Resurgence of formal series solutions of nonlinear differential and difference equations

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Abstract: We discuss the resurgence of formal series solutions of nonlinear differential and difference equations of level 1. We derive an estimate for iterated convolution products. We describe the possible location of the singularities of their Borel transforms.

Key words: Resurgent function; Borel resummation method; iterated convolution.

1. Introduction. This is an announcement of our forthcoming paper [10], the main subject of which is the resurgence of the formal series solution of

$$\frac{d\varphi}{dx} = F(x^{-1}, \varphi), \tag{1}$$

with $F(x^{-1}, \varphi) \in \mathbf{C}^n\{x^{-1}, \varphi\}$ satisfying the conditions

$$F(0,0) = 0 \quad \text{and} \quad \det(\partial_\varphi F(0,0)) \neq 0.$$

Such an equation has a unique formal solution of the form $\varphi(x) = \sum_{k\geq 1} \varphi_k x^{-k} \in x^{-1}\mathbf{C}^n[[x^{-1}]]$. As is mentioned in [5], [6] and [7], $\varphi(x)$ is resurgent, i.e., the formal Borel transform

$$\hat{\varphi}(\xi) := \sum_{j=1}^{\infty} \varphi_j \frac{\xi^{j-1}}{(j-1)!}$$

of $\varphi(x)$ is analytically continuable along any path avoiding locally discrete subsets in $\mathbf{C}$. (Cf. [5], [6], [7] and [8].) Further, in [4], finer resurgent structure of $\varphi(x)$ and transseries solutions of (1) was studied under non-resonance conditions using staircase distributions. Correspondingly, the case of difference equations was discussed in [11]. (See also [1] and [2].)

In this article, we study the singular locus of $\hat{\varphi}(\xi)$, including the resonant case, developing the method in [9] (see also [3], [13], [14] and [12]): We derive an estimate of iterated convolution products (Theorem 2.4), which reflect an iteration procedure for constructing $\hat{\varphi}(\xi)$. Since $\hat{\varphi}(\xi)$ can be written by a sum of iterated convolution products, employing the estimates, we obtain the following theorem.

Theorem 1.1. *Let* $\Omega = \{\Omega_L\}_{L\in\mathbf{R}_{\geq 0}}$ *be the discrete filtered set defined by*

$$\Omega_L = \{\xi \in \mathbf{C} \mid \det(\xi + \partial_\varphi F(0,0)) = 0, |\xi| \leq L\}. \tag{2}$$

Then, the formal series solution $\varphi(x) \in x^{-1}\mathbf{C}^n[[x^{-1}]]$ *of* (1) *is* $\Omega^{*\infty}$*-resurgent.*

For the definition of $\Omega^{*\infty}$-resurgence, the reader is referred to [9]. This means that $\hat{\varphi}(\xi)$ can be analytically continued along any path which starts from 0 and avoids $\{\omega_1 + \cdots + \omega_r \mid r \geq 1, \omega_j \in \Omega_{L_j}, L_1 + \cdots + L_r = L\}$, where L is the length of the path. See Theorem 3.2 for the case of difference equations.

2. Iterated convolution of endlessly continuable functions. In [9], we discussed analytic properties of a convolution product $\hat{\Phi}_1 * \cdots * \hat{\Phi}_k$ of endlessly continuable functions $\hat{\Phi}_1, \cdots, \hat{\Phi}_k$. However, this was not sufficient to handle actual problems (including induction procedures in particular). In this section, we systematically study convolution products described by iteration diagrams, which we call iterated convolution products.

2.1. Iteration diagram.

Definition 2.1. Let $T = (V, E)$ be a directed tree diagram, where V (resp. E) is the set of vertices (resp. edges) of T. We call T *iteration diagram* if T satisfies the condition that any vertex $v \in V$ has at most one outgoing edge. $\mathscr{T}$ denotes the set of iteration diagrams.

Since $T \in \mathscr{T}$ is connected and has no cycles, we have

Lemma 2.2. *Each* $T \in \mathscr{T}$ *has a vertex* u *such*

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that, for any vertex $v \in V \setminus \{u\}$, there exists a path $v = v_1 \to v_2 \to \cdots \to v_k = u$ $(k \geq 2)$.

Since such a vertex u in Lemma 2.2 is unique, we call it *root* of T and denote it by $\hat{v}$. We call a vertex v *leaf* of T if v has no incoming edge and denote the set of leaves of T by L. We assign each vertex v a weight w_v defined as the cardinal of $\{v' \in L \mid \exists \text{ a path } v' = v_1 \to \cdots \to v_k = v \ (k \geq 2)\}$ if $v \notin L$ and $w_v = 1$ if $v \in L$. We obtain from the definition of the weight that $w_{\hat{v}} = |L|$. We set

$$\mathscr{T}_k := \{T \in \mathscr{T} \mid |T| = k\} \quad (k \geq 1).$$

2.2. Iterated convolution. We follow the notations in [9]. (See Section 3 of [9] for the definitions of the Ω-endless Riemann surface $\mathfrak{p}_\Omega : (X_\Omega, \underline{0}_\Omega) \to (\mathbf{C}, 0)$ and $K_\Omega^{\delta,L}$.) Let Ω be a discrete filtered set and take entire functions $\{\hat{f}_v\}_{v\in V}$ and Ω-continuable functions $\{\hat{\Phi}_v\}_{v\in V}$ with $T = (V, E) \in \mathscr{T}_k$ $(k \geq 1)$. We construct $\{\hat{\Psi}_v\}_{v\in V}$ from $\{\hat{f}_v\}_{v\in V}$ and $\{\hat{\Phi}_v\}_{v\in V}$ by the following rule:

$$\hat{\Psi}_v := \hat{\Phi}_v\left(\hat{f}_v * \prod_{u\to v}^{*} \hat{\Psi}_u\right) \quad (v \in V), \tag{3}$$

where $\prod_{u\to v}^{*} \hat{\Psi}_u$ is the convolution product of $\hat{\Psi}_u$ over all the vertices $u \in V$ that have an edge $u \to v$ and, when $v \in L$, we regard it as the unit δ. We will mostly be interested in $\hat{\Psi}_{\hat{v}}$. We set

$$\Delta_T := \left\{(s_v)_{v\in V} \in \mathbf{R}^k_{\geq 0} \;\middle|\; \sum_{u\to v} s_u \leq s_v, s_{\hat{v}} \leq 1\right\}$$

and $[\Delta_T] \in \mathscr{E}_k(\mathbf{R}^k)$ denotes the corresponding integration current, where the sum $\sum_{u\to v}$ is taken over all the vertices $u \in V$ that have an edge $u \to v$. We define the orientation of Δ_T so that

$$\int_{\Delta_T} \bigwedge_{v\in V} \mathrm{d}s_v = \frac{1}{k!}$$

holds. We fix $\rho > 0$ such that $\Omega_{2\rho} = \emptyset$ and set

$$U := \{\xi \in \mathbf{C} \mid |\xi| < 2\rho\}.$$

We take a neighborhood $\underline{U}_v$ of $\underline{0}_{\Omega_v} \in X_{\Omega_v}$ so that

$$\mathfrak{p}_{\Omega_v}|_{\underline{U}_v} : \underline{U}_v \xrightarrow{\sim} U,$$

where $\Omega_v := \Omega^{*w_v}$ $(v \in V)$. Let $\mathscr{L}_v = (\mathfrak{p}_{\Omega_v}|_{\underline{U}_v})^{-1}$ and consider, for $\xi \in U$, the map

$$\vec{\mathscr{D}}(\xi) : (s_v)_{v\in V} \mapsto (\mathscr{L}_v(s_v\xi))_{v\in V} \in X_\Omega^T := \prod_{v\in V} X_{\Omega_v}$$

defined on a neighborhood of Δ_T in $\mathbf{R}^k$. We denote by $\vec{\mathscr{D}}(\xi)_{\#}[\Delta_T] \in \mathscr{E}_k(X_\Omega^T)$ the push-forward of $[\Delta_T]$ by $\vec{\mathscr{D}}(\xi)$. Then, we obtain the following representation of $\hat{\Psi}_{\hat{v}}$:

Lemma 2.3. *Let $T = (V, E) \in \mathscr{T}_k$ $(k \geq 1)$ and define*

$$\beta_T := \left(\prod_{v\in V} (\mathfrak{p}^*_{\Omega_v}\hat{\Phi}_v)(\underline{\xi}_v)\hat{f}_v\left(\xi_v - \sum_{u\to v} \xi_u\right)\right) \bigwedge_{v\in V} \mathrm{d}\underline{\xi}_v,$$

where $\bigwedge_{v\in V} \mathrm{d}\underline{\xi}_v$ is the pullback of the k-form $\bigwedge_{v\in V} \mathrm{d}\xi_v$ in X_Ω^T by $\prod_{v\in V} \mathfrak{p}_{\Omega_v} : X_\Omega^T \to \mathbf{C}^k$ and $\xi_v = \mathfrak{p}_{\Omega_v}(\underline{\xi}_v)$ $(v \in V)$. Then

$$(1 * \hat{\Psi}_{\hat{v}})(\xi) = \vec{\mathscr{D}}(\xi)_{\#}[\Delta_T](\beta_T)$$

holds for $\xi \in U$.

The following estimate is essential in the proof of the resurgence of formal series solutions of (1):

Theorem 2.4. *Let Ω be a discrete filtered set and let $\delta, L > 0$ be reals such that $\Omega_{2\delta} = \emptyset$. Then there exist $c, \delta' > 0$ such that $\delta' \leq \delta$ and, for every $T \in \mathscr{T}$ and for every entire functions $\{\hat{f}_v\}_{v\in V}$ and Ω-continuable functions $\{\hat{\Phi}_v\}_{v\in V}$, the function $\hat{\Psi}_{\hat{v}}$ defined by the rule* (3) *belongs to $\hat{\mathscr{R}}_{\Omega_{\hat{v}}}$ and satisfies the following estimates*:

$$\sup_{K^{\delta,L}_{\Omega_{\hat{v}}}} |\mathfrak{p}^*_{\Omega_{\hat{v}}}(1 * \hat{\Psi}_{\hat{v}})| \leq \frac{c^k}{k!} \sup_{\substack{(L_v/L)_{v\in V}\\ \in \Delta_T}} \prod_{v\in V} \sup_{K^{\delta',L_v}_{\Omega_v}} |\mathfrak{p}^*_{\Omega_v}\hat{\Phi}_v| \sup_{|\xi|\leq L_v} |\hat{f}_v|.$$

Remark 2.5. Since there exists a natural morphism $\mathfrak{q}_v : X_{\Omega_v} \to X_\Omega$ and $\mathfrak{q}_v(K^{\delta',L_v}_{\Omega_v}) \subset K^{\delta',L_v}_\Omega$, we have

$$\sup_{K^{\delta',L_v}_{\Omega_v}} |\mathfrak{p}^*_{\Omega_v}\hat{\Phi}_v| \leq \sup_{K^{\delta',L_v}_{\Omega}} |\mathfrak{p}^*_{\Omega}\hat{\Phi}_v|.$$

3. Resurgence of formal series solutions.

In this section, we discuss the resurgence of the formal series solution

$$\varphi = \begin{pmatrix} \varphi^{(1)} \\ \vdots \\ \varphi^{(n)} \end{pmatrix} \in x^{-1}\mathbf{C}^n[[x^{-1}]]$$

of (1). Regarding (1) as an equation for $\tilde{\varphi}(x) := x\varphi(x) - \varphi_1$, we may assume without loss of generality that

$$\partial^\ell_\varphi F(0, 0) = 0$$

for $\ell = (\ell_1, \cdots, \ell_n) \in \mathbf{Z}^n_{\geq 0}$ with $|\ell| := \ell_1 + \cdots + \ell_n \geq 2$. We decompose

$$F(x^{-1},\varphi) = F_0(x^{-1}) + \partial_\varphi F(0,0)\cdot\varphi + \sum_{|\ell|\geq 1} F_\ell(x^{-1})\varphi^\ell$$

with $F_0(0) = F_\ell(0) = 0$. We set $P(\xi) := -\xi - \partial_\varphi F(0,0)$. Applying the Borel transform, (1) is rewritten as follows:

$$P(\xi)\hat{\varphi} = \hat{F}_0 + \sum_{|\ell|\geq 1} \hat{F}_\ell * \hat{\varphi}^{*\ell}. \tag{4}$$

We inductively determine $\hat{\varphi}_k$ $(k \geq 1)$ by

$$\hat{\varphi}_1 := P^{-1}\hat{F}_0,$$

$$\hat{\varphi}_{k+1} := P^{-1}\sum_{j=1}^{k}\sum_{|\ell|=j}\hat{F}_\ell * \sum_{\substack{k_1+\cdots+k_j\\=k}} \hat{\varphi}^{(\ell)}_{k_1,\cdots,k_j},$$

where $\hat{\varphi}^{(\ell)}_{k_1,\cdots,k_j}$ is a convolution product of

$$\left\{\hat{\varphi}^{(m)}_{k_i} \;\middle|\; 1+\sum_{p=1}^{m-1}\ell_p \leq i \leq \sum_{p=1}^{m}\ell_p, 1 \leq m \leq n\right\}.$$

Then, $\hat{\varphi} = \sum_{k\geq 1}\hat{\varphi}_k$ gives a solution of (4). We use the following finer decomposition of $\hat{\varphi}$:

Lemma 3.1. *There exists a decomposition of* $\hat{\varphi}_k = \sum_{j=1}^{N_k}\hat{\varphi}_{k,j}$ *such that the following properties are satisfied*:

(a) *To each component of* $\hat{\varphi}_{k,j}$*, there exists* $T = (V,E) \in \mathscr{T}_k$ *such that* $\hat{\varphi}_{k,j}$ *is written by* $\hat{\Psi}_{\hat{v}}$ *as in Theorem* 2.4 *with* $\{\hat{\Phi}_v\}_{v\in V}$ *(resp.* $\{\hat{f}_v\}_{v\in V}$*) components of* P^{-1} *(resp.* $\hat{F}_\ell$ $(|\ell| \leq k-1)$*).*

(b) *There exists a constant* $C > 0$ *such that* $N_k \leq C^k$ $(k \geq 1)$.

Let $\Omega = \{\Omega_L\}_{L\in\mathbf{R}_{\geq 0}}$ be the discrete filtered set defined by (2). Since all the components of P^{-1} (resp. $\hat{F}_\ell$) are Ω-continuable (resp. entire), we obtain from Theorem 2.4 and Lemma 3.1 the following estimates: There exists $C > 0$ such that

$$\sup_{K^{\delta,L}_{\Omega^{*k}}} |\mathfrak{p}^*_{\Omega^{*k}}\hat{\varphi}_k| \leq \frac{C^k}{k!}$$

holds for $k \geq 1$. Therefore, $\mathfrak{p}^*_{\Omega^{*\infty}}\hat{\varphi} = \sum_{k=1}^{\infty}\mathfrak{p}^*_{\Omega^{*\infty}}\hat{\varphi}_k$ converges on $K^{\delta,L}_{\Omega^{*\infty}}$ for every $\delta, L > 0$, and hence $\Omega^{*\infty}$-resurgence of φ follows.

By totally the same discussion, we obtain the resurgence of a formal series solution $\varphi(x) \in x^{-1}\mathbf{C}^n[[x^{-1}]]$ of

$$\varphi(x+1) = F(x^{-1},\varphi(x)), \tag{5}$$

where $F(x^{-1},\varphi) \in \mathbf{C}^n\{x^{-1},\varphi\}$ satisfying the conditions

$$F(0,0) = 0 \quad\text{and}\quad \det(1-\partial_\varphi F(0,0)) \neq 0.$$

Theorem 3.2. *Let* $\Omega = \{\Omega_L\}_{L\in\mathbf{R}_{\geq 0}}$ *be a discrete filtered set defined by*

$$\Omega_L = \{\xi \in \mathbf{C} \mid \det(e^{-\xi} - \partial_\varphi F(0,0)) = 0, |\xi| \leq L\}.$$

Then, a formal series solution $\varphi(x) \in x^{-1}\mathbf{C}^n[[x^{-1}]]$ *of* (5) *is* $\Omega^{*\infty}$*-resurgent.*

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References

[1] B. Braaksma and R. Kuik, Resurgence relations for classes of differential and difference equations, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), no. 4, 479–492.

[2] B. Braaksma and R. Kuik, Asymptotics and singularities for a class of difference equations, in *Analyzable functions and applications*, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, 2005, pp. 113–135.

[3] B. Candelpergher, J.-C. Nosmas and F. Pham, *Approche de la résurgence*, Actualités Mathématiques., Hermann, Paris, 1993.

[4] O. Costin, On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations, Duke Math. J. **93** (1998), no. 2, 289–344.

[5] J. Écalle, *Les fonctions résurgentes. Tome I*, Publications Mathématiques d'Orsay, 81, 5, Univ. Paris XI, Département de Mathématiques, Orsay, 1981.

[6] J. Écalle, *Les fonctions résurgentes. Tome II*, Publications Mathématiques d'Orsay, 81, 6, Univ. Paris XI, Département de Mathématiques, Orsay, 1981.

[7] J. Écalle, *Les fonctions résurgentes. Tome III*, Publications Mathématiques d'Orsay, 85, 5, Univ. Paris XI, Département de Mathématiques, Orsay, 1985.

[8] J. Écalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actualités Mathématiques., Hermann, Paris, 1992.

[9] S. Kamimoto and D. Sauzin, Nonlinear analysis with endlessly continuable functions, RIMS Kôkyûroku Bessatsu, **B57** (2016), 235–247.

[10] S. Kamimoto, Iterated convolution of resurgent functions. (in preparation).

[11] R. Kuik, Transseries in difference and differential equations, 2003, Doctoral Thesis (University of Groningen).

[12] Y. Ou and E. Delabaere, Endless continuability and convolution product, arXiv:1410.1200.
[13] D. Sauzin, On the stability under convolution of resurgent functions, Funkcial. Ekvac. **56** (2013), no. 3, 397–413.
[14] D. Sauzin, Nonlinear analysis with resurgent functions, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 3, 667–702.