On Koyama's refinement of the prime geodesic theorem

By Muharem AVDISPAHIĆ

Department of Mathematics, University of Sarajevo,
Zmaja od Bosne 35, BA-71000 Sarajevo, Bosnia and Herzegovina

(Communicated by Kenji FUKAYA, M.J.A., Feb. 13, 2018)

Abstract: We give a new proof of the best presently-known error term in the prime geodesic theorem for compact hyperbolic surfaces, without the assumption of excluding a set of finite logarithmic measure. Stronger implications of the Gallagher-Koyama approach are derived, yielding to a further reduction of the error term outside a set of finite logarithmic measure.

Key words: Prime geodesic theorem; Selberg zeta function; hyperbolic manifolds.

1. Introduction. Let $\Gamma \subset PSL(2,\mathbf{R})$ be a strictly hyperbolic Fuchsian group acting on the upper half-plane $\mathcal{H}$ equipped with the hyperbolic metric. The quotient space $\Gamma \setminus \mathcal{H}$ can be identified with a compact Riemann surface $\mathcal{F}$ of a genus $g \geq 2$. The object of our attention is the asymptotic behaviour of the summatory von Mangoldt function

$$\psi_\Gamma(x) = \sum_{\substack{P,k \\ N(P)^k \leq x}} \log N(P)$$

where the sum is taken over primitive hyperbolic conjugacy classes P in Γ (prime geodesics on $\mathcal{F}$), $N(P) = \exp(\mathrm{length}(P))$ is the norm of a class P and k runs through positive integers.

In the recent paper [6] published in this journal, Shin-ya Koyama studied the existence of a subset E in $\mathbf{R}_{\geq 2}$ with finite logarithmic measure such that

$$\psi_\Gamma(x) = x + \sum_{\frac{3}{4}<\rho<1} \frac{x^\rho}{\rho} + O\big(x^{3/4}(\log\log x)^{1/4+\varepsilon}\big) \quad (x \to \infty, x \notin E).$$

Here and in the sequel, ρ denotes zeros of the Selberg zeta function Z_Γ. It is known that the complex zeros of Z_Γ are of the form $\rho = \frac{1}{2} \pm i\gamma$ and that Z_Γ has finitely many real zeros, all lying in the interval $[0,1]$. Koyama was motivated by Gallagher's [4] approach to the prime number theorem under the Riemann hypothesis.

We give a new proof of the following sharper result (cf. [7], [3]).

2010 Mathematics Subject Classification. Primary 11M36; Secondary 11F72, 58J50.

Theorem 1.

$$\psi_\Gamma(x) = x + \sum_{\frac{3}{4}<\rho<1} \frac{x^\rho}{\rho} + O\big(x^{3/4}\big) \quad (x \to \infty).$$

We observe that the analogue is also valid for higher dimensional hyperbolic manifolds with cusps. Applying the Gallagher-Koyama method, we further reduce the error term outside a set of finite logarithmic measure.

Theorem 2. *For $\alpha > 0$, there exists a set H of finite logarithmic measure such that*

$$\psi_\Gamma(x) = x + \sum_{\frac{3}{4}-\varepsilon<\rho<1} \frac{x^\rho}{\rho} + O\left(\frac{x^{3/4}}{(\log x)^\alpha}\right) \quad (x \to \infty, x \notin H),$$

where $\varepsilon > 0$ is arbitrarily small.

2. From Hejhal to Randol.

Proof of Theorem 1. We shall take the same starting point as in [6], i.e., Hejhal's explicit formula with an error term for the function $\psi_{1,\Gamma}(x) = \int_1^x \psi_\Gamma(x)\,dx$ (cf. [5, Theorem 6.16. on p. 110]):

$$(1) \quad \psi_{1,\Gamma}(x) = \alpha_0 x + \beta_0 x \log x + \alpha_1 + \beta_1 \log x + F\left(\frac{1}{x}\right) + \frac{x^2}{2} + \sum_{\substack{\rho \\ |\gamma|<T}} \frac{x^{\rho+1}}{\rho(\rho+1)} + O\left(\frac{x^2 \log x}{T}\right) \ (x \to \infty).$$

Recall that $F(x) = (2g-2)\sum_{k=2}^{\infty} \frac{2k+1}{k(k-1)} x^{1-k}$.

The novelty of our approach consists in integrating (1) at this point and then temporarily

doi: 10.3792/pjaa.94.21

getting rid of Hejhal's error term. Indeed, the integration of (1) firstly yields the explicit formula with an error term for $\psi_{2,\Gamma}(x) = \int_1^x \psi_{1,\Gamma}(x)\,dx$. Now, letting $T \to \infty$ in the obtained formula, we end up with

$$\begin{aligned}\psi_{2,\Gamma}(x) &= \alpha_0' x^2 + \beta_0' x^2 \log x + \alpha_1' x + \beta_1 x \log x \\ &\quad + \frac{x^3}{6} + \beta_2 + (2g-2)\sum_{k=2}^{\infty} \frac{2k+1}{k(k-1)}\frac{x^{2-k}}{(2-k)} \\ &\quad + \sum_{\frac12<\rho<1} \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)} \\ &\quad + \sum_{\mathrm{Re}(\rho)=\frac12} \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)}.\end{aligned}$$

Usually, to derive the asymptotics of $\psi_\Gamma(x)$ from the asymptotics of $\psi_{2,\Gamma}(x)$, one introduces the second-difference operators:

$$\Delta_2^+ f(x) = f(x+2h) - 2f(x+h) + f(x) \text{ and}$$
$$\Delta_2^- f(x) = f(x-2h) - 2f(x-h) + f(x),$$

where $h > 0$ is to be determined later.

Since ψ_Γ is a non-decreasing function, we have

$$\frac{1}{h^2}\Delta_2^- \psi_{2,\Gamma}(x) \le \psi_\Gamma(x) \le \frac{1}{h^2}\Delta_2^+ \psi_{2,\Gamma}(x).$$

We apply Δ_2^+ to all summands in the explicit formula for $\psi_{2,\Gamma}(x)$. E.g., $\Delta_2^+\left(\frac{x^3}{6}\right) = xh^2 + h^3$, which gives us $\frac{1}{h^2}\Delta_2^+\left(\frac{x^3}{6}\right) = x + h$, etc.

Applying $\frac{1}{h^2}\Delta_2^+$ to the sum $\sum_{\frac12<\rho<1} \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)}$, we end up with

$$\sum_{\frac12<\rho<1} \frac{x^\rho}{\rho} + O(h).$$

When dealing with the absolutely convergent series $\sum_{\mathrm{Re}(\rho)=\frac12} \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)}$, we take into account that

$$\frac{1}{h^2}\Delta_2^+ \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)} = O\left(\min\left(\frac{x^{1/2}}{|\rho|}, \frac{x^{5/2}}{h^2|\rho|^3}\right)\right).$$

Thus,

$$\begin{aligned}&\frac{1}{h^2}\Delta_2^+ \sum_{\mathrm{Re}(\rho)=\frac12} \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)} \\ &\quad = O\left(x^{1/2}\sum_{\substack{\mathrm{Re}(\rho)=\frac12\\ |\rho|<M}} \frac{1}{|\rho|}\right) + O\left(\frac{x^{5/2}}{h^2}\sum_{\substack{\mathrm{Re}(\rho)=\frac12\\ |\rho|\ge M}} \frac{1}{|\rho|^3}\right) \\ &\quad = O\big(x^{1/2}M\big) + O\left(\frac{x^{5/2}}{h^2 M}\right) \text{ for } M > 2.\end{aligned}$$

We are left to optimize the terms $O(h)$, $O\big(x^{1/2}M\big)$, $O\big(\frac{x^{5/2}}{h^2 M}\big)$. This is achieved by choosing $h = x^{3/4}$, $M = x^{1/4}$. All other ingredients are dominated by $O\big(x^{3/4}\big)$.

The same procedure works in case of $\Delta_2^-\psi_{2,\Gamma}(x)$, i.e., for estimating $\psi_{2,\Gamma}(x)$ from below.

So,

$$\psi_\Gamma(x) = x + \sum_{\frac34<\rho<1} \frac{x^\rho}{\rho} + O\big(x^{3/4}\big).$$

□

Remark 1. The error term $O\big(x^{3/4}\big)$ in Theorem 1 yields $O\big(x^{3/4}/\log x\big)$ in the prime geodesic theorem. Concerning the explicit formula for $\psi_{1,\Gamma}$, one can consult [2], where a better estimate for the logarithmic derivative of the Selberg zeta function is established.

Remark 2. The full analogue is valid for higher dimensional hyperbolic manifolds with cusps. Namely, the error term in the prime geodesic theorem in that setting reads $O\big(x^{3d_0/2}(\log x)^{-1}\big)$ where $d_0 = \frac{d-1}{2}$ and d is the dimension of a manifold [1, Theorem 1].

3. An application of the Gallagher-Koyama method.

Proof of Theorem 2. In estimating $\psi_\Gamma(x)$ we shall use explicit formula (1) and the relation $\frac1h\Delta_1^-\ \psi_{1,\Gamma}(x) \le \psi_\Gamma(x) \le \frac1h\Delta_1^+\ \psi_{1,\Gamma}(x)$, where $0 < h < \frac{x}{2}$ is to be determined later on. Here, $\Delta_1^+ f(x) = f(x+h) - f(x)$ and $\Delta_1^- f(x) = f(x) - f(x-h)$.

Let $\beta > 4\alpha + 1$. According to (1) and the relation above, we have

$$\begin{aligned}(2)\qquad \psi_\Gamma(x) &\le \frac1h\int_x^{x+h} \psi_\Gamma(t)\,dt \\ &= x + \sum_{\frac12<\rho<1} \frac{x^\rho}{\rho} + O(\log x) + O(h) \\ &\quad + O\left(\frac{x^2\log x}{hT}\right) \\ &\quad + \frac1h\left|\sum_{\substack{\mathrm{Re}(\rho)=\frac12\\ |\gamma|\le T}} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)}\right|.\end{aligned}$$

Now,

$$\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\|\gamma|\le T}} \frac{(x+h)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)} = \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\|\gamma|\le(\log T)^{\beta}}} \frac{(x+h)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)} + \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\(\log T)^{\beta}<|\gamma|\le T}} \frac{(x+h)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)}.$$

For the first sum on the right-hand side, we have

$$\frac{1}{h}\left|\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\|\gamma|\le(\log T)^{\beta}}} \frac{(x+h)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)}\right| = O\left(x^{1/2}\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\|\gamma|\le(\log T)^{\beta}}}\frac{1}{|\rho|}\right) = O\big(x^{1/2}(\log T)^{\beta}\big).$$

The second sum is to be split into

$$\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\(\log T)^{\beta}<|\gamma|\le T}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} - \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\(\log T)^{\beta}<|\gamma|\le T}} \frac{x^{\rho+1}}{\rho(\rho+1)} = \sigma_{\beta,T}(x+h)-\sigma_{\beta,T}(x).$$

Let

$$D_Y^T = \left\{x\in[T,eT) : \left|\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\Y<|\gamma|\le T}}\frac{x^{\rho+1}}{\rho(\rho+1)}\right| > \frac{x^{3/2}}{(\log x)^{2\alpha}}\right\}, \quad Y<T.$$

By Koyama's argument [6, p. 80],

$$Y^{-1} \gg \frac{1}{(\log eT)^{4\alpha}}\int_{D_Y^T}\frac{dx}{x} = \frac{1}{(1+\log T)^{4\alpha}}\,\mu^{\times}D_Y^T.$$

Hence,

$$\mu^{\times}D_Y^T \ll \frac{(1+\log T)^{4\alpha}}{Y}.$$

For $x\in[e^n,e^{n+1})$, let $T=e^n$. The error term in (2) becomes $O(x\log x/h)$. Let Y take values $Y_1=(\log\ T)^{\beta}=n^{\beta}$, $Y_2=(n-1)^{\beta}$, $Y_3=e^{n-1}$. Denote $E_n = D_{Y_1}^T$, $F_n=D_{Y_2}^T$, $G_n=D_{Y_3}^T$ and $E=\cup E_n$, $F=\cup F_n$, $G=\cup G_n$, respectively. We have

$$\mu^{\times}E \ll \sum_{n=2}^{\infty}\frac{(n+1)^{4\alpha}}{n^{\beta}}<\infty, \text{ since } \beta>4\alpha+1;$$

$$\mu^{\times}F \ll \sum_{n=2}^{\infty}\frac{(n+1)^{4\alpha}}{(n-1)^{\beta}}<\infty \text{ for the same reason;}$$

$$\mu^{\times}G \ll \sum_{n=2}^{\infty}\frac{(n+1)^{4\alpha}}{e^{n-1}}<\infty.$$

Put $H=E\cup F\cup G$. Obviously, $\mu^{\times}H<\infty$. We take $x, x+h\in\mathbf{R}_{\ge 2}\setminus H$.

For $x\in[e^n,e^{n+1})\setminus E_n$, $T=e^n$, we get

$$\sigma_{\beta,T}(x) = O\left(\frac{x^{3/2}}{(\log x)^{2\alpha}}\right).$$

Case I. If $x+h\in[e^n,e^{n+1})\setminus H$, then we also have

$$\sigma_{\beta,T}(x+h) = O\left(\frac{(x+h)^{3/2}}{(\log(x+h))^{2\alpha}}\right) = O\left(\frac{x^{3/2}}{(\log x)^{2\alpha}}\right).$$

Case II. If $x+h\in[e^{n+1},e^{n+2})\setminus H$, we shall express the sum $\sigma_{\beta,T}(x+h)$ in the form

$$\sigma_{\beta,T}(x+h) = \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\n^{\beta}<|\gamma|\le e^{n+1}}}\frac{(x+h)^{\rho+1}}{\rho(\rho+1)} - \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\e^n<|\gamma|\le e^{n+1}}}\frac{(x+h)^{\rho+1}}{\rho(\rho+1)}.$$

The first sum is $O\Big(\frac{x^{3/2}}{(\log x)^{2\alpha}}\Big)$ because $x+h\notin F_{n+1}=D_{Y_2}^{eT}$. The second sum is

$$\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2}\\e^n<|\gamma|\le e^{n+1}}}\frac{(x+h)^{\rho+1}}{\rho(\rho+1)} = O\left(\frac{x^{3/2}}{(\log x)^{2\alpha}}\right),$$

since $x+h\notin G_{n+1}=D_{Y_3}^{eT}$.

So, in both cases, the relation (2) becomes

$$\psi_{\Gamma}(x)\le x+\sum_{\frac{1}{2}<\rho<1}\frac{x^{\rho}}{\rho}+O(\log x)+O(h) + O\left(\frac{x\log x}{h}\right)+O\left(\frac{x^{3/2}}{h(\log x)^{2\alpha}}\right).$$

The optimal bound is achieved with $h=\frac{x^{3/4}}{(\log x)^{\alpha}}$. Thus,

$$\psi_{\Gamma}(x)\le x+\sum_{\frac{1}{2}<\rho<1}\frac{x^{\rho}}{\rho}+O\left(\frac{x^{3/4}}{(\log x)^{\alpha}}\right).$$

The opposite inequality is derived from $\psi_\Gamma(x) \geq \frac{1}{h}\Delta_1^- \psi_{1,\Gamma}(x)$ by the same procedure. If $\varepsilon > 0$ is arbitrarily small, then $\sum_{\frac{1}{2}<\rho<\frac{3}{4}-\varepsilon} \frac{x^\rho}{\rho}$ is obviously dominated by the error term. This completes the proof. $\square$

References

[1] M. Avdispahić and Dž. Gušić, On the error term in the prime geodesic theorem, Bull. Korean Math. Soc. **49** (2012), no. 2, 367–372.

[2] M. Avdispahić and L. Smajlović, An explicit formula and its application to the Selberg trace formula, Monatsh. Math. **147** (2006), no. 3, 183–198.

[3] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics, 106, Birkhäuser Boston, Inc., Boston, MA, 1992.

[4] P. X. Gallagher, Some consequences of the Riemann hypothesis, Acta Arith. **37** (1980), 339–343.

[5] D. A. Hejhal, *The Selberg trace formula for* PSL$(2,\mathbf{R})$. *Vol. I*, Lecture Notes in Mathematics, 548, Springer-Verlag, Berlin, 1976.

[6] S. Koyama, Refinement of prime geodesic theorem, Proc. Japan Acad. Ser. A Math. Sci. **92** (2016), no. 7, 77–81.

[7] B. Randol, On the asymptotic distribution of closed geodesics on compact Riemann surfaces, Trans. Amer. Math. Soc. **233** (1977), 241–247.