The local zeta integrals for $GL(2,\mathbf{C}) \times GL(2,\mathbf{C})$

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Abstract: In this article, for irreducible admissible infinite-dimensional representations Π and Π' of $GL(2,\mathbf{C})$, we show that the local L-factor $L(s,\Pi\times\Pi')$ can be expressed as some local zeta integral for $GL(2,\mathbf{C})\times GL(2,\mathbf{C})$.

Key words: Whittaker functions; automorphic forms; zeta integrals.

1. Introduction. Let $F=\mathbf{R}$ or $\mathbf{C}$. Let Π and Π' be irreducible admissible infinite-dimensional representations of $GL(2,F)$. We consider here the local zeta integral $Z(s,W,W',f)$ for $GL(2,F)\times GL(2,F)$, which is defined from Whittaker functions W for (Π,ψ), W' for (Π',ψ^{-1}) and a standard Schwartz function f on F^2 with the standard character ψ of F. In Theorems 17.2 (3) (for $F=\mathbf{R}$) and 18.1 (3) (for $F=\mathbf{C}$) of the lecture note [Ja], Jacquet asserts only that the associated local L-factor $L(s,\Pi\times\Pi')$ can be expressed as a finite sum of the local zeta integrals for $GL(2,F)\times GL(2,F)$, that is,

$$\sum_{i=1}^{m} Z(s,W_i,W_i',f_i)=L(s,\Pi\times\Pi')$$

with some W_i, W_i' and f_i. However, in the proof of Theorem 17.2 of [Ja], he shows a stronger result for $F=\mathbf{R}$. He gives Whittaker functions W_0, W_0' and a standard Schwartz function f_0 satisfying

$$Z(s,W_0,W_0',f_0)=L(s,\Pi\times\Pi') \tag{1.1}$$

for $F=\mathbf{R}$, explicitly. (See Proposition 2.5.2 in [Zh] for the case omitted in [Ja].) On the other hand, the proof of Theorem 18.1 in [Ja] is written with the modified zeta integrals defined from vector valued functions, and it is not clear whether the stronger assertion (1.1) for $F=\mathbf{C}$ is true or not. In this article, we give Whittaker functions W_0, W_0' and a standard Schwartz function f_0 satisfying (1.1) for $F=\mathbf{C}$, explicitly, rewriting Jacquet's calculation in [Ja] using Schur's orthogonality and explicit formulas of Whittaker functions.

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2. Whittaker functions on $GL(2,\mathbf{C})$. Let $G=GL(2,\mathbf{C})$ be the complex general linear group of degree 2, and we fix an Iwasawa decomposition $G=NAK$ with

$$N=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\;\middle|\; x\in\mathbf{C}\right\},$$

$$A=\{\mathrm{diag}(y_1y_2,y_2)\mid y_1,y_2\in\mathbf{R}_+\}$$

and the unitary group $K=U(2)$ of degree 2. Here $\mathbf{R}_+$ is the set of positive real numbers. We denote by $\mathfrak{g}_{\mathbf{C}}$ the complexification $\mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$ of the associated Lie algebra $\mathfrak{g}$ of G.

For $c\in\mathbf{C}^\times$, we define a character ψ_c of $\mathbf{C}$ by

$$\psi_c(x)=e^{2\pi\sqrt{-1}(cx+\overline{cx})}\quad (x\in\mathbf{C}),$$

and let $C^\infty(N\backslash G;\psi_c)$ be the space of smooth functions f on G satisfying

$$f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\psi_c(x)f(g)\quad (x\in\mathbf{C}, g\in G),$$

on which G acts by the right translation. Here we note that there is a G-isomorphism

$$\Xi_c\colon C^\infty(N\backslash G;\psi_1)\to C^\infty(N\backslash G;\psi_c), \tag{2.1}$$

defined by $\Xi_c(f)(g)=f(\mathrm{diag}(c,1)\,g)$ $(g\in G)$.

Let (Π,H_Π) be an irreducible admissible infinite-dimensional representation of G. We denote by $H_{\Pi,K}$ the subspace of H_Π consisting of all K-finite vectors. We define the space $\mathcal{I}_{\Pi,\psi_c}$ of homomorphisms $\Phi\colon H_{\Pi,K}\to C^\infty(N\backslash G;\psi_c)_K$ of $(\mathfrak{g}_{\mathbf{C}},K)$-modules such that $\Phi(f)$ is of moderate growth for any $f\in H_{\Pi,K}$. Theorem 6.3 in [JL] tells that the space $\mathcal{I}_{\Pi,\psi_c}$ is one dimensional. We define the space $\mathcal{W}(\Pi,\psi_c)$ of Whittaker functions for (Π,ψ_c) by

$$\mathcal{W}(\Pi,\psi_c)=\{\Phi(f)\mid f\in H_{\Pi,K},\ \Phi\in\mathcal{I}_{\Pi,\psi_c}\}.$$

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3. Irreducible representations of K. Let Λ be the set $\{\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2 \mid \lambda_1 \geq \lambda_2\}$ of dominant weights. Let V_λ be the $\mathbf{C}$-vector space of degree $(\lambda_1 - \lambda_2)$ homogeneous polynomials in z_1, z_2, for $\lambda = (\lambda_1, \lambda_2) \in \Lambda$. The group K acts on V_λ by

$$\tau_\lambda(k)p(z_1, z_2) = \det(k)^{\lambda_2} p((z_1, z_2)k)$$

for $k \in K$ and $p(z_1, z_2) \in V_\lambda$. Then the representations $(\tau_\lambda, V_\lambda)$ $(\lambda \in \Lambda)$ of K are irreducible, and exhaust the equivalence classes of irreducible representations of K.

Let $\lambda = (\lambda_1, \lambda_2) \in \Lambda$. We define $\{v_q^\lambda\}_{q=0}^{\lambda_1-\lambda_2}$ as a basis of V_λ by $v_q^\lambda = z_1^{\lambda_1-\lambda_2-q} z_2^q$. We set

$$\tilde{\lambda} = (-\lambda_2, -\lambda_1).$$

Then there is a K-invariant $\mathbf{C}$-bilinear pairing $\langle \cdot, \cdot \rangle$ on $V_{\tilde{\lambda}} \otimes_{\mathbf{C}} V_\lambda$, which is determined by

$$\langle v_{\tilde{q}}^{\tilde{\lambda}},\ v_q^\lambda \rangle = (-1)^{\lambda_1 - q} \binom{\lambda_1 - \lambda_2}{q}^{-1}$$

and $\langle v_r^{\tilde{\lambda}},\ v_q^\lambda \rangle = 0$ $(r \neq \tilde{q})$ with $\tilde{q} = \lambda_1 - \lambda_2 - q$. Here we denote by $\binom{n}{i}$ the binomial coefficient $\dfrac{n!}{i!(n-i)!}$ for $n, i \in \mathbf{Z}$ such that $n \geq i \geq 0$.

4. Principal series representations. For $\nu = (\nu_1, \nu_2) \in \mathbf{C}^2$ and $d = (d_1, d_2) \in \mathbf{Z}^2$, let $H_{(\nu,d)}^\infty$ be the space of smooth functions f on G such that, for $t_1, t_2 \in \mathbf{C}^\times$, $x \in \mathbf{C}$ and $g \in G$,

$$f\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} g\right) = \left(\frac{t_1}{|t_1|}\right)^{d_1} \left(\frac{t_2}{|t_2|}\right)^{d_2} \times |t_1|^{2\nu_1+1} |t_2|^{2\nu_2-1} f(g).$$

The group G acts on $H_{(\nu,d)}^\infty$ by the right translation

$$(\Pi_{(\nu,d)}(g)f)(h) = f(hg) \quad (g, h \in G,\ f \in H_{(\nu,d)}^\infty).$$

Let $(\Pi_{(\nu,d)}, H_{(\nu,d)})$ be a Hilbert representation of G, which is the completion of $(\Pi_{(\nu,d)}, H_{(\nu,d)}^\infty)$ relative to the L^2-inner product on K with respect to the Haar measure. We call $(\Pi_{(\nu,d)}, H_{(\nu,d)})$ a principal series representation of G.

Theorem 6.2 in [JL] tells that any irreducible admissible infinite-dimensional representation of G is isomorphic to some irreducible principal series representation of G as $(\mathfrak{g}_{\mathbf{C}}, K)$-modules. Moreover, when $\Pi_{(\nu,d)}$ is irreducible, we may assume $d \in \Lambda$ without loss of generalities, since

$$H_{(\nu,d),K} \simeq H_{((\nu_2,\nu_1),(d_2,d_1)),K} \tag{4.1}$$

as $(\mathfrak{g}_{\mathbf{C}}, K)$-modules. Here $H_{(\nu,d),K}$ is the subspace of $H_{(\nu,d)}$ consisting of all K-finite vectors. From Lemma 6.1 (ii) in [JL], we obtain the following

Proposition 4.1. *As K-modules,*

$$H_{(\nu,d),K} \simeq \bigoplus_{m=0}^{\infty} V_{(d_1+m, d_2-m)}$$

holds for $\nu = (\nu_1, \nu_2) \in \mathbf{C}^2$ and $d = (d_1, d_2) \in \Lambda$.

5. Explicit formulas of Whittaker functions. For $s \in \mathbf{C}$ and $i \in \mathbf{Z}_{\geq 0}$, we set

$$(s)_i = \frac{\Gamma(s+i)}{\Gamma(s)} = s(s+1)\cdots(s+i-1),$$
$$\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

as usual, where $\Gamma(s)$ is the Gamma function and $\mathbf{Z}_{\geq 0}$ is the set of non-negative integers. For $a_1, a_2 \in \mathbf{C}$, we define a function $\mathrm{K}(a_1, a_2)$ on $\mathbf{R}_+$ by

$$\mathrm{K}(a_1, a_2)(y_1) = 8 y_1^{a_1+a_2} K_{a_1-a_2}(4\pi y_1)$$
$$= \frac{1}{2\pi\sqrt{-1}} \int_{\alpha-\sqrt{-1}\infty}^{\alpha+\sqrt{-1}\infty} \Gamma_{\mathbf{C}}(s+a_1)\Gamma_{\mathbf{C}}(s+a_2) y_1^{-2s} ds$$

for $y_1 \in \mathbf{R}_+$. Here $K_\mu(z)$ is the modified Bessel function of the second kind (§17.71 in [WW], (6.5) in [Bu]), and α is a real number such that

$$\alpha > \max\{-\mathrm{Re}(a_1), -\mathrm{Re}(a_2)\}.$$

Let $(\Pi_{(\nu,d)}, H_{(\nu,d)})$ be an irreducible principal series representation with $\nu = (\nu_1, \nu_2) \in \mathbf{C}^2$, $d = (d_1, d_2) \in \Lambda$. Let $c \in \mathbf{C}^\times$. Let $m \in \mathbf{Z}_{\geq 0}$ and we set $\lambda = (\lambda_1, \lambda_2) = (d_1 + m, d_2 - m)$. By Proposition 4.1, there is a non-zero K-homomorphism $\phi\colon V_\lambda \to \mathcal{W}(\Pi_{(\nu,d)}, \psi_c)$, which is unique up to scalar multiple. For $g \in G$ with the Iwasawa decomposition

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 \\ 0 & y_2 \end{pmatrix} k \quad \begin{pmatrix} y_1, y_2 \in \mathbf{R}_+, \\ x \in \mathbf{C},\ k \in K \end{pmatrix}$$

and $v \in V_\lambda$, we have

$$\phi(v)(g) = \psi_c(x) y_2^{2\nu_1+2\nu_2} \phi(\tau_\lambda(k)v)(\mathrm{diag}(y_1, 1)).$$

Hence we note that ϕ is characterized by the functions $\phi(v_q^\lambda)(\mathrm{diag}(y_1, 1))$ $(0 \leq q \leq \lambda_1 - \lambda_2)$ on $\mathbf{R}_+$. We will give explicit formulas of these functions.

Because of (2.1), it suffices to consider the case of $c = 1$. We set $c = 1$ and

$$\varphi_q(y_1) = \frac{\phi(v_q^\lambda)(\mathrm{diag}(y_1, 1))}{(\sqrt{-1})^{\lambda_1 - q} y_1} \quad (0 \leq q \leq \lambda_1 - \lambda_2).$$

Translating Eq. (15) and Eq. (16) in [Po] into our notation, for $0 \leq q \leq \lambda_1 - \lambda_2$, we have

$$\{(\partial_1 - 2\nu_1 - m + q)(\partial_1 - 2\nu_2 + m - \tilde{q}) - (4\pi y_1)^2\}\varphi_q = -8\pi q y_1 \varphi_{q-1}, \tag{5.1}$$

$$(5.2)\quad \{(\partial_1-2\nu_1+m-q)(\partial_1-2\nu_2-m+\tilde{q}) - (4\pi y_1)^2\}\varphi_q = -8\pi\tilde{q}y_1\varphi_{q+1}$$

with $\tilde{q}=\lambda_1-\lambda_2-q$ and $\partial_1 = y_1\dfrac{d}{dy_1}$. Here we set $\varphi_q=0$ if $q<0$ or $q>\lambda_1-\lambda_2$.

Since the functions φ_q $(0\le q\le\lambda_1-\lambda_2)$ are determined from φ_0 by the equation (5.2), we note that $\phi\neq 0$ implies $\varphi_0\neq 0$. Taking $q=0$ in the equation (5.1) and comparing with the Bessel differential equation, Popa shows that φ_0 is a non-zero constant multiple of $\mathrm{K}\big(\nu_1+\frac{m}{2},\,\nu_2+\frac{\lambda_1-\lambda_2-m}{2}\big)$ in [Po].

If we assume $\varphi_0=\mathrm{K}\big(\nu_1+\frac{m}{2},\,\nu_2+\frac{\lambda_1-\lambda_2-m}{2}\big)$, we obtain the formula

$$\varphi_q = \sum_{i=0}^{\min\{q,m\}}\binom{q}{i}\frac{(-m)_i\big(-\nu_1+\nu_2-\frac{\lambda_1-\lambda_2}{2}\big)_i}{(2\pi)^i(-\lambda_1+\lambda_2)_i} \times \mathrm{K}\big(\nu_1+\tfrac{q+m}{2}-i,\,\nu_2+\tfrac{\tilde{q}-m}{2}\big)$$

for $0\le q\le\lambda_1-\lambda_2$, recursively, by (5.2) with

$$\frac{1}{8\pi y_1}\{(\partial_1-2b_1)(\partial_1-2b_2)-(4\pi y_1)^2\}\mathrm{K}(a_1,a_2)(y_1) = (b_1+b_2-a_1-a_2)\mathrm{K}\big(a_1+\tfrac12,a_2-\tfrac12\big)(y_1) + (2\pi)^{-1}(b_1-a_1)(b_2-a_1)\mathrm{K}\big(a_1-\tfrac12,a_2-\tfrac12\big)(y_1)$$

for $a_1,a_2,b_1,b_2\in\mathbf{C}$ and $y_1\in\mathbf{R}_+$. From the above arguments, we obtain the following proposition.

Proposition 5.1. *Let $(\Pi_{(\nu,d)},H_{(\nu,d)})$ be an irreducible principal series representation of G with $\nu=(\nu_1,\nu_2)\in\mathbf{C}^2$, $d=(d_1,d_2)\in\Lambda$. Let $c\in\mathbf{C}^\times$ and $m\in\mathbf{Z}_{\ge0}$. Set $\lambda=(\lambda_1,\lambda_2)=(d_1+m,d_2-m)$. There is a K-homomorphism $\phi^{(c)}_{[\nu,d;m]}\colon V_\lambda\to\mathcal{W}(\Pi_{(\nu,d)},\psi_c)$ such that, for $y_1\in\mathbf{R}_+$ and $0\le q\le\lambda_1-\lambda_2$,*

$$\left(\frac{c\sqrt{-1}}{|c|}\right)^{-\lambda_1+q} y_1^{-1}\phi^{(c)}_{[\nu,d;m]}(v_q^\lambda)(\mathrm{diag}(y_1,1))$$

$$(5.3)\quad = \sum_{i=0}^{\min\{q,m\}}\binom{q}{i}\frac{(-m)_i\big(-\nu_1+\nu_2-\frac{\lambda_1-\lambda_2}{2}\big)_i}{(2\pi)^i(-\lambda_1+\lambda_2)_i} \times\mathrm{K}\big(\nu_1+\tfrac{q+m}{2}-i,\,\nu_2+\tfrac{\tilde{q}-m}{2}\big)(|c|y_1)$$

$$(5.4)\quad = \sum_{i=0}^{\min\{\tilde{q},m\}}\binom{\tilde{q}}{i}\frac{(-m)_i\big(-\nu_2+\nu_1-\frac{\lambda_1-\lambda_2}{2}\big)_i}{(2\pi)^i(-\lambda_1+\lambda_2)_i} \times\mathrm{K}\big(\nu_2+\tfrac{\tilde{q}+m}{2}-i,\,\nu_1+\tfrac{q-m}{2}\big)(|c|y_1)$$

with $\tilde{q}=\lambda_1-\lambda_2-q$.

Here the second expression (5.4) is derived from the formula (7.4.4.1 in [PBM])

$${}_3F_2\left(\begin{matrix}a_1,\ a_2,\ -m\\ b_1,\ b_2\end{matrix};1\right) = \frac{(b_1+b_2-a_1-a_2)_m}{(b_1)_m} \times{}_3F_2\left(\begin{matrix}b_2-a_1,\ b_2-a_2,\ -m\\ b_1+b_2-a_1-a_2,\ b_2\end{matrix};1\right)$$

(m is a non-negative integer)

of the generalized hypergeometric series (the both sides of this equality are rational functions of a_1, a_2, b_1, b_2), and the expression

$$\left(\frac{c\sqrt{-1}}{|c|}\right)^{-\lambda_1+q} y_1^{-1}\phi^{(c)}_{[\nu,d;m]}(v_q^\lambda)(\mathrm{diag}(y_1,1)) = \frac{1}{2\pi\sqrt{-1}}\int_{\alpha-\sqrt{-1}\infty}^{\alpha+\sqrt{-1}\infty}(|c|y_1)^{-2s} \times{}_3F_2\left(\begin{matrix}-\nu_1+\nu_2-\frac{\lambda_1-\lambda_2}{2},\ -q,\ -m\\ 1-s-\nu_1-\frac{q+m}{2},\ -\lambda_1+\lambda_2\end{matrix};1\right) \times\Gamma_{\mathbf{C}}\big(s+\nu_1+\tfrac{q+m}{2}\big)\Gamma_{\mathbf{C}}\big(s+\nu_2+\tfrac{\tilde{q}-m}{2}\big)\,ds$$

obtained from (5.3).

Remark 5.2. From the explicit formulas of $\phi^{(c)}_{[\nu,d;m]}(v_q^\lambda)(\mathrm{diag}(y_1,1))$ in Proposition 5.1, we have

$$\left(\frac{c\sqrt{-1}}{|c|}\right)^{-\lambda_1+q} y_1^{-1}\phi^{(c)}_{[\nu,d;m]}(v_q^\lambda)(\mathrm{diag}(y_1,1)) = \begin{cases}\mathrm{K}\big(\nu_1+\frac{q}{2},\,\nu_2+\frac{\tilde{q}}{2}\big)(|c|y_1) & \text{if } m=0,\\ \mathrm{K}\big(\nu_1+\frac{m}{2},\,\nu_2+\frac{\lambda_1-\lambda_2-m}{2}\big)(|c|y_1) & \text{if } q=0,\\ \mathrm{K}\big(\nu_1+\frac{\lambda_1-\lambda_2-m}{2},\,\nu_2+\frac{m}{2}\big)(|c|y_1) & \text{if } \tilde{q}=0.\end{cases}$$

6. The local zeta integrals for $G\times G$. Let $\mathcal{S}(\mathbf{C}^2)$ be the space of Schwartz functions on $\mathbf{C}^2$. Let $\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}$ be the subspace of $\mathcal{S}(\mathbf{C}^2)$ consisting of all functions f of the form

$$(6.1)\quad f(z_1,z_2)=p(z_1,z_2,\overline{z_1},\overline{z_2})e^{-2\pi(|z_1|^2+|z_2|^2)}$$

for $z_1,z_2\in\mathbf{C}$, with polynomial functions p on $\mathbf{C}^4$. We call functions in $\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}$ standard Schwartz functions on $\mathbf{C}^2$.

Let $(\Pi_{(\nu,d)},H_{(\nu,d)})$ and $(\Pi_{(\nu',d')},H_{(\nu',d')})$ be irreducible principal series representations of G with $\nu=(\nu_1,\nu_2)\in\mathbf{C}^2$, $d=(d_1,d_2)\in\Lambda$, $\nu'=(\nu'_1,\nu'_2)\in\mathbf{C}^2$ and $d'=(d'_1,d'_2)\in\Lambda$. From the Langlands parameters of $\Pi_{(\nu,d)}$ and $\Pi_{(\nu',d')}$, we define the local L-factor $L(s,\Pi_{(\nu,d)}\times\Pi_{(\nu',d')})$ by

$$L(s,\Pi_{(\nu,d)}\times\Pi_{(\nu',d')}) = \prod_{1\le i,j\le 2}\Gamma_{\mathbf{C}}\Big(s+\nu_i+\nu'_j+\tfrac{|d_i+d'_j|}{2}\Big).$$

For $W\in\mathcal{W}(\Pi_{(\nu,d)},\psi_\varepsilon)$, $W'\in\mathcal{W}(\Pi_{(\nu',d')},\psi_{-\varepsilon})$ $(\varepsilon\in\{\pm1\})$ and $f\in\mathcal{S}(\mathbf{C}^2)$, we define the local zeta integral $Z(s,W,W',f)$ for $G\times G$ by

$$Z(s,W,W',f) = \int_{N\backslash G} W(g)W'(g)f((0,1)g)|\det g|^{2s}\,d\dot g,$$

where $d\dot g$ is the right invariant measure on $N\backslash G$. In this article, we normalize $d\dot g$ so that, for any compactly supported continuous function F on $N\backslash G$,

$$\int_{N\backslash G} F(g)\,d\dot g = \int_0^\infty\int_0^\infty\left(\int_K F(yk)\,dk\right)\frac{dy_1}{y_1^3}\frac{dy_2}{y_2}$$

with $y=\mathrm{diag}(y_1y_2,y_2)\in A$ and dk is the Haar measure on K such that $\int_K dk=1$. The local zeta integral $Z(s,W,W',f)$ converges for $\mathrm{Re}(s)\gg 0$.

The group K acts on $\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}$ by

$$(\tau(k)f)(z_1,z_2)=f((z_1,z_2)k)$$

for $k\in K$ and $f\in\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}$. For non-negative integers l, r, let $\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}_{l,r}$ be the subspace of $\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}$ consisting of all functions of the form (6.1) with polynomial functions $p(w_1,w_2,w_3,w_4)$ which are degree l homogeneous with respect to w_1,w_2, and degree r homogeneous with respect to w_3,w_4. Then it is easy to see that $\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}=\bigoplus_{l,r\geq 0}\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}_{l,r}$ and

$$\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}_{l,r}\simeq V_{(l,0)}\otimes_{\mathbf{C}}V_{(0,-r)}. \tag{6.2}$$

For $\mathbf{n}=(n_1,n_2,n_3,n_4)\in(\mathbf{Z}_{\geq 0})^4$, we define a function $f_{\mathbf{n}}=f_{(n_1,n_2,n_3,n_4)}$ on $\mathbf{C}^2$ by

$$f_{\mathbf{n}}(z_1,z_2)=z_1^{n_1}z_2^{n_2}\overline{z_1}^{n_3}\overline{z_2}^{n_4}e^{-2\pi(|z_1|^2+|z_2|^2)}.$$

Then we note that $f_{(n_1,n_2,n_3,n_4)}\in\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}_{n_1+n_2,n_3+n_4}$.

Let $\varepsilon\in\{\pm 1\}$. We take K-homomorphisms $\phi^{(\varepsilon)}_{[\nu,d;m]}$ and $\phi^{(-\varepsilon)}_{[\nu',d';m']}$ as in Proposition 5.1. Let m and m' be non-negative integers. Set

$$\lambda=(\lambda_1,\lambda_2)=(d_1+m,d_2-m),$$
$$\lambda'=(\lambda_1',\lambda_2')=(d_1'+m',d_2'-m').$$

For each $s\in\mathbf{C}$ such that $\mathrm{Re}(s)\gg 0$, we note that

$$v\otimes v'\otimes f\mapsto Z(s,\phi^{(\varepsilon)}_{[\nu,d;m]}(v),\phi^{(-\varepsilon)}_{[\nu',d';m']}(v'),f)$$

defines a K-homomorphism from the tensor product $V_\lambda\otimes_{\mathbf{C}}V_{\lambda'}\otimes_{\mathbf{C}}\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}_{l,r}$ to $V_{(0,0)}=\mathbf{C}$, and this homomorphism vanishes unless

$$\mathrm{Hom}_K(V_\lambda\otimes_{\mathbf{C}}V_{\lambda'}\otimes_{\mathbf{C}}\mathcal{S}(\mathbf{C}^2)^{\mathrm{std}}_{l,r},V_{(0,0)})\neq\{0\}. \tag{6.3}$$

By (6.2) and the decomposition law

$$V_\mu\otimes_{\mathbf{C}}V_{\mu'}\simeq\bigoplus_{i=0}^{\min\{\mu_1-\mu_2,\mu_1'-\mu_2'\}}V_{\mu+\mu'+(-i,i)}$$

of K-modules for $\mu=(\mu_1,\mu_2),\mu'=(\mu_1',\mu_2')\in\Lambda$, we know that (6.3) holds if and only if the non-negative integers m, m', l and r satisfy

$$\lambda_1+\lambda_1'\geq 0\geq\lambda_2+\lambda_2', \tag{6.4}$$

$$r=\lambda_1+\lambda_1'+\lambda_2+\lambda_2'+l, \tag{6.5}$$

$$l\geq\max\{-\lambda_1-\lambda_2',-\lambda_2-\lambda_1'\}. \tag{6.6}$$

The inequality (6.4) implies that $m+m'\geq m_0$ with

$$m_0=\max\{0,-d_1-d_1',d_2+d_2'\}. \tag{6.7}$$

Assume $m=m_0$ and $m'=0$. Then the smallest non-negative integers l, r satisfying (6.5) and (6.6) are given by $l=l_1+l_2$ and $r=r_1+r_2$ with

$$\begin{cases} l_i=-\lambda_i-d_{3-i}',\ r_i=0 & \text{if } \lambda_i+d_{3-i}'\leq 0,\\ l_i=0,\ r_i=\lambda_i+d_{3-i}' & \text{if } \lambda_i+d_{3-i}'\geq 0.\end{cases} \tag{6.8}$$

Theorem 6.1. *Retain the notation. Then*

$$Z(s,W_0,W_0',f_0)=L(s,\Pi_{(\nu,d)}\times\Pi_{(\nu',d')}) \tag{6.9}$$

holds for $s\in\mathbf{C}$ such that $\mathrm{Re}(s)\gg 0$, where

$$W_0=\phi^{(\varepsilon)}_{[\nu,d;m_0]}(v_0^\lambda),\quad W_0'=\phi^{(-\varepsilon)}_{[\nu',d';0]}(v^{d'}_{d_1'-d_2'}),$$

$$f_0=(-1)^{d_2'}\frac{4(\lambda_1+d_1'+l_1+l_2+1)!}{(\lambda_1-\lambda_2)!(d_1'-d_2')!}\times(\lambda_1+d_1'-r_1-r_2)!\,f_{(l_1,l_2,r_1,r_2)}.$$

7. The proof of Theorem 6.1. First, we prepare two lemmas.

Lemma 7.1. (i) *Let $a_1,a_2,b_1,b_2\in\mathbf{C}$. Then for $s\in\mathbf{C}$ such that $\mathrm{Re}(s)\gg 0$, it holds that*

$$\int_0^\infty \mathrm{K}(a_1,a_2)(y_1)\mathrm{K}(b_1,b_2)(y_1)y_1^{2s}\frac{dy_1}{y_1} = \frac{\Gamma_{\mathbf{C}}(s+a_1+b_1)\Gamma_{\mathbf{C}}(s+a_1+b_2)}{\Gamma_{\mathbf{C}}(2s+a_1+a_2+b_1+b_2)}\times\Gamma_{\mathbf{C}}(s+a_2+b_1)\Gamma_{\mathbf{C}}(s+a_2+b_2).$$

(ii) *For $s\in\mathbf{C}$ such that $\mathrm{Re}(s)>0$, it holds that*

$$\int_0^\infty e^{-2\pi t^2}t^{2s}\frac{dt}{t}=\frac{1}{4}\Gamma_{\mathbf{C}}(s).$$

(iii) *For $z_1,z_2\in\mathbf{C}$, $n\in\mathbf{Z}_{\geq 0}$ such that $\mathrm{Re}(z_1)>0$, $\mathrm{Re}(z_2)>n$, it holds that*

$$\sum_{j=0}^n\frac{\Gamma_{\mathbf{C}}(z_1+j)\Gamma_{\mathbf{C}}(z_2-j)}{j!(n-j)!} = \frac{\Gamma_{\mathbf{C}}(z_1)\Gamma_{\mathbf{C}}(z_1+z_2)\Gamma_{\mathbf{C}}(z_2-n)}{n!\Gamma_{\mathbf{C}}(z_1+z_2-n)}.$$

Proof. The statement (i) is derived from Barnes' lemma (§14.52 in [WW]) and the Mellin inversion formula (see for example, §1.5 in [Bu]). The statement (ii) is immediately follows from

Euler's integral form of the Gamma function. The statement (iii) is derived from the formula of the value of the Gaussian hypergeometric series at 1 (§14.11 in [WW]). □

Lemma 7.2. *Let $y=\mathrm{diag}(y_1y_2,y_2)\in A$. For $\lambda=(\lambda_1,\lambda_2)$, $\lambda'=(\lambda_1',\lambda_2')\in\Lambda$, $0\le q\le\lambda_1-\lambda_2$, $0\le q'\le\lambda_1'-\lambda_2'$ and $\mathbf{n}=(n_1,n_2,n_3,n_4)\in(\mathbf{Z}_{\ge0})^4$, the integral*

$$\int_K\langle v_{\tilde q}^{\tilde\lambda},\tau_\lambda(k)v_0^\lambda\rangle\langle v_{\tilde q'}^{\tilde\lambda'},\tau_{\lambda'}(k)v_{\lambda_1'-\lambda_2'}^{\lambda'}\rangle f_{\mathbf{n}}((0,1)yk)dk$$

is equal to

$$\frac{(-1)^{n_1+n_3-q}(q+n_1)!(q'+n_2)!}{(\lambda_1+\lambda_1'+n_1+n_2+1)!}y_2^{n_1+n_2+n_3+n_4}e^{-2\pi y_2^2}$$

if $\lambda_1+\lambda_2'+n_1-n_3=\lambda_2+\lambda_1'+n_2-n_4=\lambda_1+\lambda_1'-q-q'=0$, and is equal to 0 if otherwise. Here we set $\tilde q=\lambda_1-\lambda_2-q$ and $\tilde q'=\lambda_1'-\lambda_2'-q'$.

Proof. By direct computation, we have

$$\langle v_{\tilde q}^{\tilde\lambda},\tau_\lambda(k)v_0^\lambda\rangle=(-1)^{\lambda_1-q}(\det k)^{\lambda_2}k_{11}^{\tilde q}k_{21}^{q},$$
$$\langle v_{\tilde q'}^{\tilde\lambda'},\tau_{\lambda'}(k)v_{\lambda_1'-\lambda_2'}^{\lambda'}\rangle=(-1)^{\lambda_1'-q'}(\det k)^{\lambda_2'}k_{12}^{\tilde q'}k_{22}^{q'},$$
$$f_{\mathbf{n}}((0,1)yk)=(-1)^{n_3}(\det k)^{-n_3-n_4}k_{21}^{n_1}k_{22}^{n_2}k_{12}^{n_3}k_{11}^{n_4}\times y_2^{n_1+n_2+n_3+n_4}e^{-2\pi y_2^2}$$

for $k=\begin{pmatrix}k_{11}&k_{12}\\k_{21}&k_{22}\end{pmatrix}\in K$. Hence, we have

$$\begin{aligned}&\int_K\langle v_{\tilde q}^{\tilde\lambda},\tau_\lambda(k)v_0^\lambda\rangle\langle v_{\tilde q'}^{\tilde\lambda'},\tau_{\lambda'}(k)v_{\lambda_1'-\lambda_2'}^{\lambda'}\rangle f_{\mathbf{n}}((0,1)yk)dk\\&=(-1)^{n_3}y_2^{n_1+n_2+n_3+n_4}e^{-2\pi y_2^2}\\&\quad\times\int_K\langle v_{\tilde q+n_4}^{\tilde\lambda+(n_4,-n_1)},\tau_{\lambda+(n_1,-n_4)}(k)v_0^{\lambda+(n_1,-n_4)}\rangle\\&\quad\times\langle v_{\tilde q'+n_3}^{\tilde\lambda'+(n_3,-n_2)},\tau_{\lambda'+(n_2,-n_3)}(k)v_{\lambda_1'-\lambda_2'+n_2+n_3}^{\lambda'+(n_2,-n_3)}\rangle dk.\end{aligned}$$

Applying Schur's orthogonality relations (Proposition 4.4 in [BD])

$$\int_K\langle w,\tau_\mu(k)v\rangle\langle w',\tau_{\mu'}(k)v'\rangle\,dk=\begin{cases}\dfrac{\langle v,v'\rangle\langle w,w'\rangle}{\mu_1-\mu_2+1}&\text{if }\tilde\mu=\mu',\\0&\text{otherwise}\end{cases}$$

$$\left(\begin{array}{lll}\mu=(\mu_1,\mu_2)\in\Lambda,&w\in V_{\tilde\mu},&v\in V_\mu,\\\mu'=(\mu_1',\mu_2')\in\Lambda,&w'\in V_{\tilde\mu'},&v'\in V_{\mu'}\end{array}\right)$$

to the right-hand side of this equality, we obtain the assertion. □

Let us prove Theorem 6.1. Let $\varepsilon\in\{\pm1\}$. Let $(\Pi_{(\nu,d)},H_{(\nu,d)})$ and $(\Pi_{(\nu',d')},H_{(\nu',d')})$ be irreducible principal series representations of G with $\nu=(\nu_1,\nu_2)\in\mathbf{C}^2$, $d=(d_1,d_2)\in\Lambda$, $\nu'=(\nu_1',\nu_2')\in\mathbf{C}^2$ and $d'=(d_1',d_2')\in\Lambda$. Let m_0 be the integer defined by (6.7), and put $\lambda=(\lambda_1,\lambda_2)=(d_1+m_0,d_2-m_0)$. Let l_1, l_2, r_1 and r_2 be the integers defined by (6.8). In order to simplify the notation, hereafter, we set

$$\phi=\phi_{[\nu,d;m_0]}^{(\varepsilon)},\quad\phi'=\phi_{[\nu',d';0]}^{(-\varepsilon)},\quad f=f_{(l_1,l_2,r_1,r_2)}.$$

For $s\in\mathbf{C}$ such that $\mathrm{Re}(s)\gg0$, we have

$$\begin{aligned}&Z(s,\phi(v_0^\lambda),\phi'(v_{d_1'-d_2'}^{d'}),f)\\&=\int_{N\backslash G}\phi(v_0^\lambda)(g)\phi'(v_{d_1'-d_2'}^{d'})(g)f((0,1)g)|\det g|^{2s}\,d\dot g\\&=\int_0^\infty\int_0^\infty\Bigl(\int_K\phi(\tau_\lambda(k)v_0^\lambda)(y)\phi'(\tau_{d'}(k)v_{d_1'-d_2'}^{d'})(y)\\&\qquad\times f((0,1)yk)\,dk\Bigr)y_1^{2s}y_2^{4s}\frac{dy_1}{y_1^3}\frac{dy_2}{y_2}\end{aligned}$$

with $y=\mathrm{diag}(y_1y_2,y_2)\in A$. Since

$$\tau_\lambda(k)v_0^\lambda=\sum_{q=0}^{\lambda_1-\lambda_2}(-1)^{\lambda_1-q}\binom{\lambda_1-\lambda_2}{q}\langle v_{\tilde q}^{\tilde\lambda},\tau_\lambda(k)v_0^\lambda\rangle v_q^\lambda,$$

$$\tau_{d'}(k)v_{d_1'-d_2'}^{d'}=\sum_{q'=0}^{d_1'-d_2'}(-1)^{d_1'-q'}\binom{d_1'-d_2'}{q'}\times\langle v_{\tilde q'}^{\tilde d'},\tau_{d'}(k)v_{d_1'-d_2'}^{d'}\rangle v_{q'}^{d'}$$

with $\tilde q=\lambda_1-\lambda_2-q$ and $\tilde q'=d_1'-d_2'-q'$, we have

$$\begin{aligned}&Z(s,\phi(v_0^\lambda),\phi'(v_{d_1'-d_2'}^{d'}),f)\\&=\sum_{q=0}^{\lambda_1-\lambda_2}\sum_{q'=0}^{d_1'-d_2'}(-1)^{\lambda_1+d_1'-q-q'}\binom{\lambda_1-\lambda_2}{q}\binom{d_1'-d_2'}{q'}\\&\quad\times\int_0^\infty\int_0^\infty\Bigl(\int_K\langle v_{\tilde q}^{\tilde\lambda},\tau_\lambda(k)v_0^\lambda\rangle\,\langle v_{\tilde q'}^{\tilde d'},\tau_{d'}(k)v_{d_1'-d_2'}^{d'}\rangle\\&\quad\times f((0,1)yk)\,dk\Bigr)\phi(v_q^\lambda)(y)\phi'(v_{q'}^{d'})(y)y_1^{2s}y_2^{4s}\frac{dy_1}{y_1^3}\frac{dy_2}{y_2}.\end{aligned}$$

Applying Lemma 7.2 and Lemma 7.1 (ii), successively, we find that

$$\begin{aligned}&Z(s,\phi(v_0^\lambda),\phi'(v_{d_1'-d_2'}^{d'}),f)\\&=\frac{\Gamma_{\mathbf{C}}\bigl(2s+\nu_1+\nu_2+\nu_1'+\nu_2'+\frac{l_1+l_2+r_1+r_2}{2}\bigr)}{4(\lambda_1+d_1'+l_1+l_2+1)!}\\&\quad\times\sum_{q=r_1}^{\lambda_1+d_1'-r_2}\frac{(-1)^{\lambda_1+d_2'-q}(\lambda_1-\lambda_2)!(d_1'-d_2')!}{(q-r_1)!(\lambda_1+d_1'-r_2-q)!}\\&\quad\times\int_0^\infty\phi(v_q^\lambda)(y')\phi'(v_{\lambda_1+d_1'-q}^{d'})(y')y_1^{2s}\frac{dy_1}{y_1^3}\end{aligned}$$

with $y' = \mathrm{diag}(y_1, 1) \in A$.

Let us consider the case of $m_0 = 0$, that is, $\lambda = d$. By Remark 5.2, Lemma 7.1 (i) and the above equality, we have

$$Z(s, \phi(v_0^{\lambda}), \phi'(v_{d_1'-d_2'}^{d'}), f)$$
$$= (-1)^{d_2'} \frac{\Gamma_{\mathbf{C}}\left(2s+\nu_1+\nu_2+\nu_1'+\nu_2'+\frac{l_1+l_2+r_1+r_2}{2}\right)}{4(d_1+d_1'+l_1+l_2+1)!}$$
$$\times \frac{(d_1-d_2)!(d_1'-d_2')!\Gamma_{\mathbf{C}}\left(s+\nu_1+\nu_1'+\frac{d_1+d_1'}{2}\right)}{\Gamma_{\mathbf{C}}\left(2s+\nu_1+\nu_2+\nu_1'+\nu_2'+\frac{d_1-d_2+d_1'-d_2'}{2}\right)}$$
$$\times \Gamma_{\mathbf{C}}\left(s+\nu_2+\nu_2'+\frac{-d_2-d_2'}{2}\right)$$
$$\times \sum_{q=r_1}^{d_1+d_1'-r_2} \frac{\Gamma_{\mathbf{C}}\left(s+\nu_1+\nu_2'+\frac{-d_1-d_2'}{2}+q\right)}{(q-r_1)!(d_1+d_1'-r_2-q)!}$$
$$\times \Gamma_{\mathbf{C}}\left(s+\nu_2+\nu_1'+\frac{2d_1-d_2+d_1'}{2}-q\right).$$

Replacing $q \to j + r_1$ and applying Lemma 7.1 (iii), we obtain (6.9) in this case.

Let us consider the case of $m_0 = -d_1 - d_1'$. In this case, we note that

$$l_1 = -\lambda_1 - d_2', \quad l_2 = -\lambda_2 - d_1', \quad r_1 = r_2 = 0$$

and $\lambda_1 = -d_1'$. Hence, we have

$$Z(s, \phi(v_0^{\lambda}), \phi'(v_{d_1'-d_2'}^{d'}), f) = \frac{(\lambda_1-\lambda_2)!(d_1'-d_2')!}{4(-\lambda_2-d_2'+1)!}$$
$$\times (-1)^{\lambda_1+d_2'}\Gamma_{\mathbf{C}}\left(2s+\nu_1+\nu_2+\nu_1'+\nu_2'+\frac{-\lambda_2-d_2'}{2}\right)$$
$$\times \int_0^{\infty} \phi(v_0^{\lambda})(y')\phi'(v_0^{d'})(y')y_1^{2s}\,\frac{dy_1}{y_1^3}.$$

By Remark 5.2 and Lemma 7.1 (i), we obtain (6.9) in this case.

Let us consider the case of $m_0 = d_2 + d_2'$. In this case, we note that

$$l_1 = l_2 = 0, \quad r_1 = \lambda_1 + d_2', \quad r_2 = \lambda_2 + d_1'$$

and $\lambda_2 = -d_2'$. Hence, we have

$$Z(s, \phi(v_0^{\lambda}), \phi'(v_{d_1'-d_2'}^{d'}), f) = \frac{(\lambda_1-\lambda_2)!(d_1'-d_2')!}{4(\lambda_1+d_1'+1)!}$$
$$\times \Gamma_{\mathbf{C}}\left(2s+\nu_1+\nu_2+\nu_1'+\nu_2'+\frac{\lambda_1+d_1'}{2}\right)$$
$$\times \int_0^{\infty} \phi(v_{\lambda_1-\lambda_2}^{\lambda})(y')\phi'(v_{d_1'-d_2'}^{d'})(y')y_1^{2s}\,\frac{dy_1}{y_1^3}.$$

By Remark 5.2 and Lemma 7.1 (i), we obtain (6.9) in this case.

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