Taylor series for the reciprocal gamma function and multiple zeta values

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Abstract: We give a purely algebraic proof of a formula for Taylor coefficients of the reciprocal gamma function. The formula expresses each coefficient in terms of multiple zeta values. Our proof uses Hoffman's harmonic algebra of multiple zeta values.

Key words: Multiple zeta value; gamma function.

1. Introduction. In [1], the following formula for the Taylor series of the function $\exp\left(\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\zeta(n)x^n\right)$ is given:

$$(1)\quad \exp\left(\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\zeta(n)x^n\right) = 1+\sum_{k=2}^{\infty}(-1)^k\sum_{\substack{k_1+\cdots+k_r=k\\ r\geq 1,\forall k_i\geq 2}}(-1)^r\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!} \times\zeta(k_1,\ldots,k_r)x^k.$$

Here, $\zeta(k_1,\ldots,k_r)$ is the multiple zeta value (MZV) defined by

$$\zeta(k_1,\ldots,k_r)=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

for each index set $\mathbf{k}=(k_1,\ldots,k_r)$ of positive integers k_i, with the last entry $k_r>1$ for convergence. We note that the function on the left-hand side of (1) is equal to $e^{-\gamma x}/\Gamma(1+x)$, where γ is Euler's constant and $\Gamma(x)$ is the gamma function. In [3] and [4], this function plays an important role in the theory of regularization of MZVs. The formula (1) also gives an explicit formula for the Taylor series for the reciprocal gamma function:

$$\frac{1}{\Gamma(x)} = x+\gamma x^2+\sum_{n=2}^{\infty}\left(\frac{\gamma^n}{n!}+\sum_{k=0}^{n-2}\frac{(-1)^{n-k}\gamma^k}{k!}\times\sum_{\substack{k_1+\cdots+k_r=n-k\\ r\geq 1,\forall k_i\geq 2}}(-1)^r\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}\zeta(k_1,\ldots,k_r)\right)x^{n+1}.$$

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Arakawa and Kaneko proved (1) by using the Weierstrass infinite product of the gamma function. The aim of this paper is to give an alternative, purely algebraic proof of the formula in the setting of abstract algebra of MZVs.

We recall the algebraic setup of MZVs that was introduced by Hoffman [2]. We work with indices directly, rather than with non-commutative polynomials as in [2]. Let $\mathscr{R}$ be the $\mathbf{Q}$-vector space

$$\mathscr{R}=\bigoplus_{r=0}^{\infty}\mathbf{Q}[\mathbf{N}^r]$$

spanned by a finite $\mathbf{Q}$-linear combination of symbols $[\mathbf{k}]=[k_1,\ldots,k_r]$ with $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbf{N}^r$ for some r. We understand $\mathbf{Q}[\mathbf{N}^0]=\mathbf{Q}[\phi]$ for $r=0$. Further let $\mathscr{R}^0$ denote the subspace of $\mathscr{R}$ spanned by the *admissible* symbols, i.e., by $[\phi]$ and the symbols $[k_1,\ldots,k_r]$ with $k_r\geq 2$. On $\mathscr{R}$, we consider the $\mathbf{Q}$-bilinear harmonic (stuffle) product $*$ which is defined inductively as:

(a) for any index $\mathbf{k}$, $[\phi]*[\mathbf{k}]=[\mathbf{k}]*[\phi]=[\mathbf{k}]$;

(b) for any indices $\mathbf{k}=(k_1,\ldots,k_r)$ and $\mathbf{l}=(l_1,\ldots,l_s)$ with $r,s\geq 1$,

$$[\mathbf{k}]*[\mathbf{l}]=[[\mathbf{k}_-]*[\mathbf{l}],k_r]+[[\mathbf{k}]*[\mathbf{l}_-],l_s] +[[\mathbf{k}_-]*[\mathbf{l}_-],k_r+l_s],$$

where $\mathbf{k}_-=(k_1,\ldots,k_{r-1}),\mathbf{l}_-=(l_1,\ldots,l_{s-1})$.

Hoffman proved that $\mathscr{R}_*:=(\mathscr{R},*)$ is a commutative and associative $\mathbf{Q}$-algebra and that $\mathscr{R}^0_*$ is a subalgebra of $\mathscr{R}_*$ [2]. Moreover, he proved that the evaluation map $\zeta:\mathscr{R}^0_*\ni[k_1,\ldots,k_r]\mapsto\zeta(k_1,\ldots,k_r)\in\mathbf{R}$, being extended $\mathbf{Q}$-linearly, is an algebra homomorphism from $\mathscr{R}^0_*$ to $\mathbf{R}$.

Our result is the following

Theorem 1. *Let $\mathscr{R}_*[[x]]$ be the ring of formal power series over $\mathscr{R}_*$. The equality*

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$$(2)\quad \exp_*\left(\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}[n]x^n\right) = 1+\sum_{k=2}^{\infty}(-1)^k \sum_{\substack{k_1+\cdots+k_r=k\\ r\geq 1,\forall k_i\geq 2}}(-1)^r\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!} \times [k_1,\ldots,k_r]\,x^k$$

holds in $\mathscr{R}_*[[x]]$. *Where* $\exp_*$ *is the exponential* $\exp_*(f)=\sum_{n=0}^{\infty}\frac{f^n}{n!}$ *with* f^n *being the power in the ring* $\mathscr{R}_*[[x]]$.

Applying the evaluation map ζ coefficient-wise to both sides of (2), we obtain (1).

2. Proof. Set

$$S(k):=\sum_{\substack{k_1+\cdots+k_r=k\\ r\geq 1,\forall k_i\geq 2}}(-1)^r\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}[k_1,\ldots,k_r]$$

for $k\geq 2$, and put $S(0)=1, S(1)=0$. By taking the $\log_*$ of both sides of the equation (2) and then taking $x\cdot\partial/\partial x$, we see that (since both sides are 1 for $x=0$) equation (2) is equivalent to

$$\left(\sum_{n=2}^{\infty}(-1)^{n-1}[n]x^n\right)*\left(\sum_{m=0}^{\infty}(-1)^m S(m)x^m\right) = \sum_{n=2}^{\infty}(-1)^n nS(n)x^n,$$

which in turn is equivalent to

$$\sum_{m=2}^{n}[m]*S(n-m)=-nS(n)$$

for $n\geq 2$. We compute $[m]*[k_1,\ldots,k_r]$ in $[m]*S(n-m)$ by the harmonic product:

L.H.S.

$$=\sum_{m=2}^{n}\sum_{\substack{k_1+\cdots+k_r=n-m\\ r\geq 1,\forall k_i\geq 2}}(-1)^r\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}[m]*[k_1,\ldots,k_r]$$

$$=\sum_{r=1}^{\left[\frac{n-2}{2}\right]}(-1)^r\sum_{m=2}^{n-2r}\sum_{\substack{k_1+\cdots+k_r=n-m\\ \forall k_i\geq 2}}\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!} \times\left(\sum_{l=1}^{r}[k_1,\ldots,k_l+m,\ldots,k_r] + \sum_{l=1}^{r+1}[k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{m}},\ldots,k_r]\right).$$

On the other hand,

R.H.S.

$$=-n\sum_{r=1}^{\left[\frac{n}{2}\right]}(-1)^r\sum_{\substack{k_1+\cdots+k_r=n\\ \forall k_i\geq 2}}\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}[k_1,\ldots,k_r].$$

Therefore, the proof is completed if we show that the terms of length r on both sides coincide for each r, i.e.,

$$(3)\quad \sum_{m=2}^{n-2r}\sum_{\substack{k_1+\cdots+k_r=n-m\\ \forall k_i\geq 2}}\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!} \times\sum_{l=1}^{r}[k_1,\ldots,k_l+m,\ldots,k_r] - \sum_{m=2}^{n-2r+2}\sum_{\substack{k_1+\cdots+k_{r-1}=n-m\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!} \times\sum_{l=1}^{r}[k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{m}},\ldots,k_{r-1}] = -n\sum_{\substack{k_1+\cdots+k_r=n\\ \forall k_i\geq 2}}\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}[k_1,\ldots,k_r].$$

Let us compute the first sum on the left-hand side of the last equation (3) by putting $k_l+m=h$. Then, we have

$$\sum_{m=2}^{n-2r}\sum_{\substack{k_1+\cdots+k_r=n-m\\ \forall k_i\geq 2}}\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!} \times\sum_{l=1}^{r}[k_1,\ldots,k_l+m,\ldots,k_r]$$

$$=\sum_{l=1}^{r}\sum_{m=2}^{n-2r}\sum_{h=m+2}^{n-2r+2} \times\sum_{\substack{k_1+\cdots+k_{r-1}=n-h\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!}\frac{(h-m-1)}{(h-m)!} \times[k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}]$$

$$=\sum_{l=1}^{r}\sum_{h=4}^{n-2r+2}\sum_{\substack{k_1+\cdots+k_{r-1}=n-h\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!} \times\left(\sum_{m=2}^{h-2}\frac{(h-m-1)}{(h-m)!}\right)[k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}].$$

Since $\sum_{m=2}^{h-2}\frac{(h-m-1)}{(h-m)!}=\sum_{m=2}^{h-2}\left(\frac{1}{(h-m-1)!}-\frac{1}{(h-m)!}\right)=-\frac{1}{(h-2)!}+1$, the first sum of (3) is equal to

$$(4)\qquad -\sum_{l=1}^{r}\sum_{h=2}^{n-2r+2}\sum_{\substack{k_1+\cdots+k_{r-1}=n-h\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!}\times\left(h\frac{(h-1)}{h!}\right)[k_1,\ldots,\underset{l\text{-th}}{\hat{h}},\ldots,k_{r-1}]$$

$$(5)\qquad +\sum_{l=1}^{r}\sum_{h=2}^{n-2r+2}\sum_{\substack{k_1+\cdots+k_{r-1}=n-h\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!}\times[k_1,\ldots,\underset{l\text{-th}}{\hat{h}},\ldots,k_{r-1}].$$

Because the terms of $h=2,3$ in (4) cancel out the terms of $h=2,3$ in (5) respectively, we include the terms of $h=2,3$ in the sums. Replacing h in the first sum by k_l, we have

$$-\sum_{l=1}^{r}\sum_{\substack{k_1+\cdots+k_r=n\\ \forall k_i\geq 2}}k_l\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}[k_1,\ldots,k_l,\ldots,k_r]$$
$$+\sum_{l=1}^{r}\sum_{h=2}^{n-2r+2}\sum_{\substack{k_1+\cdots+k_{r-1}=n-h\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!}\times[k_1,\ldots,\underset{l\text{-th}}{\hat{h}},\ldots,k_{r-1}]$$
$$=-n\sum_{\substack{k_1+\cdots+k_r=n\\ \forall k_i\geq 2}}\prod_{j=1}^{r}\frac{(k_j-1)}{k_j!}[k_1,\ldots,k_r]$$
$$+\sum_{h=2}^{n-2r+2}\sum_{\substack{k_1+\cdots+k_{r-1}=n-h\\ \forall k_i\geq 2}}\prod_{j=1}^{r-1}\frac{(k_j-1)}{k_j!}\times\sum_{l=1}^{r}[k_1,\ldots,\underset{l\text{-th}}{\hat{h}},\ldots,k_{r-1}].$$

This gives equation (3) and hence completes the proof. □

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