A p-analogue of Euler's constant and congruence zeta functions

By Nobushige KUROKAWA and Yuichiro TAGUCHI

Department of Mathematics, School of Science, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan

(Communicated by Heisuke HIRONAKA, M.J.A., Jan. 12, 2018)

Abstract: A p-analogue of a formula of Euler on the Euler constant is given, and it is interpreted in terms of the absolute zeta functions of tori.

Key words: Euler constant; p-analogue; congruence zeta function; absolute zeta function.

1. Introduction. We show the following result:

Theorem 1. *For a prime number*[1] p, *we have*

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(n) = \frac{\log p}{p-1}\gamma(p).$$

Here, $\zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}$ *denotes the congruence zeta function of the direct product of* $n-1$ *copies of the multiplicative group scheme* $\mathbf{G}_{\mathrm{m}}$ *over* $\mathbf{F}_p$, *and* $\gamma(p)$ *is the* p*-analogue of the Euler constant*[2] γ, *which is defined by*

$$\gamma(p) := \sum_{m=1}^{\infty} \frac{1}{[m]_p}$$

with

$$[m]_p := \frac{p^m - 1}{p-1} = \sum_{k=0}^{m-1} p^k,$$

or, what amounts to the same, by the Jackson integral

$$\gamma(p) = \int_0^1 \frac{1}{1-x}\, d_p x := \sum_{m=1}^{\infty} \frac{1}{1-p^{-m}} (p^{-m+1} - p^{-m}).$$

The constant $\gamma(p)$ appears naturally as the Euler constant for a p-analogue of the Riemann zeta function (Kurokawa-Wakayama [10] (2004)), with the slight difference that our $\gamma(p)$ equals their $\gamma(q) - \dfrac{(q-1)\log(q-1)}{\log q} + \dfrac{q-1}{2}$ with $q = p$.

Note also that

$$\gamma(p) = (p-1)\sum_{m=1}^{\infty} \frac{d(m)}{p^m},$$

where $d(m)$ denotes the number of positive divisors of m, and that $\gamma(p)$ is known (essentially by Erdös [3] (1948)) to be an irrational number for any p (cf. [10], Thm. 2.4). Some numerical examples of the values of $\gamma(p)$ are as follows:

$$\gamma(2) = 1.6066951524152917637833015231909245804\ldots,$$
$$\gamma(3) = 1.3643070052104761335225263724532480199\ldots,$$
$$\gamma(5) = 1.2069354143918898317926486375759647706\ldots,$$
$$\gamma(7) = 1.1454603744615694692131978665061470124\ldots,$$
$$\gamma(11) = 1.0916034921693992068064574193097607991\ldots,$$
$$\gamma(13) = 1.0773482332373439813248280328001992315\ldots,$$
$$\gamma(17) = 1.0590164284716950207914174528974338014\ldots,$$
$$\gamma(19) = 1.0527704668263105664864363142395871465\ldots,$$

whereas the classical Euler constant is

$$\gamma = 0.5772156649015328606065120900824024310\ldots.$$

Euler [5] (1776) proved the formula

$$\sum_{n=1}^{\infty} \frac{1}{n} \log\left(\prod_{k=1}^{n} k^{(-1)^k \binom{n-1}{k-1}} \right) = \gamma$$

for the original Euler constant γ. From the point of view of zeta functions over $\mathbf{F}_1$ of Soulé [11] (2004) (see also Kurokawa [7] (2005), Deitmar [2] (2006), Connes-Consani [1] (2010) and Kurokawa-Ochiai [8] (2013)), the equality of Euler is nothing but

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(n) = \gamma;$$

we explain the proof of it in §3. Our Theorem 1 is a p-analogue of Euler's result using the congruence zeta function $\zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(s)$.

We prove Theorem 1 in a bit stronger form:

2010 Mathematics Subject Classification. Primary 11M06; Secondary 11M35, 11M41, 14G10.

[1] Although p here and elsewhere can be a prime power, it would be natural to say "prime p" in such a context where we consider the absolute limit $p \to 1$.

[2] Recall that the Euler constant $\gamma := \lim_{n\to\infty}(\sum_{m=1}^{n} \frac{1}{m} - \log n)$ is a renormalization of the divergent series $\sum_{m=1}^{\infty} \frac{1}{m}$, and is also expressed ([4]) by the integral $\int_0^1 (\frac{1}{1-x} + \frac{1}{\log x})dx$.

doi: 10.3792/pjaa.94.13

Theorem 2. *For a prime number p and a complex number s with* $\mathrm{Re}(s) > 0$*, we have*

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(s+n-1) = \frac{\log p}{p-1} \sum_{m=0}^{\infty} \frac{1}{[s+m]_p}.$$

Theorem 1 is obtained by letting $s = 1$ in Theorem 2.

2. Proof of Theorems. It is sufficient to prove Theorem 2. A direct calculation shows

$$\zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(s+n-1) = \prod_{k=1}^{n} (1-p^{-k-s+1})^{(-1)^k \binom{n-1}{k-1}}.$$

Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(s+n-1) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \log(1-p^{-k-s+1}) \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \sum_{m=1}^{\infty} \frac{1}{m} p^{-mk-ms+m} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sum_{m=1}^{\infty} \frac{1}{m} p^{-mk-ms} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m} \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} p^{-mk} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m} (1-p^{-m})^{n-1} \\
&= -\sum_{m=1}^{\infty} \frac{p^{-ms}}{m} \cdot \frac{\log(1-(1-p^{-m}))}{1-p^{-m}},
\end{aligned}$$

where we used the formula

$$\sum_{n=1}^{\infty} \frac{1}{n} x^{n-1} = \frac{-\log(1-x)}{x}$$

for $0 < x = 1 - p^{-m} < 1$. Thus we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(s+n-1) = \log p \sum_{m=1}^{\infty} \frac{p^{-ms}}{1-p^{-m}}.$$

Now, we use the formula

$$\sum_{m=1}^{\infty} \frac{u^m}{1-v^m} = \sum_{l=0}^{\infty} \frac{1}{u^{-1}v^{-l}-1}$$

for $0 < |u|, |v| < 1$, which is shown as follows:

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{u^m}{1-v^m} &= \sum_{m=1}^{\infty} u^m \left(\sum_{l=0}^{\infty} v^{ml} \right) \\
&= \sum_{l=0}^{\infty} \left(\sum_{m=1}^{\infty} (uv^l)^m \right) \\
&= \sum_{l=0}^{\infty} \frac{uv^l}{1-uv^l} \\
&= \sum_{l=0}^{\infty} \frac{1}{u^{-1}v^{-l}-1}.
\end{aligned}$$

By using the above formula for $u = p^{-s}$ and $v = p^{-1}$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_p}(s+n-1) &= \log p \sum_{l=0}^{\infty} \frac{1}{p^{s+l}-1} \\
&= \frac{\log p}{p-1} \sum_{l=0}^{\infty} \frac{1}{[s+l]_p}.
\end{aligned}$$

□

3. Variation of Euler's formula. In this section, we first explain that Euler's formula

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \left(\prod_{k=1}^{n} k^{(-1)^k \binom{n-1}{k-1}} \right) = \gamma$$

is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(n) = \gamma. \tag{3.1}$$

This follows from the equality

$$\zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(n) = \prod_{k=1}^{n} k^{(-1)^k \binom{n-1}{k-1}}$$

for all integers $n \geq 1$. In fact, we show:

Theorem 3. *For any integer $n \geq 1$ and a complex variable s, we have*

$$\zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(s) = \prod_{k=1}^{n} (s-n+k)^{(-1)^k \binom{n-1}{k-1}}.$$

This is Theorem C (2) of [8]. We shall summarize the proof below for the convenience of the reader. Let us begin by recalling the definition of absolute zeta functions from [8] and [9]. For a function

$$f: \mathbf{R}_{>0} \to \mathbf{R},$$

we define the absolute zeta function $\zeta_f(s)$ of f by

$$\zeta_f(s) := \exp\left(\frac{\partial}{\partial w} Z_f(w,s) \bigg|_{w=0} \right),$$

where

$$Z_f(w,s) := \frac{1}{\Gamma(w)} \int_1^{\infty} f(x) x^{-s-1} (\log x)^{w-1} dx,$$

if the integral exists. For a scheme X of finite type over $\mathbf{Z}$, we define the absolute zeta function $\zeta_{X/\mathbf{F}_1}(s)$ of X by

$$\zeta_{X/\mathbf{F}_1}(s) := \zeta_{f_X}(s),$$

if there exists a polynomial function $f_X : \mathbf{R}_{>0} \to \mathbf{R}$ such that $f_X(x)$ equals the number $|X(\mathbf{F}_x)|$ of $\mathbf{F}_x$-valued points of X whenever x is a prime power. Note that f_X for some typical X (including $\mathbf{G}_{\mathrm{m}}^{n-1}$) are known to be absolute automorphic forms ([9], §3).

Proof of Theorem 3. For $X = \mathbf{G}_{\mathrm{m}}^{n-1}$, we have

$$f_X(x) = |\mathbf{G}_{\mathrm{m}}^{n-1}(\mathbf{F}_x)| = (x-1)^{n-1}.$$

Hence

$$\begin{aligned} &Z_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(w,s) \\ &\quad = \frac{1}{\Gamma(w)}\int_1^\infty (x-1)^{n-1}x^{-s-1}(\log x)^{w-1}dx \\ &\quad = \sum_{k=1}^n (-1)^{k-1}\binom{n-1}{k-1} \\ &\qquad \times \frac{1}{\Gamma(w)}\int_1^\infty x^{-(s-n+k)-1}(\log x)^{w-1}dx. \end{aligned}$$

Since

$$\frac{1}{\Gamma(w)}\int_1^\infty x^{-s-1}(\log x)^{w-1}dx = s^{-w},$$

it follows that

$$Z_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(w,s) = \sum_{k=1}^n (-1)^{k-1}\binom{n-1}{k-1}(s-n+k)^{-w},$$

and hence

$$\begin{aligned} \zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(s) &= \exp\left(\frac{\partial}{\partial w} Z_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(w,s)\bigg|_{w=0}\right) \\ &= \prod_{k=1}^n (s-n+k)^{(-1)^k\binom{n-1}{k-1}}. \end{aligned}$$

□

Next we consider a variation of the formula (3.1). Hasse proved the following identity ([6], p. 451)[3]

$$\text{(3.2)}\quad \begin{aligned} &w\zeta(w+1,s) \\ &\quad = \sum_{n=1}^\infty \frac{1}{n}\left(\sum_{k=1}^n (-1)^{k-1}\binom{n-1}{k-1}(s+k-1)^{-w}\right) \end{aligned}$$

for the Hurwitz zeta function defined by

$$\zeta(w,s) = \sum_{n=0}^\infty \frac{1}{(s+n)^w} \quad (\mathrm{Re}(w) > 1)$$

for $s \in \mathbf{R} \setminus \mathbf{Z}_{<0}$, and, using this, proved the meromorphic continuation of $\zeta(w,s)$ to the whole w-plane. The Euler constant $\gamma(s)$ for $\zeta(w,s)$ should be defined by

$$\begin{aligned} \gamma(s) :&= \lim_{w\to 0}\left(\zeta(w+1,s) - \frac{1}{w}\right) \\ &= \frac{\partial}{\partial w}(w\zeta(w+1,s))\bigg|_{w=0}. \end{aligned}$$

It follows from Hasse's formula (3.2) that

$$\text{(3.3)}\quad \gamma(s) = \sum_{n=1}^\infty \frac{1}{n}\sum_{k=1}^n (-1)^k\binom{n-1}{k-1}\log(s+k-1).$$

Here, the term-by-term differentiation is allowed because the right-hand side of (3.2) converges uniformly on each compact subset of $\mathbf{C}$ by Hasse ([6], p. 452).

On the other hand, by Lerch's formula

$$\zeta(w+1,s) = \frac{1}{w} - \frac{\Gamma'}{\Gamma}(s) + O(w) \quad \text{as } w \to 0,$$

we have

$$\text{(3.4)}\qquad \lim_{w\to 0}\left(\zeta(w+1,s) - \frac{1}{w}\right) = -\frac{\Gamma'}{\Gamma}(s).$$

Putting (3.3) and (3.4) together, we obtain the formula

$$\sum_{n=1}^\infty \frac{1}{n}\sum_{k=1}^n (-1)^k\binom{n-1}{k-1}\log(s+k-1) = -\frac{\Gamma'}{\Gamma}(s).$$

According to Theorem 3, this result can be interpreted as:

Theorem 4. *For $s \in \mathbf{R} \setminus \mathbf{Z}_{<0}$, we have*

$$\sum_{n=1}^\infty \frac{1}{n}\log\zeta_{\mathbf{G}_{\mathrm{m}}^{n-1}/\mathbf{F}_1}(s+n-1) = -\frac{\Gamma'}{\Gamma}(s).$$

This reduces to (3.1) when $s=1$, because $\Gamma'(1) = -\gamma$ and $\Gamma(1) = 1$.

Acknowledgments. The authors thank Prof. Hidekazu Tanaka for providing them with some numerical data of the values of $\gamma(p)$, which he calculated by Pari/GP.

References

[1] A. Connes and C. Consani, Schemes over $\mathbf{F}_1$ and zeta functions, Compos. Math. **146** (2010), no. 6, 1383–1415.

[3] This formula is proved in [6] only for the Riemann zeta function, but it is remarked that the same holds for the Hurwitz zeta function and Dirichlet L-functions as well.

[2] A. Deitmar, Remarks on zeta functions and K-theory over $\mathbf{F}_1$, Proc. Japan Acad. Ser. A Math. Sci. **82** (2006), no. 8, 141–146.
[3] P. Erdös, On arithmetical properties of Lambert series, J. Indian Math. Soc. (N.S.) **12** (1948), 63–66.
[4] L. Euler, De numero memorabili in summatione progressionis harmonicae naturalis occurrente, Acta Academiae Scientiarum Imperialis Petropolitanae **5** (1785), 45–75. (E583, dated February 22, 1776, Collected Works I-15, 569–603).
[5] L. Euler, Evolutio formulae integralis $\int \partial x(\frac{1}{1-x} + \frac{1}{lx})$ a termino $x = 0$ usque ad $x = 1$ extensae, Nova Acta Academiae Scientiarum Petropolitanae **4** (1789), 3–16. (E629, dated February 29, 1776, Collected Works I-18, 318–334).
[6] H. Hasse, Ein Summierungsverfahren für die Riemannsche ξ-Reihe, Math. Z. **32** (1930), no. 1, 458–464.
[7] N. Kurokawa, Zeta functions over $\mathbf{F}_1$, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 10, 180–184.
[8] N. Kurokawa and H. Ochiai, Dualities for absolute zeta functions and multiple gamma functions, Proc. Japan Acad. Ser. A Math. Sci. **89** (2013), no. 7, 75–79.
[9] N. Kurokawa and H. Tanaka, Absolute zeta functions and the automorphy, Kodai Math. J. **40** (2017), no. 3, 584–614.
[10] N. Kurokawa and M. Wakayama, On q-analogues of the Euler constant and Lerch's limit formula, Proc. Amer. Math. Soc. **132** (2004), no. 4, 935–943.
[11] C. Soulé, Les variétés sur le corps à un élément, Mosc. Math. J. **4** (2004), no. 1, 217–244.