Mod 3 Chern classes and generators

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Abstract: We show the non-triviality of the mod 3 Chern class of degree 324 of the adjoint representation of the exceptional Lie group E_8.

Key words: Chern class; exceptional Lie group; complex representation.

1. Introduction. Let p be a prime number. In the study of mod p cohomology of the classifying space of a simply-connected, simple, compact connected Lie group G, Stiefel-Whitney classes and Chern classes play an important role. For example, the mod 2 cohomology of the classifying space of the exceptional Lie group E_6 is generated by two generators of degree 4 and of degree 32 as an algebra over the mod 2 Steenrod algebra, and Toda pointed out that the generator of degree 32 could be given as the Chern class of an irreducible representation $\rho_6 : E_6 \to SU(27)$ in [12]. Mimura and Nishimoto [8], Kono [7] and the author [5] proved that Stiefel-Whitney classes $w_{16}(\rho_4)$, $w_{128}(\rho_8)$ and Chern classes $c_{16}(\rho_6)$, $c_{32}(\rho_7)$ are algebra generators of the mod 2 cohomology of the classifying space BG for $G = F_4, E_8, E_6, E_7$, where ρ_4, ρ_8 are real irreducible representations of dimension 26, 248, and ρ_6, ρ_7 are complex irreducible representations of dimension 27, 56, respectively. For $G = F_4, E_6, E_7$, the mod 2 cohomology of the classifying space is generated by two elements, that is, one is the element of degree 4 and the other is $w_{16}(\rho_4)$, $c_{16}(\rho_6)$, $c_{32}(\rho_7)$, respectively. In the case $G = E_8$ and $p = 2, 3$, the mod p cohomology of the classifying space is not yet computed. Since the non-triviality of the Stiefel-Whitney class $w_{128}(\rho_8)$ tells us that the differentials in the spectral sequence vanish on the corresponding element, we expect that it not only gives us a nice description for the generator but also helps us in the computation of the mod 2 cohomology of BE_8.

This paper is the sequel of [5] in the sense that we consider the mod 3 analogue of the above results. In particular, we prove the non-triviality of the mod 3 Chern class $c_{162}(\rho_8)$ of degree 324. For an odd prime number p and for a simply-connected, simple, compact connected Lie group, the Rothenberg-Steenrod spectral sequence collapses at the E_2-level and so at least additively the mod p cohomology is isomorphic to the cotorsion product of the mod p cohomology of G except for the case $p = 3$, $G = E_8$. In [6], we proved that there exists an algebra generator of degree greater than or equal to 324 in the mod 3 cohomology ring of BE_8. On the other hand, in [9,10], Mimura and Sambe proved that the E_2-term of the Rothenberg-Steenrod spectral sequence is generated as an algebra by elements of degree less than or equal to 168. Hence the spectral sequence must not collapse at the E_2-level. We expect that, in the mod 3 cohomology, the mod 3 Chern class $c_{162}(\rho_8)$ plays an important role similar to that of the Stiefel-Whitney class $w_{128}(\rho_8)$ in the mod 2 cohomology.

Now, we state our main theorem. Let T be a fixed maximal torus of the exceptional Lie group F_4. We choose a maximal non-toral elementary abelian 3-subgroup A of F_4 so that $T \cap A$ is nontrivial. We refer the reader to the paper of Andersen, Grodal, Møller and Viruel [2, Section 8] for the details of non-toral elementary abelian p-subgroups of exceptional Lie groups and their Weyl groups. Let μ be a subgroup of $T \cap A$ of order 3. The group μ is the cyclic group of order 3. We consider the following diagram of inclusion maps.

$$\begin{array}{ccccc} T & \longrightarrow & F_4 & \longrightarrow & G \\ \uparrow & & \uparrow & & \\ \mu & \longrightarrow & A. & & \end{array}$$

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We denote by $\iota : \mu \to G$ the inclusion map of μ to $G = F_4, E_6, E_7, E_8$. The mod 3 cohomology $H^*(B\mu; \mathbf{Z}/3)$ of the classifying space $B\mu$ is isomorphic to

$$\mathbf{Z}/3[u_2] \otimes \Lambda(u_1),$$

where u_2 is the image of the mod 3 Bockstein homomorphism of a generator u_1 of $H^1(B\mu; \mathbf{Z}/3) = \mathbf{Z}/3$. From now on, we consider complex representations only and we denote complexifications of real representations ρ_4, ρ_8 by the same symbols ρ_4, ρ_8, respectively.

Theorem 1.1. *The total Chern classes $c(\iota^*(\rho_i))$ of the above induced representations $\iota^*(\rho_i)$, where $i = 4, 6, 7, 8$, are as follows*:

$$\begin{aligned} c(\iota^*(\rho_4)) &= 1 - u_2^{18}, \\ c(\iota^*(\rho_6)) &= 1 - u_2^{18}, \\ c(\iota^*(\rho_7)) &= (1 - u_2^{18})^2 = 1 + u_2^{18} + u_2^{36}, \\ c(\iota^*(\rho_8)) &= (1 - u_2^{18})^9 = 1 - u_2^{162}. \end{aligned}$$

As a corollary of this theorem, using Lemma 3.1, we have the following

Corollary 1.2. *The Chern classes $c_{18}(\rho_4)$, $c_{18}(\rho_6)$, $c_{18}(\rho_7)$, $c_{162}(\rho_8)$ are nontrivial in $H^*(BF_4; \mathbf{Z}/3)$, $H^*(BE_6; \mathbf{Z}/3)$, $H^*(BE_7; \mathbf{Z}/3)$, $H^*(BE_8; \mathbf{Z}/3)$, respectively. Moreover, the Chern classes $c_{18}(\rho_4)$, $c_{18}(\rho_6)$, $c_{18}(\rho_7)$ are indecomposable, so that they are algebra generators.*

This paper is organized as follows: In Section 2, we recall complex representations $\rho_4, \rho_6, \rho_7, \rho_8$ and their restrictions to Spin(8). In Section 3, we prove Theorem 1.1. We end this paper by showing the non-triviality of the mod 5 Chern class $c_{100}(\rho_8)$ of BE_8 in the appendix.

2. Complex representations.

In this section, we consider complex representations $\rho_4, \rho_6, \rho_7, \rho_8$ in Theorem 1.1 and the complexification ρ_4' of the adjoint representation of F_4 and their restrictions to Spin(8). For the details of representation rings of Spin groups and cyclic groups, we refer the reader to standard textbooks on representation theory, e.g. Husemoller's book [4] and/or the book of Bröcker and tom Dieck [3].

First, we recall the complex representation ring of Spin$(2n)$. Let us consider the following pull-back diagram.

$$\begin{array}{ccc} \tilde{T}^n & \xrightarrow{\tilde{k}_n} & \mathrm{Spin}(2n) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T^n & \xrightarrow{k_n} & SO(2n), \end{array}$$

where $SO(2n)$ is the special orthogonal group, $\pi : \mathrm{Spin}(2n) \to SO(2n)$ is the universal covering, T^n is the maximal torus of $SO(2n)$ consisting of matrices of the form

$$\left(\begin{array}{c|c|c} \begin{matrix}\cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1\end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix}\cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n\end{matrix} \end{array}\right),$$

k_n is the inclusion map and $\tilde{T}^n$ is a maximal torus of Spin$(2n)$. The complex representation ring of

$$S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

is $R(S^1) = \mathbf{Z}[z, z^{-1}]$ where z is represented by the canonical complex line bundle. Considering T^n as the product of n copies of S^1's, let $p_i : T^n \to S^1$ be the projection to the i-th factor. We denote by z_i the element $p_i^*(z)$, $\pi^*(p_i^*(z))$ in $R(T^n)$, $R(\tilde{T}^n)$, respectively, so that $\pi^*(z_i) = z_i$. Then, we have

$$\begin{aligned} R(T^n) &= \mathbf{Z}[z_1, \dots, z_n, (z_1 \cdots z_n)^{-1}], \\ R(\tilde{T}^n) &= \mathbf{Z}[z_1, \dots, z_n, (z_1 \cdots z_n)^{-1/2}] \end{aligned}$$

and the complex representation ring of Spin$(2n)$ is

$$\mathbf{Z}[\lambda_1, \dots, \lambda_{n-1}, \Delta^+, \Delta^-]$$

where

$$\begin{aligned} \tilde{k}_n^*(\lambda_1) &= \sum_{i=1}^n (z_i + z_i^{-1}), \\ \tilde{k}_n^*(\lambda_2) &= \sum_{1 \le i < j \le n} (z_i + z_i^{-1})(z_j + z_j^{-1}), \\ \tilde{k}_n^*(\Delta^+) &= \sum_{\varepsilon_1 \cdots \varepsilon_n = 1} (z_1^{\varepsilon_1} \cdots z_n^{\varepsilon_n})^{1/2}, \\ \tilde{k}_n^*(\Delta^-) &= \sum_{\varepsilon_1 \cdots \varepsilon_n = -1} (z_1^{\varepsilon_1} \cdots z_n^{\varepsilon_n})^{1/2}, \end{aligned}$$

and $\varepsilon_i \in \{\pm 1\}$. For the sake of notational simplicity, from now on, we write Δ for $\Delta^+ + \Delta^-$. Let $i : \mu \to S^1$ be the inclusion map. We denote by z the generator $i^*(z)$ of $R(\mu)$. Then, it is also known that $R(\mu) = \mathbf{Z}[z]/(z^3)$.

Next, we recall complex representations $\rho_4, \rho_6, \rho_7, \rho_8$ of dimension $26, 27, 56, 248$ in Section 1 and the complexification ρ_4' of the adjoint representation of F_4. We consider the following commutative diagram.

$$\begin{array}{ccccccc}
\mathrm{Spin}(8) & \xrightarrow{i_8} & \mathrm{Spin}(10) & \xrightarrow{i_{10}} & \mathrm{Spin}(12) & \xrightarrow{i_{14}\circ i_{12}} & \mathrm{Spin}(16) \\
\downarrow{\scriptstyle j_8} & & \downarrow{\scriptstyle j_{10}} & & \downarrow{\scriptstyle j_{12}} & & \downarrow{\scriptstyle j_{16}} \\
F_4 & \xrightarrow{i_4} & E_6 & \xrightarrow{i_6} & E_7 & \xrightarrow{i_7} & E_8,
\end{array}$$

where

$$i_{2n-2} : \mathrm{Spin}(2n-2) \to \mathrm{Spin}(2n)$$

is the obvious inclusion map. For ρ_4, ρ_4', we refer the reader to Yokota's paper [14]. For ρ_6, ρ_7, we refer the reader to Adams' book [1, Corollaries 8.3, 8.2]. For E_8, from the construction of E_8 in Adams [1, Section 7] and the fact that the adjoint representation of $\mathrm{Spin}(2n)$ is the second exterior power of the standard representation, we have the following proposition.

Proposition 2.1. *We have*

$$\begin{aligned}
j_8^*(\rho_4) &= 2 + \lambda_1 + \Delta, \\
j_8^*(\rho_4') &= 4 + \lambda_1 + \Delta + \lambda_2, \\
j_{10}^*(\rho_6) &= 1 + \lambda_1 + \Delta^+, \\
j_{12}^*(\rho_7) &= 2\lambda_1 + \Delta^-, \\
j_{16}^*(\rho_8) &= 8 + \lambda_2 + \Delta^+,
\end{aligned}$$

in $R(\mathrm{Spin}(8))$, $R(\mathrm{Spin}(8))$, $R(\mathrm{Spin}(10))$, $R(\mathrm{Spin}(12))$, $R(\mathrm{Spin}(16))$*, respectively.*

Since the induced homomorphism i_{2n-2}^* maps $\lambda_1, \lambda_2, \Delta^+, \Delta^-, \Delta$ to $2+\lambda_1, 2\lambda_1+\lambda_2, \Delta, \Delta, 2\Delta$, respectively, we have the following proposition.

Proposition 2.2. *For* $G = F_4, E_6, E_7, E_8$*, let* $j : \mathrm{Spin}(8) \to G$ *be the inclusion map. In* $R(\mathrm{Spin}(8))$*, we have*

$$\begin{aligned}
j^*(\rho_4) &= 2 + \lambda_1 + \Delta, \\
j^*(\rho_6) &= 3 + \lambda_1 + \Delta, \\
j^*(\rho_7) &= 8 + 2\lambda_1 + 2\Delta, \\
j^*(\rho_8) &= 32 + 8\lambda_1 + 8\Delta + \lambda_2.
\end{aligned}$$

3. Mod 3 Chern classes. In this section, we prove Theorem 1.1. We consider the following diagram of inclusion maps.

$$\begin{array}{ccccc}
\tilde{T}^4 & \xrightarrow{\tilde{k}_4} & \mathrm{Spin}(8) & \xrightarrow{j_8} & F_4 \\
\uparrow{\scriptstyle \iota_0} & & & & \uparrow \\
\mu & & \xrightarrow{\iota_1} & & A.
\end{array}$$

The maximal torus $\tilde{T}^4$ of $\mathrm{Spin}(8)$ is the maximal torus T of F_4 we mentioned in Section 1. By abuse of notation, we denote both the inclusion map of μ to $\tilde{T}^4$ and its composition with $\tilde{k}_4$ by the same symbol ι_0. Let $\sqrt{0}$ be the nilradical of $H^*(BA; \mathbf{Z}/3)$ and $H^*(B\mu; \mathbf{Z}/3)$, so that we have the induced homomorphism

$$\iota_1^* : H^*(BA; \mathbf{Z}/3)/\sqrt{0} \to H^*(B\mu; \mathbf{Z}/3)/\sqrt{0} = \mathbf{Z}/3[u_2].$$

Lemma 3.1. *The image of the induced homomorphism*

$$\iota^* : H^*(BF_4; \mathbf{Z}/3) \to H^*(B\mu; \mathbf{Z}/3)/\sqrt{0}$$

is in $\mathbf{Z}/3[u_2^{18}]$*, i.e.* $\mathrm{Im}\,\iota^* \subset \mathbf{Z}/3[u_2^{18}] \subset \mathbf{Z}/3[u_2]$.

Proof. It is well-known that the Weyl group $W(A) = N(A)/C(A)$ of A in F_4 is isomorphic to the special linear group $SL_3(\mathbf{Z}/3)$. See the paper of Andersen, Grodal, Møller and Viruel [2, Section 8]. Moreover, $H^*(BA; \mathbf{Z}/3)/\sqrt{0}$ is a polynomial algebra with 3 variables of degree 2 and $SL_3(\mathbf{Z}/3)$ acts in the usual manner. The ring of invariants is also a polynomial algebra

$$(H^*(BA; \mathbf{Z}/3)/\sqrt{0})^{W(A)} = \mathbf{Z}/3[e_3, c_{3,1}, c_{3,2}].$$

The invariants $e_3^2 = c_{3,0}, c_{3,1}, c_{3,2}$ are known as Dickson invariants and their degrees are $52, 48, 36$, respectively. Moreover, the induced homomorphism ι_1^* maps $c_{3,0}, c_{3,1}, c_{3,2}$ to $0, 0, u_2^{18}$, respectively. See Wilkerson's paper [13, Corollary 1.4] for the details. Since the induced homomorphism ι^* factors through

$$(H^*(BA; \mathbf{Z}/3)/\sqrt{0})^{W(A)} \to H^*(B\mu; \mathbf{Z}/3)/\sqrt{0},$$

the lemma follows. □

Next, we compute the total Chern class $c(\iota_0^*(\lambda_1 + \Delta))$.

Proposition 3.2. *The total Chern class* $c(\iota_0^*(\lambda_1 + \Delta))$ *is equal to* $1 - u_2^{18}$.

Proof. Since $\dim(\lambda_1 + \Delta) = 24$, and since $c(\iota_0^*(\lambda_1 + \Delta)) = c(\iota^*(\rho_4)) \in \mathbf{Z}/3[u_2^{18}]$ by Lemma 3.1, $c(\iota_0^*(\lambda_1 + \Delta))$ is equal to $1 + \alpha u_2^{18}$ for some $\alpha \in \mathbf{Z}/3$. On the other hand, ι_0^* maps z_i to z^{α_i} for some $\alpha_i \in \mathbf{Z}/3$ and, since ι_0 is the inclusion map, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (0, 0, 0, 0)$. So,

$$c(\iota_0^*(\lambda_1)) = \prod_{i=1}^{4}(1-\alpha_i^2u_2^2)$$

and $\alpha_i \neq 0$ for some i. Hence, $c(\iota_0^*(\lambda_1))$ is divisible by $1-u_2^2$. Therefore,

$$c(\iota_0^*(\lambda_1+\Delta)) = c(\iota_0^*(\lambda_1))c(\iota_0^*(\Delta))$$

is also divisible by $1-u_2^2$ and so $\alpha = -1$ in $\mathbf{Z}/3$. □

Next, we compute the total Chern class $c(\iota_0^*(\lambda_2))$.

Proposition 3.3. *The total Chern class* $c(\iota_0^*(\lambda_2))$ *is equal to* $1-u_2^{18}$.

Proof. As in the proof of the previous proposition, assume that $\iota_0^*(z_i) = z^{\alpha_i}$. Let

$$f_{ij} = (1-(\alpha_i+\alpha_j)u_2)(1-(\alpha_i-\alpha_j)u_2)$$
$$(1-(-\alpha_i+\alpha_j)u_2)(1-(-\alpha_i-\alpha_j)u_2).$$

Then,

$$c(\iota_0^*(\lambda_2)) = \prod_{1\leq i<j\leq 4} f_{ij}$$

and

$$f_{ij} = 1-2(\alpha_i^2+\alpha_j^2)u_2^2+(\alpha_i^2-\alpha_j^2)^2u_2^4.$$

For $(\alpha_i^2,\alpha_j^2) = (1,1)$, we have

$$f_{ij} = 1-u_2^2.$$

For $(\alpha_i^2,\alpha_j^2) = (1,0)$ or $(0,1)$, we have

$$f_{ij} = 1-2u_2^2+u_2^4 = (1-u_2^2)^2.$$

Since $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \neq (0,0,0,0)$, there exists (i,j) such that $(\alpha_i,\alpha_j) \neq (0,0)$. Hence the total Chern class $c(\iota_0^*(\lambda_2))$ is not trivial and it is divisible by $1-u_2^2$.

Let us consider the total Chern class $c(\iota^*(\rho_4'))$. By Lemma 3.1, it is in $\mathbf{Z}/3[u_2^{18}]$ and by Proposition 3.2, we have

$$c(\iota^*(\rho_4')) = c(\iota_0^*(\lambda_2))c(\iota_0^*(\lambda_1+\Delta))$$
$$= c(\iota_0^*(\lambda_2))(1-u_2^{18}).$$

So, $c(\iota_0^*(\lambda_2))$ is also in $\mathbf{Z}/3[u_2^{18}]$. Since $\dim\lambda_2 = 24$, $c(\iota_0^*(\lambda_2)) = 1+\alpha u_2^{18}$ for some $\alpha \in \mathbf{Z}/3$. Since $c(\iota_0^*(\lambda_2))$ is divisible by $1-u_2^2$, $\alpha = -1$ as in the proof of the previous proposition. □

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. Using Propositions 2.1, 2.2 and using Propositions 3.2, 3.3 above, we have

$$c(\iota^*(\rho_4)) = c(\iota_0^*(\lambda_1+\Delta)) = 1-u_2^{18},$$
$$c(\iota^*(\rho_6)) = c(\iota_0^*(\lambda_1+\Delta)) = 1-u_2^{18},$$
$$c(\iota^*(\rho_7)) = c(\iota_0^*(\lambda_1+\Delta))^2 = (1-u_2^{18})^2,$$
$$c(\iota^*(\rho_8)) = c(\iota_0^*(\lambda_1+\Delta))^8c(\iota_0^*(\lambda_2))$$
$$= (1-u_2^{18})^9.$$

□

A. Mod 5 Chern classes. Let p be an odd prime number. Let G be a simply-connected, simple, compact connected Lie group. If the integral homology of G has no p-torsion, then the mod p cohomology ring of its classifying space is a polynomial algebra and it is well-known. See, for example, the book of Mimura and Toda [11]. The integral homology of G has p-torsion if and only if (G,p) is one of $(F_4,3),(E_6,3),(E_7,3),(E_8,3)$ and $(E_8,5)$. We dealt with the cases for $p=3$ in this paper. For completeness, in this appendix, we deal with the remaining case, $p=5$, $G=E_8$, that is, we prove the non-triviality of the mod 5 Chern class $c_{100}(\rho_8)$ of the complexification of the adjoint representation ρ_8 of the exceptional Lie group E_8.

The mod 5 analogue of Corollary 1.2 is as follows:

Theorem A.1. *The mod* 5 *Chern class* $c_{100}(\rho_8)$ *is non-trivial. Moreover, the mod* 5 *Chern class* $c_{100}(\rho_8)$ *is indecomposable in* $H^*(BE_8;\mathbf{Z}/5)$.

To prove this theorem, we need the mod 5 analogue of Lemma 3.1. As in the case $p=3$, $G=F_4$, there exists a non-toral maximal elementary abelian 5-subgroup of rank 3 in the exceptional Lie group E_8. We choose the maximal torus T of E_8. If necessary, by replacing A by its conjugate, we may assume that $A\cap T$ is non-trivial. We choose a subgroup μ of $A\cap T$ of order 5. Indeed, it is the cyclic group of order 5. We denote by $\iota:\mu\to E_8$ the inclusion map. The mod 5 cohomology of $B\mu$ is

$$H^*(B\mu;\mathbf{Z}/5) = \mathbf{Z}/5[u_2]\otimes\Lambda(u_1),$$

where u_1 is a generator of $H^1(B\mu;\mathbf{Z}/5) = \mathbf{Z}/5$ and u_2 is its image by the mod 5 Bockstein homomorphism. As in the previous section, we denote the nilradical by $\sqrt{0}$ and we denote the inclusion map of μ to A by $\iota_1:\mu\to A$.

Lemma A.2. *The image of the induced homomorphism*

$$\iota^*: H^*(BE_8;\mathbf{Z}/5)\to H^*(B\mu;\mathbf{Z}/5)/\sqrt{0}$$

is in $\mathbf{Z}/5[u_2^{100}]\subset H^*(B\mu;\mathbf{Z}/5)/\sqrt{0}$.

Proof. Since the induced homomorphism ι^* factors through

$$\iota_1^* : (H^*(BA; \mathbf{Z}/5)/\sqrt{0})^{W(A)} \to H^*(B\mu; \mathbf{Z}/5)/\sqrt{0},$$

all we need to do is to recall the fact that the Weyl group $W(A)$ of A in E_8 is $SL_3(\mathbf{Z}/5)$, that

$$(H^*(BA; \mathbf{Z}/5)/\sqrt{0})^{W(A)} = \mathbf{Z}/5[e_3, c_{3,2}, c_{3,1}]$$

and that the above induced homomorphism ι_1^* maps $e_3, c_{3,1}, c_{3,2}$ to $0, 0, u_2^{100}$, respectively. We find these facts in [2, Section 8] and in [13, Corollary 1.4]. □

To compute $\iota^*(\rho_8)$, we need the following commutative diagram similar to the diagram in Section 3. However, in this case, the map j_{16} : Spin(16) $\to E_8$ is not injective.

$$\begin{array}{ccccc} \tilde{T}^8 & \xrightarrow{\tilde{k}_8} & \mathrm{Spin}(16) & \xrightarrow{j_{16}} & E_8 \\ \uparrow{\scriptstyle \iota_0} & & & & \uparrow \\ \mu & & \xrightarrow{\iota_1} & & A. \end{array}$$

We choose the maximal torus T of E_8 so that $j_{16}(\tilde{T}^8) = T$. Then, since $\tilde{T}^8 \to T$ is a double cover and since μ is of order 5, there exists a map ι_0 : $\mu \to \tilde{T}^8$ such that the above diagram commutes.

We use the following propositions to prove Theorem A.1.

Proposition A.3. *The total mod* 5 *Chern class of* $\iota_0^*(\lambda_2)$ *is a product of copies of* $1 - u_2^2$ *and* $1 + u_2^2$. *Moreover, it is non-trivial.*

Proof. Let

$$f_{ij} = (1 - (\alpha_i + \alpha_j)u_2)(1 - (-\alpha_i + \alpha_j)u_2)$$
$$(1 - (\alpha_i - \alpha_j)u_2)(1 - (-\alpha_i - \alpha_j)u_2).$$

Then, we have

$$c(\iota_0^*(\lambda_2)) = \prod_{1 \le i < j \le 8} f_{ij}$$

and

$$f_{ij} = 1 - 2(\alpha_i^2 + \alpha_j^2)u_2^2 + (\alpha_i^2 - \alpha_j^2)^2 u_2^4.$$

In $\mathbf{Z}/5$, $\alpha_i^2 = 0$ or ± 1. So,

$$\begin{aligned} f_{ij} &= 1 + u_2^2 && \text{for } (\alpha_i^2, \alpha_j^2) = (1, 1), \\ f_{ij} &= 1 - u_2^2 && \text{for } (\alpha_i^2, \alpha_j^2) = (-1, -1), \\ f_{ij} &= (1 - u_2^2)^2 && \text{for } (\alpha_i^2, \alpha_j^2) = (1, 0), (0, 1), \\ f_{ij} &= (1 + u_2^2)^2 && \text{for } (\alpha_i^2, \alpha_j^2) = (-1, 0), (0, -1), \\ f_{ij} &= 1 && \text{for } (\alpha_i^2, \alpha_j^2) = (0, 0). \end{aligned}$$

Since μ is a non-trivial subgroup of $\tilde{T}^8$, α_i is non-zero for some i. So, the total Chern class is not equal to 1 and so we have the proposition. □

Proposition A.4. *The total mod* 5 *Chern class of* $\iota_0^*(\Delta^+)$ *is also a product of copies of* $1 - u_2^2$ *and* $1 + u_2^2$.

Proof. Suppose that $i_0^* : R(\mathrm{Spin}(16)) \to R(\mu)$ maps $(z_1^{\varepsilon_1} \cdots z_8^{\varepsilon_8})^{1/2}$ to $z^{\alpha_{\varepsilon_1 \dots \varepsilon_8}}$. Then, it maps $(z_1^{\varepsilon_1'} \cdots z_8^{\varepsilon_8'})^{1/2}$ to $z^{-\alpha_{\varepsilon_1 \dots \varepsilon_8}}$, where $\varepsilon_i' = -\varepsilon_i$, and we have

$$c(\iota_0^*(\Delta^+)) = \prod_{\varepsilon_1 = 1, \varepsilon_1\varepsilon_2 \cdots \varepsilon_8 = 1} (1 - \alpha^2_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_8} u_2^2).$$

Since $\alpha^2_{\varepsilon_1 \dots \varepsilon_8} = 0$ or ± 1, we have the desired result. □

Now we complete the proof of Theorem A.1.

Proof of Theorem A.1. By Propositions A.3, A.4, the total Chern class $c(\iota^*(\rho_8))$ is a product of copies of $1 - u_2^2$ and $1 + u_2^2$ and it is non-trivial. On the other hand, by Lemma A.2, since $\dim(\lambda_2 + \Delta^+) = 240$,

$$c(\iota^*(\rho_8)) = 1 + \alpha u_2^{100} + \beta u_2^{200}$$

for some $\alpha, \beta \in \mathbf{Z}/5$ and $(\alpha, \beta) \neq (0, 0)$. Since it is divisible by $1 - u^2$ or $1 + u_2^2$, we have $1 + \alpha + \beta = 0$ in $\mathbf{Z}/5$ and

$$\begin{aligned} c(\iota^*(\rho_8)) &= 1 + (-\beta - 1)u_2^{100} + \beta u_2^{200} \\ &= (1 - u_2^{100})(1 - \beta u_2^{100}). \end{aligned}$$

Since it is a product of copies of $1 - u_2^2$ and $1 + u_2^2$, $1 + \beta u_2^{100}$ is also divisible by $1 - u_2^2$ or $1 + u_2^2$ if $\beta \neq 0$. So, $\beta = 0$ or -1 and we have that $c(\iota^*(\rho_8))$ is equal to $1 - u_2^{100}$ or $(1 - u_2^{100})^2$. In particular, $c_{100}(\rho_8) = -u_2^{100}$ or $-2u_2^{100}$ and by Lemma A.2, it is indecomposable in $H^*(BE_8; \mathbf{Z}/5)$. □

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