisfying $1 \le i_1 < \cdots < i_k \le \ell$. Now write $L = L(A_\ell)$ for the intersection lattice of A_ℓ (all intersections among the hyperplanes H_{ij}) and L_k for those intersections of codimension k. The kth *local mixed product* is $\mathrm{LMP}(k) = \sum_{X \in L_k} d_1^X d_2^X \cdots d_k^X$, where $d_1^X, \ldots, d_k^X$ are the (non-zero) exponents of the closed rank k sub-arrangement $(A_\ell)_X$ (this is the arrangement consisting of all hyperplanes containing X). By [3, Corollary 4.6], if $(A_\ell, \mathbf{m})$ is free then $GMP(k) = LMP(k)$ for every $2 \le k \le \ell$.

Recall that if $\{v_i, v_j, v_k\}$ are vertices of $K_{\ell+1}$ so that $m_{ij} + m_{ik} + m_{jk}$ is odd then we call the cycle traversing these vertices an *odd three-cycle* of $\mathbf{m}$.

Proposition 3.1. *Let $(A_\ell, \mathbf{m})$ be a multi-braid arrangement with $\mathbf{m} \in \Lambda^b_\ell$. Set $m_{ijk} = m_{ij} + m_{jk} + m_{ik}$ and write q for the number of odd three-cycles of $\mathbf{m}$. Then*

$$\begin{aligned}&LMP(2) \\ &= \sum_{0 \le i < j < k \le \ell} (m_{ijk}/2)^2 + \sum_{\{i,j\} \cap \{s,t\} = \emptyset} m_{ij} m_{st} - q/4\end{aligned}$$

and

$$GMP(2) \le \binom{\ell}{2} \frac{|\mathbf{m}|^2}{\ell^2}.$$

Proof. We prove the formula for LMP(2) first. If $X \in L_2$, then either $(1) : X = H_{ij} \cap H_{st}$ for a pair of non-adjacent edges $\{i, j\}$ and $\{s, t\}$ or $(2) : X = H_{ij} \cap H_{jk} \cap H_{ik}$ for a three-cycle on $\{v_i, v_j, v_k\}$. In the first case the arrangement is boolean with (non-zero) exponents (m_{ij}, m_{st}), contributing $m_{ij} m_{st}$ to LMP(2). In the second case the arrangement is an A_2 braid arrangement and a result of Wakamiko [9] shows the exponents are $(m_{ijk}/2, m_{ijk}/2)$ if m_{ijk} is even and $((m_{ijk} + 1)/2, (m_{ijk} - 1)/2)$ if m_{ijk} is odd (since $\mathbf{m} \in \Lambda^b_\ell$). The former contributes $m_{ijk}^2/4$ to LMP(2) while the latter contributes $m_{ijk}^2/4 - 1/4$. This yields the expression for LMP(2).

The inequality for GMP(2) is immediate since the real-valued function $\sum_{1 \le i < j \le \ell} x_i x_j$ subject to the restrictions $x_i \ge 0$ for $i = 1, \ldots, \ell$ and $\sum_{i=1}^{\ell} x_i = |\mathbf{m}|$

attains an absolute maximum of $\binom{\ell}{2}\frac{|\mathbf{m}|^2}{\ell^2}$ when $x_1 = \cdots = x_\ell = \frac{|\mathbf{m}|}{\ell}$. $\square$

In the statement of the next result, $\mathrm{DV}(\mathbf{m})$ denotes $\mathrm{DV}(\mathbf{m}_U)$ when $U = \{v_0, \ldots, v_\ell\}$.

Theorem 3.2. *Suppose $(A_\ell, \mathbf{m})$ is a multibraid arrangement with $\mathbf{m} \in \Lambda_\ell^b$ and write q for the number of odd three-cycles of $\mathbf{m}$. If $DV(\mathbf{m}) > q\ell$, then $\mathbf{m}$ is not a free multiplicity.*

Remark 3.3. Theorem 3.2 generalizes [5, Theorem 4.12] to higher braid arrangements.

Proof. By Proposition 3.1, we know that

$$\mathrm{LMP}(2) - \mathrm{GMP}(2) \geq \sum_{0\leq i<j<k\leq\ell} (m_{ijk}/2)^2 + \sum_{\{i,j\}\cap\{s,t\}=\emptyset} m_{ij}m_{st} - \binom{\ell}{2}\frac{|\mathbf{m}|^2}{\ell^2} - q/4.$$

Our primary claim is

$$\mathrm{DV}(\mathbf{m}) = 4\ell \sum_{0\leq i<j<k\leq\ell} (m_{ijk}/2)^2 + 4\ell \sum_{\{i,j\}\cap\{s,t\}=\emptyset} m_{ij}m_{st} - 4\ell\binom{\ell}{2}\frac{|\mathbf{m}|^2}{\ell^2}. \tag{1}$$

Once Eq. (1) is proved, notice that

$$4\ell(\mathrm{LMP}(2) - \mathrm{GMP}(2)) \geq \mathrm{DV}(\mathbf{m}) - q\ell.$$

Then [3, Corollary 4.6] immediately yields Theorem 3.2. So we prove Eq. (1). By definition $\mathrm{DV}(\mathbf{m}) = \sum_C \mathbf{m}(C)^2$; expanding this yields

$$\mathrm{DV}(\mathbf{m}) = 2\binom{\ell-1}{2}\sum_{0\leq i<j\leq\ell} m_{ij}^2 + 4\sum_{\{i,j\}\cap\{s,t\}=\emptyset} m_{ij}m_{st} - 2(\ell-2)\sum_{0\leq i<j<k\leq\ell}(m_{ij}m_{ik} + m_{ij}m_{jk} + m_{ik}m_{jk}). \tag{2}$$

Re-writing the right-hand side of Eq. (1) using the two expressions

$$|\mathbf{m}|^2 = \sum_{0\leq i<j\leq\ell} m_{ij}^2 + 2\sum_{\{i,j\}\cap\{s,t\}=\emptyset} m_{ij}m_{st} + 2\sum_{0\leq i<j<k\leq\ell}(m_{ij}m_{ik} + m_{ij}m_{jk} + m_{ik}m_{jk})$$

and

$$\sum_{0\leq i<j<k\leq\ell} m_{ijk}^2 = (\ell-1)\sum_{0\leq i<j\leq\ell} m_{ij}^2 + 2\sum_{0\leq i<j<k\leq\ell}(m_{ij}m_{ik} + m_{ij}m_{jk} + m_{ik}m_{jk})$$

now yields the right-hand side of Eq. (2). $\square$

4. From deviations to ANN multiplicities.

Recall that $\mathbf{m}$ is an ANN multiplicity on A_ℓ if $\mathbf{m} \in \Lambda_\ell^b$ and there exist non-negative integers $n_0, \ldots, n_\ell$ and $\epsilon_{ij} \in \{-1, 0, 1\}$ so that $m_{ij} = n_i + n_j + \epsilon_{ij}$ for $0 \leq i < j \leq \ell$. In this section we prove the first part of the implication (2) $\Longrightarrow$ (3) in Theorem 1.1. Namely, we prove that if $\mathbf{m} \in \Lambda_\ell^b$ then the inequalities $\mathrm{DV}(\mathbf{m}) \leq q_U(|U| - 1)$ guarantee that $\mathbf{m}$ is an ANN multiplicity.

Lemma 4.1. *Suppose $\mathbf{m}$ is a multiplicity on A_3 with q odd three-cycles and $DV(\mathbf{m}) \leq 3q$. Then $\mathbf{m}(C) \leq 2$ for each four-cycle C in K_4.*

Proof. There are three four-cycles. Set $T_1 = m_{01} - m_{12} + m_{23} - m_{03}$, $T_2 = m_{13} - m_{01} + m_{02} - m_{23}$, and $T_3 = m_{13} - m_{12} + m_{02} - m_{03}$. Notice $T_1 + T_2 = T_3$, and $\mathrm{DV}(\mathbf{m}) = T_1^2 + T_2^2 + T_3^2$. Now, suppose without loss that $|T_3| \geq 3$. Then either $|T_1| \geq 2$ or $|T_2| \geq 2$. But then $\mathrm{DV}(\mathbf{m}) \geq 13$, contradicting that $\mathrm{DV}(\mathbf{m}) \leq 3q \leq 12$ (since $q \leq 4$). $\square$

Proposition 4.2. *Let $(A_\ell, \mathbf{m})$ be a multibraid arrangement so that $\mathbf{m} \in \Lambda_\ell^b$ and $DV(\mathbf{m}_U) \leq 3q_U$ for every subset $U \subset \{v_0, \ldots, v_\ell\}$ with $|U| = 4$. Then $\mathbf{m}$ is an ANN multiplicity.*

Proof. We need to show that there exist non-negative integers n_i for $i = 0, \ldots, \ell$ and integers $\epsilon_{ij} \in \{-1, 0, 1\}$ for $0 \leq i < j \leq \ell$ so that $m_{ij} = n_i + n_j + \epsilon_{ij}$. By Lemma 4.1, we must have $\mathbf{m}(C) \leq 2$ for every four-cycle $C \in C_4(K_{\ell+1})$. Using this condition, we give an inductive algorithm to construct the integers $n_0, \ldots, n_\ell$.

If $\ell = 2$, set $n_0 = \lceil (m_{01} + m_{02} - m_{12})/2 \rceil$, $n_1 = \lceil (m_{01} + m_{12} - m_{02})/2 \rceil$, and $n_2 = \lceil (m_{02} + m_{12} - m_{01})/2 \rceil$. Since $\mathbf{m} \in \Lambda_\ell^b$, $n_i \geq 0$ for $i = 0, 1, 2$. Moreover, $m_{ij} = n_i + n_j + \epsilon_{ij}$, where $\epsilon_{ij} \in \{-1, 0\}$.

Now assume $\ell > 2$. We make an initial guess at what the non-negative integers $n_0, \ldots, n_\ell$ and ϵ_{ij} should be, and then adjust as necessary. By induction on ℓ, there exist non-negative integers $\tilde{n}_0, \ldots, \tilde{n}_{\ell-1}$ and $\tilde{\epsilon}_{ij} \in \{-1, 0, 1\}$ such that $m_{ij} = \tilde{n}_i + \tilde{n}_j + \tilde{\epsilon}_{ij}$ for $0 \leq i < j \leq \ell - 1$. Let $\tilde{n}_\ell$ be a non-negative integer satisfying $\tilde{n}_\ell + \tilde{n}_i \geq m_{i\ell} - 1$ and set $\tilde{\epsilon}_{i\ell} = m_{i\ell} - (\tilde{n}_i + \tilde{n}_\ell)$ for every $i < \ell$, so $m_{i\ell} = \tilde{n}_i + \tilde{n}_\ell + \tilde{\epsilon}_{i\ell}$. By the choice of $\tilde{n}_\ell$, we have $\tilde{\epsilon}_{i\ell} \leq 1$ for all $i < \ell$.

Now suppose there is an index $0 \leq j < \ell$ so that $\tilde{\epsilon}_{j\ell} \leq -2$. Our goal is to decrease either $\tilde{n}_\ell$ or $\tilde{n}_j$ by one, thereby increasing $\tilde{\epsilon}_{j\ell}$, without disturbing any of the hypotheses made so far, namely

$$\tilde{n}_i + \tilde{n}_j + \tilde{\epsilon}_{ij} = m_{ij} \text{ for all } 0 \leq i < j \leq \ell,$$

$$(\star)\quad \begin{aligned} &\tilde{n}_i \geq 0 \text{ for all } 0 \leq i \leq \ell, \\ &\tilde{\epsilon}_{i\ell} \leq 1 \text{ for all } i < \ell, \\ &\tilde{\epsilon}_{st} \in \{-1, 0, 1\} \text{ for all } 0 \leq s < t \leq \ell - 1. \end{aligned}$$

First we assume $\tilde{n}_\ell > 0$ and try to decrease $\tilde{n}_\ell$ by one. We can do this without disturbing assumptions $(\star)$ provided there is no index s so that $\epsilon_{s\ell} = 1$. So, assume that there is an index $0 \leq s < \ell$ so that $\epsilon_{s\ell} = 1$. We claim that in this situation, $\epsilon_{st} \geq 0$ for every $t \neq s$. Suppose to the contrary that there is an index t so that $\epsilon_{st} = -1$ and consider the four-cycle $C : v_\ell \to v_s \to v_t \to v_j \to v_\ell$. Then

$$\mathbf{m}(C) = |\tilde{\epsilon}_{s\ell} - \tilde{\epsilon}_{j\ell} + \tilde{\epsilon}_{jt} - \tilde{\epsilon}_{st}| \geq 1 + 2 + \tilde{\epsilon}_{jt} + 1 \geq 3,$$

since $\tilde{\epsilon}_{jt} \in \{-1, 0, 1\}$ by the inductive hypothesis. This contradicts our assumption that $\mathbf{m}(C) \leq 2$. So it follows that $\tilde{\epsilon}_{st} \in \{0, 1\}$ for all t. Thus we may increase $\tilde{n}_s$ by one, thereby decreasing $\tilde{\epsilon}_{st}$ by one for every $t \neq s$, without disturbing the hypothesis that $\tilde{\epsilon}_{st} \in \{-1, 0, 1\}$. Since we can apply this argument at every index s so that $\tilde{\epsilon}_{s\ell} = 1$, we may assume $\tilde{\epsilon}_{s\ell} \leq 0$ for every $0 \leq s < \ell$. Hence, if $\tilde{n}_\ell > 0$, it is now clear that we can decrease $\tilde{n}_\ell$ by one without disturbing assumptions $(\star)$.

Now assume that $\tilde{n}_\ell = 0$. Then, for any $s < \ell$,

$$\begin{aligned} m_{s\ell} + m_{j\ell} - m_{js} &= (\tilde{n}_s + \tilde{\epsilon}_{s\ell}) + (\tilde{n}_j + \tilde{\epsilon}_{j\ell}) \\ &\quad - (\tilde{n}_j + \tilde{n}_s + \tilde{\epsilon}_{js}) \\ &= \tilde{\epsilon}_{s\ell} + \tilde{\epsilon}_{j\ell} - \tilde{\epsilon}_{js} \\ &\leq 0 - 2 - \tilde{\epsilon}_{js} \leq -1, \end{aligned}$$

since $\tilde{\epsilon}_{js} \in \{-1, 0, 1\}$ by the inductive hypothesis. Since $\mathbf{m} \in \Lambda^b_\ell$, we must have an equality for all of these, so $\epsilon_{js} = -1$ for every $s \neq j$, $s < \ell$. If $\tilde{n}_j = 0$ as well, then $m_{j\ell} = \tilde{n}_j + \tilde{n}_\ell + \epsilon_{j\ell} \leq -2$, contradicting that $m_{j\ell}$ is non-negative. Hence $\tilde{n}_j > 0$ and we can decrease $\tilde{n}_j$ by one without disturbing any of assumptions $(\star)$.

In either case, we have shown how to increase $\tilde{\epsilon}_{j\ell}$ if $\tilde{\epsilon}_{j\ell} \leq -2$ without disturbing assumptions $(\star)$. So we iterate the above arguments until $\tilde{\epsilon}_{j\ell} \geq -1$ for every $j < \ell$, then set $n_i = \tilde{n}_i$ for $0 \leq i \leq \ell$ and $\tilde{\epsilon}_{ij} = \epsilon_{ij}$ for $0 \leq i < j \leq \ell$. This completes the algorithm and the proof. □

5. Detecting signed-eliminable graphs. In this section we finish the proof of the implication (2) $\Longrightarrow$ (3) in Theorem 1.1. We will use the characterization of free ANN multiplicities given in [2], which involves *signed-eliminable graphs*. We follow the presentation from [2,8].

A *signed* graph G on $\ell + 1$ vertices is a graph whose edge set E_G is the disjoint union of a set E_G^+ of edges assigned $+$ and a set E_G^- of edges assigned $-$. Clearly if G is a signed graph on $\ell + 1$ vertices and $n_0, \ldots, n_\ell \in \mathbf{Z}_{\geq 0}$ then we can define a multiplicity $\mathbf{m} = \mathbf{m}_G(n_0, \ldots, n_\ell)$ on A_ℓ by $m_{ij} = n_i + n_j + \epsilon_{ij}$, where

$$\epsilon_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E_G^+ \\ -1 & \{v_i, v_j\} \in E_G^- \\ 0 & \{v_i, v_j\} \notin E_G. \end{cases}$$

The graph G is *signed-eliminable* with *signed-elimination ordering* $\nu : V(G) \to \{0, \ldots, \ell\}$ if ν is bijective and, for every three vertices $v_i, v_j, v_k \in V(G)$ with $\nu(v_i), \nu(v_j) < \nu(v_k)$, the induced subgraph $G|_{v_i,v_j,v_k}$ satisfies:

- For $\sigma \in \{+, -\}$, if $\{v_i, v_k\}$ and $\{v_j, v_k\}$ are edges in E_G^σ then $\{v_i, v_j\} \in E_G^\sigma$.
- For $\sigma \in \{+, -\}$, if $\{v_k, v_i\} \in E_G^\sigma$ and $\{v_i, v_j\} \in E_G^{-\sigma}$ then $\{v_k, v_j\} \in E_G$.

According to [2, Theorem 0.3], an ANN multiplicity of the form $\mathbf{m}_G(n_0, \ldots, n_\ell)$ is free if and only if G is signed-eliminable.

Remark 5.1. In [2, Theorem 0.3], it is not stated that $\mathbf{m} \in \Lambda^b_\ell$. However, this is a necessary and sufficient condition for the proofs; see Appendix A of [1]. It is straightforward to check that condition (3) in the revised statement of Theorem 0.3 in Appendix A of [1] coincides with $\mathbf{m} \in \Lambda^b_\ell$.

Definition 5.2. A graph with $(\ell + 1)$ vertices $v_0, v_1, \ldots, v_\ell$ and $\ell \geq 3$ is a

(1) σ-mountain, where $\sigma \in \{+, -\}$, if $\{v_0, v_i\} \in E_G^\sigma$ for $i = 2, \ldots, \ell - 1$, $\{v_i, v_{i+1}\} \in E_G^{-\sigma}$ for $i = 1, \ldots, \ell - 1$, and no other pair of vertices is joined by an edge;

(2) a σ-hill, where $\sigma \in \{+, -\}$, if $\{v_0, v_1\} \in E_G^\sigma, \{v_0, v_i\} \in E_G^\sigma$ for $i = 2, \ldots, \ell - 1$, $\{v_1, v_i\} \in E_G^\sigma$ for $i = 3, \ldots, \ell$, $\{v_i, v_{i+1}\} \in E_G^{-\sigma}$ for $i = 2, \ldots, \ell - 1$, and no other pair of vertices is joined by an edge;

(3) a σ-cycle if $\{v_i, v_{i+1}\} \in E_G^\sigma$ for $i = 0, \ldots, \ell - 1$, $\{v_0, v_\ell\} \in E_G^\sigma$, and no other pair of vertices is joined by an edge.

Theorem 5.3 ([8, Theorem 5.1])**.** *A signed graph is signed-eliminable if and only if it has no induced sub-graph which is a graph on four vertices which is not signed-eliminable, a σ-cycle, a σ-mountain, or a σ-hill.*

Remark 5.4. Theorem 5.3 is not precisely Nuida's characterization, but it is easily deduced from it (this is implicit in [2]).

Proposition 5.5. *Suppose $n_0, \dots, n_\ell$ are non-negative integers, G is a signed graph on $v_0, \dots, v_\ell$, and $\mathbf{m} = \mathbf{m}_G(n_0, \dots, n_\ell)$. If G is not signed-eliminable, then there is a subset $U \subset \{0, \dots, \ell\}$ so that $DV(\mathbf{m}_U) > q_U \cdot (|U| - 1)$.*

Proof. It follows from the definitions that $\mathrm{DV}(\mathbf{m}_U)$ and $q_U\ell$ may be determined after replacing m_{ij} by ϵ_{ij}. We write $\mathrm{DV}(G_U)$ for $\mathrm{DV}(\mathbf{m}_U)$ to emphasize dependence only on G and the subset U. If G is not signed eliminable then by Theorem 5.3 G contains an induced sub-graph H which is a signed graph on four vertices which is not signed-eliminable, a σ-cycle of length > 3, a σ-hill, or a σ-mountain. We assume $G = H$ and show that $\mathrm{DV}(G) > q\ell$ in each of these cases.

The inequality $\mathrm{DV}(G) > 3q$ can easily be verified by hand for each graph on four vertices which is not signed-eliminable (this is also done explicitly in [5, Corollary 6.2]). If G is a σ-cycle, σ-mountain, or σ-hill on $(\ell + 1)$ vertices then a straightforward but tedious computation yields that $\mathrm{DV}(G) = \ell^3 - 2\ell^2 - \ell + 2$ and $q = \ell^2 - 2\ell - 3$. Notice that this proves the result since $\mathrm{DV}(G) = q\ell + 2(\ell + 1) > q\ell$. Since it is long and not particularly enlightening, we will not give further details here for the computation of $DV(G)$ and q for the σ-cycle, σ-hill, and σ-mountain. The interested reader can find more details on the author's website, https://math.okstate.edu/~mdipasq, under the Research tab in the section headed 'Free Multiplicities on Braid and Graphic Arrangements.' Click on the link titled 'Supplemental computations for sigma cycles, mountains, and hills.' □

Remark 5.6. Theorem 1.1 implicitly gives an additional characterization of signed-eliminable graphs. Namely, a signed graph G is signed-eliminable if and only if $\mathrm{DV}(G_U) \le q_U(|U| - 1)$ for every subset U of $V(G)$ of size at least four.

Proof of Theorem 1.1. The direction (3) $\Longrightarrow$ (1) is trivial, so we prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

(1) $\Longrightarrow$ (2): Suppose $\mathbf{m} \in \Lambda_\ell^b$ and $(\mathcal{A}, \mathbf{m})$ is free. Let $U \subset \{v_0, \dots, v_\ell\}$ with $|U| \ge 4$. Write A_U for the closed sub-arrangement of A_ℓ with hyperplanes $\{H_{ij} : \{v_i, v_j\} \subset U\}$. By [4, Proposition 1.7], $(A_U, \mathbf{m}_U)$ is free. Hence by Theorem 3.2, $\mathrm{DV}(\mathbf{m}_U) \le q_U(|U| - 1)$. Since U was arbitrary, we are done.

(2) $\Longrightarrow$ (3): Suppose that $\mathbf{m} \in \Lambda_\ell^b$ and $\mathrm{DV}(\mathbf{m}_U) \le q_U(|U| - 1)$ for every $U \subset \{v_0, \dots, v_\ell\}$ with $|U| \ge 4$. By Proposition 4.2, $\mathbf{m}$ is an ANN multiplicity. By Proposition 5.5, $\mathbf{m} = \mathbf{m}_G(n_0, \dots, n_\ell)$ for some signed-eliminable graph G. By [1, Appendix A, Theorem 0.3], $\mathbf{m}$ is a free multiplicity on A_ℓ (see Remark 5.1). □

6. Free vertices and a conjecture. In this final section we discuss *free vertices* of a multiplicity on a graphic arrangement and present a conjecture on the structure of free multiplicities on braid arrangements. Given a graph $G = (V_G, E_G)$ on $\ell + 1$ vertices (in bijection with the variables $x_0, \dots, x_\ell$), the corresponding *graphic arrangement* is $\mathcal{A}_G = \cup_{\{v_i, v_j\} \in E_G} H_{ij}$. Just as for the braid arrangement, an edge $\{v_i, v_j\}$ of G corresponds to the form $H_{ij} = V(x_i - x_j)$, and the graphic multi-arrangement $(\mathcal{A}_G, \mathbf{m})$ can be identified with the edge-labeled graph $(G, \mathbf{m})$ where the label on $\{v_i, v_j\}$ is $\mathbf{m}(H_{ij}) = m_{ij}$. If $H \subset G$ is a subgraph, then we denote by $\mathbf{m}_H$ the restriction of $\mathbf{m}$ to E_H.

Definition 6.1. Suppose G is a graph. A vertex $v_i \in V_G$ is a *simplicial vertex* if the sub-graph of G induced by v_i and its neighbors is a complete graph. Given a multi-arrangement $(\mathcal{A}_G, \mathbf{m})$ and the corresponding edge-labeled graph $(G, \mathbf{m})$, a vertex v_i is a *free vertex* of $(G, \mathbf{m})$ if it is a simplicial vertex and for every triangle with vertices v_i, v_j, v_k we have $m_{ij} + m_{ik} \le m_{jk} + 1$.

Theorem 6.2. *Suppose G is a graph, v_i is a free vertex of $(G, \mathbf{m})$, and G' is the induced sub-graph on the vertex set $V_G \setminus \{v_i\}$. Then $(\mathcal{A}_G, \mathbf{m})$ is free if and only if $(\mathcal{A}_{G'}, \mathbf{m}_{G'})$ is free.*

Proof of Theorem 6.2. We use a result whose proof we omit since it is virtually identical to the proof of [4, Theorem 5.10]. Recall that a flat $X \in L$ is called *modular* if $X + Y \in L$ for every $Y \in L$, where $X + Y$ is the linear span of X, Y considered as linear sub-spaces of $V \cong \mathbf{K}^{\ell+1}$.

Theorem 6.3. *Suppose $(\mathcal{A}, \mathbf{m})$ is a central multi-arrangement of rank $\ell \ge 3$ and X is a modular flat of rank $\ell - 1$. Suppose $(\mathcal{A}_X, \mathbf{m}_X)$ is free with exponents $(d_1, \dots, d_{\ell-1}, 0)$ and for all $H \in \mathcal{A} \setminus \mathcal{A}_X$ and $H' \in \mathcal{A}_X$, set $Y := H \cap H'$. If one of the following two conditions is satisfied:*

(1) $\mathcal{A}_Y = H \cup H'$ *or*
(2) $\mathbf{m}(H') \ge \sum_{H \in \mathcal{A} \setminus \mathcal{A}'} \mathbf{m}(H) - 1.$

Then $(\mathcal{A}, \mathbf{m})$ is free with exponents $(d_1, \dots, d_{\ell-1}, |\mathbf{m}| - |\mathbf{m}'|)$.

Now suppose G is a graph on $\ell + 1$ vertices $\{v_0, \dots, v_\ell\}$ and $\mathcal{A}_G$ is the associated graphic arrangement. Further suppose that v_i is a free vertex of $(G, \mathbf{m})$, and G' is the induced sub-graph on the vertex set $V_G \setminus \{v_i\}$, along with the isolated vertex v_i. Set $\mathbf{m}' = \mathbf{m}_{G'}$. By [4, Proposition 1.7], if $(\mathcal{A}_{G'}, \mathbf{m}')$ is not free, then neither is $(\mathcal{A}_G, \mathbf{m})$.

Suppose now that $(\mathcal{A}_{G'}, \mathbf{m}')$ is free. We show that $(\mathcal{A}_G, \mathbf{m})$ is free using Theorem 6.3. Write $H_{ij} = V(x_i - x_j)$. Since v_i is a simplicial vertex of G, the flat $X = \cap_{v_j, v_k \neq v_i} H_{jk}$ is modular and has rank $\ell - 1$. The sub-arrangement $(\mathcal{A}_G)_X$ is the graphic arrangement $\mathcal{A}_{G'}$. Suppose $H = H_{ij} \in \mathcal{A}_G \setminus \mathcal{A}_{G'}$, $H' = H_{st} \in \mathcal{A}_{G'}$, and set $Y = H_{ij} \cap H_{st}$. If $\{s,t\} \cap \{i,j\} = \emptyset$, then $\mathcal{A}_Y = H_{ij} \cup H_{st}$. Otherwise, suppose $s = j$. Since v_i is a simplicial vertex, $\{i,t\} \in E_G$, so $\mathcal{A}_Y = H_{ij} \cup H_{it} \cup H_{jt}$. Since v_i is a free vertex, $m_{ij} + m_{it} \leq m_{jt} + 1$, which is condition (2) from Theorem 6.3. Hence $(\mathcal{A}_G, \mathbf{m})$ is free by Theorem 6.3. □

We use Theorem 6.2 to inductively construct two types of free multiplicities. Given a graph G, an *elimination ordering* is an ordering $v_0, \ldots, v_\ell$ of the vertices V_G so that v_i is a simplicial vertex of the induced sub-graph on $v_0, \ldots, v_i$ for every $i = 1, \ldots, \ell$. It is known that V_G admits an elimination ordering if and only if G is chordal [6].

Corollary 6.4. *Suppose $(G, \mathbf{m})$ is an edge-labeled chordal graph with elimination ordering $v_0, \ldots, v_\ell$ satisfying that v_i is a free vertex of the induced sub-graph on $\{v_0, \ldots, v_i\}$ for every $i \geq 2$. Then $(\mathcal{A}_G, \mathbf{m})$ is free.*

Corollary 6.5. *Let $(A_\ell, \mathbf{m})$ be a multi-braid arrangement corresponding to the complete graph $K_{\ell+1}$ on $(\ell+1)$ vertices. Suppose that $K_{\ell+1}$ admits an ordering $\{v_0, \ldots, v_\ell\}$ so that:*

(1) *For some integer $0 \leq k \leq \ell$, the induced subgraph G' on $\{v_0, \ldots, v_k\}$ satisfies that $\mathbf{m}_{G'}$ is a free ANN multiplicity.*
(2) *For $k+1 \leq i \leq \ell$, v_i is a free vertex of the induced graph on $\{v_0, \ldots, v_i\}$.*

Then $(A_\ell, \mathbf{m})$ is free.

We conjecture that all free multi-braid arrangements take the form of Corollary 6.5.

Conjecture 6.6. The multi-braid arrangement $(A_\ell, \mathbf{m})$ is free if and only if it is one of the multi-braid arrangements constructed in Corollary 6.5. Equivalently, by Theorem 6.2, if $(A_\ell, \mathbf{m})$ is free then either $\mathbf{m}$ is a free ANN multiplicity or $\mathbf{m}$ has a free vertex. Using Theorem 1.1, this is equivalent to the following statement: if $\mathbf{m}$ is a free multiplicity and $\mathbf{m} \notin \Lambda_\ell^b$, then $\mathbf{m}$ has a free vertex.

Remark 6.7. Conjecture 6.6 is proved for the A_3 braid arrangement in [5]. Using Macaulay2 [7], we have verified Conjecture 6.6 for many multiplicities on the A_4 arrangement.

Acknowledgements. I would especially like to thank Drs. Jeff Mermin, Chris Francisco, and Jay Schweig for their collaboration on the analysis of free multiplicities on the A_3 braid arrangement. I am very grateful to Dr. Takuro Abe for freely corresponding and offering many suggestions. Computations in Macaulay2 [7] were indispensable for this project.

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