On the product of Hurwitz zeta-functions

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Abstract: Corresponding to the lattice point problem for a random sphere Kendall and Rankin [8], Nakajima [9] considered the summatory function of the coefficients of the product of two Hurwitz zeta-functions and obtained the Bessel series expression. In this note we treat the case of the product of $\varkappa$ Hurwitz zeta-functions for an arbitrary integer $\varkappa \geq 2$ and obtain the expression in terms of the Voronoï-Steen function. This amounts to a refinement of corrected Nakajima's formula with streamlining of the ambiguous argument.

Key words: Hurwitz zeta-function; Dirichlet divisor problem; Reisz sums.

1. Introduction. Counting the lattice points in a domain has been a fascinating subject and already Gauss considered the lattice points in a circle with radius r, say, and enunciated the asymptotic result with the main term the area πr^2 and the error term of the order of r. Dirichlet considered the corresponding problem for a hyperbola $xy = r$ and succeeded in obtaining an asymptotic formula with the error term of the order of r. Estimating the error term has been known as the Gauss circle problem and Dirichlet's divisor problem, respectively. It was Voronoï [13] who introduced a new phase not only into the lattice point problem but also into the fields where there is a zeta-function, as expressing the error term in terms of special functions, and in particular Bessel functions. Some of the generalizations are in higher dimensions, such as the $\varkappa$-dimensional sphere problem associated to the Epstein zeta-functions and the Piltz divisor problem associated to $\zeta(s)^{\varkappa}$, where $\zeta(s)$ indicates the Riemann zeta-function.

As a generalization of the Gauss circle problem, Kendall and Rankin [8] considered the lattice points in a random sphere and obtained the Bessel series expression. Since the generating function is the $\varkappa$-dimensional Epstein zeta-function, Chandrasekharan and Narasimhan [4] naturally obtained the Bessel series expression for the $\varkappa$-dimensional random sphere problem.

The corresponding random Piltz divisor problem was considered by Nakajima [9] who obtained a Bessel series expression for the summatory function in the case of the product of two Hurwitz zeta-functions. Here the Hurwitz zeta-functions $\zeta(s, \alpha)$ are defined for $0 < \alpha \leq 1$ by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \tag{1.1}$$

for $\sigma := \operatorname{Re} s > 1$ and then continued meromorphically over the whole plane by the functional equation. The Riemann zeta-function is a special case of the Hurwitz zeta-function:

$$\zeta(s) = \zeta(s,1).$$

Let $\varkappa$ be a positive integer ≥ 2 and let $\{\lambda_n\}$ denote the strictly increasing sequence of numbers of the form

$$\lambda_n = (n_1+\alpha_1)\cdots(n_{\varkappa}+\alpha_{\varkappa}), \quad n_j \in \mathbf{N}\cup\{0\} \tag{1.2}$$

with multiplicity $\tilde{d}(\lambda_n)$, so that

$$\tilde{d}(\lambda_n) = \tilde{d}_{\varkappa}(\lambda_n) = \sum_{\substack{(n_1+\alpha_1)\cdots(n_{\varkappa}+\alpha_{\varkappa})=\lambda_n \\ n_j\in\mathbf{N}\cup\{0\}}} 1. \tag{1.3}$$

Then

$$\varphi(s) := \zeta(s,\alpha_1)\cdots\zeta(s,\alpha_{\varkappa}) = \sum_{\lambda_n} \frac{\tilde{d}(\lambda_n)}{\lambda_n^s} \tag{1.4}$$

for $\sigma > 1$.

We write

$$\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_{\varkappa}) \in \mathbf{R}^{\varkappa} \in (0,1)^{\varkappa}. \tag{1.5}$$

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As a generalization of the Piltz divisor problem, one may consider the summatory function

$$(1.6)\qquad D(x;\boldsymbol{\alpha}) = {\sum_{\lambda_n\le x}}' \tilde{d}(\lambda_n) = {\sum_{\substack{(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)\le x\\ n,j\in\mathbf{N}\cup\{0\}}}}' 1$$
$$= \mathrm{P}(x) + \Delta(x;\boldsymbol{\alpha}),$$

say, where $\mathrm{P}(x) = \mathrm{P}(x;\boldsymbol{\alpha})$ is the residual function which is the sum of the residues of the generating function, $\varphi(s)$ to be defined below, with weights and we are to express the error term $\Delta(x;\boldsymbol{\alpha})$ in terms of special functions and where the prime on the summation sign means that the term corresponding to $(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa) = x$ is halved, which arises from the discontinuous integral.

Corollary 1 (Corrected version of Nakajima's theorem)**.** *For $\boldsymbol{\alpha} = (\alpha,\beta)\in(0,1)^2$ the error term* (1.6) *may be expressed as*

$$(1.7)\quad \Delta(x;\alpha,\beta)$$
$$= \frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}\,Y_1(-4\pi\sqrt{nx})$$
$$- \frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{-\alpha,\beta}(n)+d_{\alpha,-\beta}(n)}{\sqrt{n}}\,K_1(4\pi\sqrt{nx})$$
$$- \frac{i\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}\,J_1(-4\pi\sqrt{nx}),$$

where the coefficients are to be defined below.

Even in this corrected form, there seems to be no rule of appearance of the Bessel functions and there is little hope of considering the product of $\varkappa$ Hurwitz-zeta functions. Our method gives the proper result for the general case at a stretch.

Moreover, Nakajima applied the 0th order Perron's formula (a special case of (1.16) below) to express $D(x;\alpha,\beta)$ as

$$(1.8)\qquad D(x;\alpha,\beta) = \frac{1}{2\pi i}\int_{(c)}\zeta(s,\alpha)\zeta(s,\beta)\,\frac{x^s}{s}\,\mathrm{d}s \qquad (c>1).$$

It is warned, however, e.g. in Davenport [5, pp. 104–105] that applying the 0th order Perron's formula is problematic because there is no guarantee that the interchange of summation and integration is legitimate and that to stick to the 0th order Perron's formula, one has to apply the truncated formula as can be found in many textbooks. Nakajima made the same mistake in his earlier paper [10] which was corrected and improved by Banerjee and Mehta [1].

The common procedure is to apply higher order Riesz sums as in many previous investigations including Val′fiš [12] and Chandrasekharan and Narasimhan [3, pp. 106–111]. Then the final result for the 0th Riesz sum can be obtained by differencing or in most cases by differentiating as long as the differentiated series is uniformly convergent. Indeed, in our case all the differentiated series are absolutely convergent and the differentiating procedure is valid.

Preliminaries. • The Hurwitz zeta-functions (1.1) are related with the Lerch zeta-function $l_s(\alpha)$ by the functional equation (cf. [2]):

$$(1.9)\ \zeta(1-s,\alpha) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\frac{\pi i}{2}s}l_s(\alpha) + e^{\frac{\pi i}{2}s}l_s(1-\alpha)\},$$

where the Lerch zeta-function (or also known as the polylogarithm function) is defined by

$$l_s(\alpha) = \sum_{n=1}^{\infty}\frac{e^{2\pi i n\alpha}}{n^s}$$

for $\sigma>1$, $\alpha\in\mathbf{R}$ (or $s=1, 0<\alpha<1$).

Using (1.9) and the fact that $l_s(1-\alpha) = l_s(-\alpha)$, we obtain

$$(1.10)\quad \varphi(s) = \zeta(s,\alpha_1)\zeta(s,\alpha_2)\cdots\zeta(s,\alpha_\varkappa)$$
$$= \frac{\Gamma(1-s)^\varkappa}{(2\pi)^{\varkappa(1-s)}}\Big[(e^{-\frac{\pi i}{2}(1-s)})^\varkappa l_{1-s}(\alpha_1)$$
$$\times l_{1-s}(\alpha_2)\cdots l_{1-s}(\alpha_\varkappa)$$
$$+ (e^{-\frac{\pi i}{2}(1-s)})^{\varkappa-1}e^{\frac{\pi i}{2}(1-s)}$$
$$\times\{l_{1-s}(-\alpha_1)l_{1-s}(\alpha_2)\cdots l_{1-s}(\alpha_\varkappa)$$
$$+ l_{1-s}(\alpha_1)l_{1-s}(-\alpha_2)\cdots l_{1-s}(\alpha_\varkappa)$$
$$+\cdots + l_{1-s}(\alpha_1)l_{1-s}(\alpha_2)\cdots l_{1-s}(-\alpha_\varkappa)\}$$
$$+\cdots\cdots\cdots$$
$$+ (e^{\frac{\pi i}{2}(1-s)})^\varkappa l_{1-s}(-\alpha_1)s$$
$$\times l_{1-s}(-\alpha_2)\cdots l_{1-s}(-\alpha_\varkappa)\Big].$$

The case $\varkappa = 2$ reads as

$$\zeta(s,\alpha)\zeta(s,\beta) = \frac{\Gamma(1-s)^2}{(2\pi)^{2(1-s)}}\{e^{-\pi i(1-s)}l_{1-s}(\alpha)l_{1-s}(\beta)$$
$$+ l_{1-s}(\alpha)l_{1-s}(-\beta) + l_{1-s}(-\alpha)l_{1-s}(\beta)$$
$$+ e^{\pi i(1-s)}l_{1-s}(-\alpha)l_{1-s}(-\beta)\}.$$

For $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_\varkappa)\in\mathbf{R}^\varkappa$ we write

$$\ell_s(\alpha_1)\cdots\ell_s(\alpha_\varkappa)=\sum_{n=1}^{\infty}\frac{d_{\boldsymbol{\alpha}}(n)}{n^s},\quad \sigma>1 \tag{1.11}$$

where

$$\begin{aligned} d_{\boldsymbol{\alpha}}(n)&=d_{\alpha_1,\cdots,\alpha_\varkappa}(n)\\ &=\sum_{\substack{m_j\in\mathbf{N}\\ m_1\cdots m_\varkappa=n}} e^{2\pi i(\alpha_1m_1+\cdots+\alpha_\varkappa m_\varkappa)}. \end{aligned} \tag{1.12}$$

We write $d_{+(\varkappa-r),-r}(n)$ to indicate $d_{\boldsymbol{\alpha}}(n)$ with $(\varkappa-r)$ α_i's are positive and (r) α_i's are negative, e.g.

$$\begin{aligned} d_{+2,-1}&=d_{+2,-1}(n)\\ &=\sum_{\substack{m_j\in\mathbf{N}\\ m_1m_2m_3=n}} e^{2\pi i(\alpha_1m_1+\alpha_2m_2-\alpha_3m_3)}. \end{aligned}$$

• The Meijer G-function is

$$\begin{aligned} &G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)\\ &=\frac{1}{2\pi i}\int_c\frac{\prod_{j=1}^{m}\Gamma(b_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)}\\ &\quad\times\frac{\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j-s)}x^s\mathrm{d}s. \end{aligned} \tag{1.13}$$

• The Voronoï-Steen function $V=V(x;a_1,\cdots,a_n)$ (cf. [11]) is defined by

$$\begin{aligned} &\frac{1}{2\pi i}\int_0^\infty x^sV(x;a_1,\cdots,a_n)\,\frac{\mathrm{d}x}{x}\\ &=\Gamma(s+a_1)\cdots\Gamma(s+a_n). \end{aligned} \tag{1.14}$$

It is a special case of the G-function:

$$V(x;a_1,\cdots,a_n)=G^{n,0}_{0,n}\left(x\,\middle|\,\begin{matrix}-\\ a_1,\dots,a_n\end{matrix}\right). \tag{1.15}$$

The general Perron's formula can be found in Hardy-Riesz [6] to the effect that

$$\begin{aligned} &\frac{1}{\Gamma(\rho+1)}\sideset{}{'}\sum_{\lambda_k\le x}\alpha_k(x-\lambda_n)^\rho\\ &=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s+\rho+1)}\,\mathrm{d}s, \end{aligned} \tag{1.16}$$

where the left-hand side sum is the Riesz sum of order ρ, $\varphi(s)=\sum_{k=1}^\infty\frac{\alpha_k}{k^s}$ and c is bigger than the abscissa of absolute convergence of $\varphi(s)$.

The general formula for the difference operator of order $\rho\in\mathbf{N}$ with difference $y\ge0$ is given by

$$\Delta_y^\rho f(x)=\sum_{\nu=0}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}f(x+\nu y). \tag{1.17}$$

If f has the ρ-th derivative $f^{(\rho)}$, then

$$\begin{aligned} &\Delta_y^\rho f(x)\\ &=\int_x^{x+y}\mathrm{d}t_1\int_{t_1}^{t_1+y}\mathrm{d}t_2\cdots\int_{t_{\rho-1}}^{t_{\alpha-1}+y}f^{(\rho)}(t_\rho)\,\mathrm{d}t_\rho. \end{aligned} \tag{1.18}$$

If the order $\rho\in\mathbf{N}$, then the Riesz sum amounts to the $\varkappa$ times integration of the original sum $A_\lambda(x)$. Thus Landau's differencing is an analogue of the integration and differentiation.

The estimate is known that

$$\zeta(s,\alpha)=O\big(|t|^{\tau(\sigma)}\log|t|\big), \tag{1.19}$$

where

$$\begin{aligned} \tau(\sigma)&=\frac12(1-\sigma),\quad 0\le\sigma\le1,\\ \tau(\sigma)&=\frac12-\sigma,\quad \sigma\le0. \end{aligned} \tag{1.20}$$

Then instead of (1.22) we consider the ρth Riesz sum with

$$\mathbf{N}\ni\rho>\frac12\varkappa(1+a)+1, \tag{1.21}$$

$$\begin{aligned} D^\rho(x;\boldsymbol{\alpha})&=\frac{1}{\Gamma(\rho+1)}\sum_{\lambda_n\le x}(x-\lambda_n)^\rho\tilde d(\lambda_n)\\ &=\frac{1}{2\pi i}\int_{(c)}\varphi(s)\,\frac{1}{s(s+1)\cdots(s+\rho)}\,x^{s+\rho}\,\mathrm{d}s, \end{aligned} \tag{1.22}$$

where (c) indicates the Bromwich integral along the vertical line $\sigma=c$, $-\infty<t<\infty$ and $\varphi(s)$ is defined by (1.4).

2. A multi-dimensional divisor problem. For the summatory function (1.6) of the $\varkappa$-dimensional shifted divisor function we have the following theorem.

Theorem 2.1. *For*

$$\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_\varkappa)\in(0,1)^\varkappa,$$

we have

$$\begin{aligned} D(x;\boldsymbol{\alpha})&=\mathrm{P}(x)-\frac{(e^{-\frac{i\pi}{2}})^\varkappa}{(2\pi)^\varkappa}\sum_{n=1}^\infty\frac{d_{+\varkappa,-0}(n)}{n}\\ &\quad\times V\Big((2\pi)^\varkappa(e^{\frac{\pi i}{2}})^\varkappa nx\Big|1,\cdots,1,0\Big)\\ &\quad-\binom{\varkappa}{1}\frac{(e^{-\frac{i\pi}{2}})^{\varkappa-2}}{(2\pi)^\varkappa}\sum_{n=1}^\infty\frac{d_{+(\varkappa-1),-1}(n)}{n} \end{aligned} \tag{2.1}$$

$$\times V\Big((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa-2}nx\Big|1,\cdots,1,0\Big)$$
$$-\binom{\varkappa}{2}\frac{(e^{-\frac{i\pi}{2}})^{\varkappa-4}}{(2\pi)^{\varkappa}}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-2),-2}(n)}{n}$$
$$\times V\Big((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa-4}nx\Big|1,\cdots,1,0\Big)$$
$$-\cdots$$
$$-\frac{(e^{\frac{i\pi}{2}})^{\varkappa}}{(2\pi)^{\varkappa}}\sum_{n=1}^{\infty}\frac{d_{+0,-\varkappa}(n)}{n}$$
$$\times V\Big((2\pi)^{\varkappa}(e^{-\frac{\pi i}{2}})^{\varkappa}nx\Big|1,\cdots,1,0\Big).$$

Here $\mathrm{P}(x)=\mathrm{P}_{\boldsymbol{\alpha}}(x)$ *is the residual function which is the sum of the residues of the weighted generating function*

$$\varphi(s)\,\frac{x^s}{s}=\zeta(s,\alpha_1)\cdots\zeta(s,\alpha_\varkappa)\,\frac{x^s}{s} \tag{2.2}$$

at $s=0$ *and* 1, *or*

$$\frac{1}{2\pi i}\int_C\varphi(s)\,\frac{x^s}{s}\,\mathrm{d}s \tag{2.3}$$

where C *is a curve lying in the strip* $-a<\sigma<1$ *and encircling* $s=0$ *and* 1.

Proof. We apply the Cauchy residue theorem as follows: Take a rectangle with vertices at $s=c+it, -T<t<T,\quad s=\sigma+iT, -a<\sigma<c,\quad s=-a+it, -T<t<T$ and $s=\sigma-iT, -a<\sigma<c$, where $T>0$ is to mean $T_1>0$ and $T_2>0$ tending to ∞ independently but we usually use this convention. By (1.19), the integrand is

$$T^{\frac{1}{2}\varkappa(1+a)}.$$

Hence since the order ρ of the Riesz sum satisfies (1.21), the horizontal integrals are estimated by

$$T^{-1}\to 0$$

as $T\to\infty$. At the same time, this assures absolute convergence of the vertical integral along $s=-a+it$.

Thus we may let $T\to\infty$ and express the initial integral by the residual function and the resulting vertical integral, which is the meaning of ‘moving the line of integration’ from (c) to $(-a)$:

$$D(x;\boldsymbol{\alpha})=P(x)+\Delta(x;\boldsymbol{\alpha}) \tag{2.4}$$

where $P(x)=P_{\boldsymbol{\alpha}}(x)$ is the residual function (2.2) and

$$\Delta(x;\boldsymbol{\alpha})=\frac{1}{2\pi i}\int_{(-a)}\zeta(s,\alpha_1)\zeta(s,\alpha_2)\cdots\zeta(s,\alpha_\varkappa)\times\frac{1}{s(s+1)\cdots(s+\rho)}\,x^{s+\rho}\mathrm{d}s \tag{2.5}$$

is the error term.

At this stage one can apply the standard procedure of applying the functional equation (1.10) followed by the change of summation and integration. What arises is the ρ times integration of the Voronoï-Steen function. I.e. we have sums with the coefficient

$$G_{1,\varkappa}^{\varkappa+1,1}\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa}nx\,\middle|\,\begin{matrix}1\\1,1,\cdots,1,-\rho\end{matrix}\right). \tag{2.6}$$

It would be of some interest to express the resulting G-functions in explicit form as in [7] but here we stick to the 0th order Riesz sum.

Hence we apply the differentiation under the integral sign ρ times since we may assure the uniform convergence of the integral of the form $I_1:\frac{1}{2\pi i}\int_{(-a)}\Gamma(1-s)^{\varkappa}\,\frac{w^s}{s}\,\mathrm{d}s$. We denote the resulting integrals by $I_1, I_2,\cdots$. We may also differentiate the residual function

$$\frac{1}{2\pi i}\int_C\varphi(s)\,\frac{1}{s(s+1)\cdots(s+\rho)}\,x^{s+\rho}\,\mathrm{d}s \tag{2.7}$$

under the integral sign to arrive at (2.3).

Using

$$\frac{\Gamma(1-s)^{\varkappa}}{s}=-\Gamma(1-s)^{\varkappa-1}\Gamma(-s), \tag{2.8}$$

we simplify the G-function into $G_{0,\varkappa}^{\varkappa,0}$ or the Voronoï-Steen function.

E.g. the first integral is

$$I_1=\frac{1}{2\pi i}\int_{(-a)}\frac{\Gamma(1-s)^{\varkappa}}{(2\pi)^{\varkappa(1-s)}}\,(e^{-\frac{\pi i}{2}(1-s)})^{\varkappa}l_{1-s}(\alpha_1)\cdots l_{1-s}(\alpha_\varkappa)\,\frac{x^s}{s}\,\mathrm{d}s$$
$$=-\frac{1}{(2\pi)^{\varkappa}}\,e^{-\frac{\pi i}{2}\varkappa}\sum_{n=1}^{\infty}\frac{d_{+\varkappa,-0}(n)}{n}\times G_{0,\varkappa}^{\varkappa,0}\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa}nx\,\middle|\,\begin{matrix}-\\1,1,\cdots,1,0\end{matrix}\right)$$
$$=-\frac{(e^{-\frac{\pi i}{2}})^{\varkappa}}{(2\pi)^{\varkappa}}\sum_{n=1}^{\infty}\frac{d_{+\varkappa,-0}(n)}{n}\times V\Big((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa}nx\Big|1,1,\cdots,1,0\Big).$$

The second integral is

$$I_2 = \frac{1}{2\pi i}\int_{(-a)} \frac{\Gamma(1-s)^{\varkappa}}{(2\pi)^{\varkappa(1-s)}} (e^{-\frac{\pi i}{2}(1-s)})^{\varkappa-2}$$
$$\{l_{1-s}(-\alpha_1)\cdots l_{1-s}(\alpha_\varkappa)+\cdots$$
$$+ l_{1-s}(\alpha_1)\cdots l_{1-s}(-\alpha_\varkappa)\}\frac{x^s}{s}\,\mathrm{d}s$$
$$= -\binom{\varkappa}{1}\frac{1}{(2\pi)^{\varkappa}} e^{-\frac{\pi i}{2}(\varkappa-2)}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-1),-1}(n)}{n}$$
$$\times G^{\varkappa,0}_{0,\varkappa}\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa-2}nx\,\middle|\, \begin{matrix} - \\ 1,1,\cdots,1,0\end{matrix}\right)$$
$$= -\binom{\varkappa}{1}\frac{(e^{-\frac{\pi i}{2}})^{\varkappa-2}}{(2\pi)^{\varkappa}}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-1),-1}(n)}{n}$$
$$\times V\Big((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa-2}nx\Big|1,1,\cdots,1,0\Big).$$

Procedure being the same, we conclude the result thereby covering the case of $\varkappa=2$. □

In particular, Theorem 2.1 with $\varkappa = 2,3$ amounts to

Corollary 2. (i) *For* $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in(0,1)^2$, *we have*

$$D(x;\boldsymbol{\alpha}) = \mathrm{P}(x) + \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d_{\alpha_1,\alpha_2}(n)}{n}V\Big((2\pi)^2e^{\pi i}nx\Big|1,0\Big)$$
$$-\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d_{\alpha_1,-\alpha_2}(n)+d_{-\alpha_1,\alpha_2}(n)}{n}V\Big((2\pi)^2nx\Big|1,0\Big)$$
$$+\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d_{-\alpha_1,-\alpha_2}(n)}{n}V\Big((2\pi)^2e^{-\pi i}nx\Big|1,0\Big),$$

where

$$\mathrm{P}(x)=\mathrm{P}_{\alpha_1,\alpha_2}(x) = x\log x + \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha_1)\right) + \left(-\frac{\Gamma'}{\Gamma}(\alpha_2)\right)-1\right\}x + \left(\frac12-\alpha_1\right)\left(\frac12-\alpha_2\right).$$

(ii) *For* $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\in(0,1)^3$, *we have*

$$D(x;\boldsymbol{\alpha}) = \mathrm{P}(x)$$
$$-\frac{(e^{-\frac{i\pi}{2}})^3}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+3,-0}(n)}{n}V\Big((2\pi)^3(e^{\frac{\pi i}{2}})^3nx\Big|1,1,0\Big)$$
$$-3\frac{e^{-\frac{i\pi}{2}}}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+2,-1}(n)}{n}V\Big((2\pi)^3e^{\frac{\pi i}{2}}nx\Big|1,1,0\Big)$$
$$-3\frac{e^{\frac{i\pi}{2}}}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+1,-2}(n)}{n}V\Big((2\pi)^3e^{-\frac{\pi i}{2}}nx\Big|1,1,0\Big)$$
$$-\frac{(e^{\frac{i\pi}{2}})^3}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+0,-3}(n)}{n}V\Big((2\pi)^3(e^{-\frac{\pi i}{2}})^3nx\Big|1,1,0\Big)$$

with $\mathrm{P}(x)=\mathrm{P}_{\boldsymbol{\alpha}}(x)=\mathrm{P}_{\alpha_1,\alpha_2,\alpha_3}(x)$ *being the residual function* (2.2).

3. Two-dimensional case. Recall the inverse Heaviside integral

$$G^{2,0}_{0,2}\left(z\,\middle|\,\begin{matrix}-\\ a,b\end{matrix}\right) = 2\,z^{\frac12(a+b)}K_{a-b}\big(2\sqrt{z}\big), \tag{3.1}$$

where $K_\kappa(z)$ indicates the modified Bessel function of the third kind which is often referred to as the K-Bessel function. In view of this, Corollary 2 (i) entails

Theorem 3.1. *For* $0<\alpha,\beta<1$,

$$D(x;\alpha,\beta)=\mathrm{P}(x)+\Delta(x;\alpha,\beta) \tag{3.2}$$

where

$$\mathrm{P}(x)=\mathrm{P}_{\alpha,\beta}(x) = x\log x + \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right) + \left(-\frac{\Gamma'}{\Gamma}(\beta)\right)-1\right\}x+\zeta(0,\alpha)\zeta(0,\beta) \tag{3.3}$$

and

$$\Delta(x;\alpha,\beta) = \frac{i\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}K_1(4\pi i\sqrt{nx}) - \frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{-\alpha,\beta}(n)+d_{\alpha,-\beta}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}). \tag{3.4}$$

Remark 1. The K-Bessel function is related to other Bessel functions via

$$Y_\nu(iz) = e^{\frac{\pi i(\nu+1)}{2}}I_\nu(z) - \frac{2}{\pi}e^{-\frac{\pi i\nu}{2}}K_\nu(z) \quad \left(-\pi<\arg z\le\frac{\pi}{2}\right). \tag{3.5}$$

Let $J_\kappa(z)$ denote the Bessel function of the first kind. Then

$$J_\nu(iz)=e^{\frac{\pi i\nu}{2}}I_\nu(z).$$

Now for $\nu=1$, this gives

$$K_1(-iz)=\frac{\pi}{2}(-J_1(z)-iY_1(z)),$$
$$K_1(iz)=\frac{\pi}{2}(-J_1(-z)-iY_1(-z)).$$

Hence Corollary 1 is a restatement of Theorem 3.1.

It is of some interest to consider the case where one of the perturbation parameters is 1:

Corollary 3. *For* $0 < \alpha < 1$*, we have*

$$\begin{aligned} D(x;\alpha,1) &= \mathrm{P}_{\alpha,1}(x) \\ &\quad + \frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_\alpha(n)+d_{-\alpha}(n)}{\sqrt{n}}Y_1(-4\pi\sqrt{nx}) \\ &\quad - \frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_\alpha(n)+d_{-\alpha}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}) \\ &\quad - \frac{i\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_\alpha(n)+d_{-\alpha}(n)}{\sqrt{n}}J_1(-4\pi\sqrt{nx}) \end{aligned}$$

with $d_\alpha(n) = d_{\alpha,1}(n) = \sum_{l|n} e^{2\pi i l\alpha}$ *defined by* (1.12) *and*

$$\begin{aligned} P_{\alpha,1}(x) &= x\log x + x\left[\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right)+\gamma-1\right] \\ &\quad + \frac{1}{2}\alpha - \frac{1}{4}, \end{aligned}$$

γ *being the Euler constant.*

Proof. As in (2.5), the generating function is $\zeta(s,\alpha)\zeta(s)$.

Now $\zeta(s,\alpha)\zeta(s)$ satisfies

$$\begin{aligned} \text{(3.6)}\ \zeta(s,\alpha)\zeta(s) &= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\pi^{-\frac{1}{2}+s} \\ &\quad \times \{e^{-\frac{\pi i}{2}(1-s)}l_{1-s}(\alpha)+e^{\frac{\pi i}{2}(1-s)}l_{1-s}(-\alpha)\}\zeta(1-s). \end{aligned}$$

Thus the treatment is verbatim to that of Theorem 2.1 and we complete the proof of the corollary. □

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