On the Iwasawa μ-invariants of branched $\mathbf{Z}_p$-covers

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Abstract: Following the analogies between knots and primes, we establish relative genus theory for a branched cover of rational homology 3-spheres. Then we formulate analogues of Iwasawa's theorems on μ-invariants for branched $\mathbf{Z}_p$-covers of rational homology 3-spheres, by using relative genus theory.

Key words: Link; rational homology 3-sphere; branched covering; relative genus theory; Iwasawa theory; arithmetic topology.

1. Introduction. In this article, following the analogies between knots and primes ([Mor12]), we establish relative genus theory for a branched cover of rational homology 3-spheres ($\mathbf{Q}\mathrm{HS}^3$). Then we formulate analogues of Iwasawa's theorems on μ-invariants ([Iwa73]) in 3-dimensional topology by using relative genus theory.

Genus theory for number fields was first studied for quadratic, abelian, and Galois extensions over $\mathbf{Q}$ by Hasse, Iyanaga–Tamagawa and Leopoldt, and Fröhlich. The case over a general number field k was formulated by Furuta in [Fur67] and is called relative genus theory. A role of the co-invariant group was also discussed in [Yok67]. Genus theory for 3-manifolds, on the other hand, was formulated in [Mor01] and [Mor12] for the cyclic case over an integral homology 3-sphere ($\mathbf{Z}\mathrm{HS}^3$), and was also discussed in [Uek14]. In this paper, we generalize these results and establish relative genus theory for a branched Galois cover of oriented, connected, and closed 3-manifolds. In addition, by employing Niibo's idèle ([Nii14], [NU]), we give an alternative proof which is parallel to the one for number fields.

Next, we recall Iwasawa theory. Let p be a prime number and let $\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}$ denote the ring of p-adic integers. A field k_∞ obtained as a $\mathbf{Z}_p$-extension of a number field is called a *$\mathbf{Z}_p$-field.* Such k_∞ is a limit of cyclic extensions k_n/k of degree p^n. As an analogue of $\mathbf{Z}_p$-extension, we consider an inverse system of cyclic branched p-covers $h_n : M_n \to M$ of $\mathbf{Q}\mathrm{HS}^3$ which are branched over a link L in M, and call it a *branched $\mathbf{Z}_p$-cover.* For these objects, the Iwasawa invariants λ, μ, ν are defined and studied ([Iwa59], [HMM06], [KM08], [KM13], [Uek]), and they describe the behaviors of the orders of p-parts of the ideal class groups $\mathrm{Cl}(k_n)$ and $H_1(M_n)$. Moreover, as an analogue of an extension of $\mathbf{Z}_p$-fields, the notion of a morphism (*branched Galois cover*) of branched $\mathbf{Z}_p$-covers was introduced in [Uek]. It is a compatible system of branched covers on each layer.

In [Iwa73], Iwasawa studied the behavior of the μ-invariants in a p-extension of $\mathbf{Z}_p$-fields, by employing relative genus theory. He gave a construction of a $\mathbf{Z}_p$-extension with arbitrary large μ, and also proved that there are infinitely many $\mathbf{Z}_p$-fields with $\mu = 0$. We formulate their analogues.

Notation. For a group G and a G-module A, let A^G and $A_G = A/I_G A$ denote the G-invariant subgroup and the G-co-invariant quotient, where $I_G = (g-1 \mid g \in G) < \mathbf{Z}[G]$ is the augmentation ideal. If G is a finite cyclic group, then $\#A^G = \#A_G$ holds.

2. Relative genus theory for number fields. First, we recall the case for number fields. We assume that algebraic extensions of $\mathbf{Q}$ are contained in $\mathbf{C}$, and number fields are finite over $\mathbf{Q}$.

Definition 2.1. Let k'/k be an abelian extension of number fields. The *relative genus field* k'^{g} of k'/k is the maximal unramified extension of k' abelian over k. The degree $g_{k'/k} = (k'^{\mathrm{g}} : k')$ is called the *relative genus number.*

In addition, for a Galois extension k'/k, we define the same notions by considering the maximal unramified extension k'^{g} of k' which is obtained as a composite of k' and an abelian extension of k instead.

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Theorem 2.2 ([Fur67]). *Let k'/k be a finite Galois extension of a number field, and let k'_0/k denote its maximal abelian subextension. Then,*

$$g_{k'/k} = \frac{\#\mathrm{Cl}(k)\prod_{\mathfrak{p}} e'_{\mathfrak{p}}}{(k'_0 : k)[\varepsilon : \eta]}$$

where $\mathfrak{p}$ runs through all the primes of k, $e'_{\mathfrak{p}}$ denotes the ramification index of the maximal abelian subextension of $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ for a prime $\mathfrak{P}$ of K dividing $\mathfrak{p}$, ε the unit group of k, and η the group of elements in ε everywhere locally norm.

Theorem 2.3 ([Yok67, Proposition 1]). *Let k'/k be a cyclic extension of number fields with $G = \mathrm{Gal}(k'/k) = \langle\sigma\rangle$. Then $g_{k'/k} = \#\mathrm{Cl}(k')_G = \#(\mathrm{Cl}(k')/\mathrm{Cl}(k')^{1-\sigma}) = \#\mathrm{Cl}(k')^G$ holds.*

By combining these two theorems, we can estimate the increase of class numbers in extensions.

3. Relative genus theory for rational homology 3-spheres. In this section, we formulate analogues of the two theorems in §2. They generalize the results of [Mor01] and [Mor12] originally for a branched cyclic cover over a $\mathbf{Z}\mathrm{HS}^3$. We also give an alternative proof by employing Niibo's idèle.

In the following, we assume that 3-manifolds are oriented, connected, and closed, and that branched covers of 3-manifolds are branched over links and are equipped with base points. In order to discuss analogues of class numbers, we sometimes assume that 3-manifolds are $\mathbf{Q}\mathrm{HS}^3$'s. A 3-manifold M is a $\mathbf{Q}\mathrm{HS}^3$ if and only if $H_1(M) < \infty$.

Definition 3.1. For a finite branched abelian cover $h : N \to M$ of 3-manifolds, the *relative genus cover* of h is the maximal unbranched cover $N^{\mathrm{g}} \to N$ abelian over M, and $g_h := \deg(N^{\mathrm{g}} \to N) \in \mathbf{N} \cup \{\infty\}$ is called the *relative genus number*.

In addition, for a finite branched Galois cover $h : N \to M$, we define the same notions by considering the maximal unbranched cover $N^{\mathrm{g}} \to N$ obtained as a composite (in the sense of Galois theory) of h and a branched abelian cover of M instead.

Now the first theorem is presented as follows:

Theorem 3.2. *Let $h : N \to M$ be a finite branched Galois cover of 3-manifolds branched over $L = \sqcup K_i$, and let $h_0 : N_0 \to M$ denote the maximal abelian subcover of h. Then the branch indices e_i of K_i in h satisfy*

$$g_h = \frac{\#H_1(M)\prod_i e_i}{\deg(h_0)}.$$

Proof. Let $(N^{\mathrm{g}})_0 \to N_0 \to M$ denote the maximal subcovers of $N^{\mathrm{g}} \to N \to M$ abelian over M. Then $g_h = \deg((N^{\mathrm{g}})_0 \to N_0)$. Indeed, let $Y^{\mathrm{g}} \to Y \to X$ and $(Y^{\mathrm{g}})_0 \to Y_0 \to X$ denote their restrictions to the exteriors of the branch links, let $D(\pi_1(X))$ denote the commutator group of $\pi_1(X)$, and put $A := \mathrm{Ker}(\pi_1(Y) \to \pi_1(N))$. Then by definition, $\pi_1(Y^{\mathrm{g}})$ is the smallest subgroup of $\pi_1(X)$ satisfying $\pi_1(Y^{\mathrm{g}}) > A$ and $\pi_1(Y^{\mathrm{g}}) = \pi_1(Y) \cap P$ for some $P > D(\pi_1(X))$. Thus $\pi_1(Y^{\mathrm{g}}) = \pi_1(Y) \cap (D(\pi_1(X)) \cdot A)$, $\pi_1(Y_0) = \pi_1(Y) \cdot D(\pi_1(X))$, and $\pi_1((Y^{\mathrm{g}})_0) = \pi_1(Y^{\mathrm{g}}) \cdot D(\pi_1(X))$. Hence $\pi_1(Y^{\mathrm{g}}) = \pi_1((Y^{\mathrm{g}})_0) \cap \pi_1(Y)$, $\pi_1(Y_0) = \pi_1(Y) \cdot \pi_1((Y^{\mathrm{g}})_0)$ and $\mathrm{Gal}((N^{\mathrm{g}})_0 \to N_0) = \pi_1(Y_0)/\pi_1((Y^{\mathrm{g}})_0) \cong \pi_1(Y)/\pi_1(Y^{\mathrm{g}}) = g_h$.

The set of meridians of $h^{-1}(L)$ generates $B := \mathrm{Ker}(H_1(Y) \twoheadrightarrow H_1(N))$. By the definition of the relative genus cover, the covers $(N^{\mathrm{g}})_0 \to N_0 \to M$ correspond to the subgroups $h_*(B) < h_*(H_1(Y)) < H_1(X)$. Since $\mathrm{Gal}(h_0) \cong H_1(X)/h_*(H_1(Y))$, we have $\mathrm{Gal}(N^{\mathrm{g}}/N) \cong h_*(H_1(Y))/h_*(B)$ and $g_h = \#(h_*(H_1(Y))/h_*(B)) = \#(H_1(X)/h_*(B))/\deg(h_0)$.

Now suppose that L is a t-component link, and let $\langle\mu_L\rangle < H_1(X)$ denote the meridian group. If M is not a $\mathbf{Q}\mathrm{HS}^3$, then the formula is clear by a surjection $H_1(X)/h_*(B) \twoheadrightarrow H_1(M)$. If M is a $\mathbf{Q}\mathrm{HS}^3$, then the Mayer–Vietoris long exact sequence yields the exact sequence $0 \to \langle\mu_L\rangle \to H_1(X) \to H_1(M) \to 0$. Let V_{K_i} denote the tubular neighborhood of K_i. Then $\pi_1(\partial V_{K_i}) \cong \mathbf{Z}^2$ is abelian, and so is the decomposition group. Since $\prod_i e_i\mathbf{Z} \cong h_*(B) < \langle\mu_L\rangle \cong \mathbf{Z}^t$, we have an exact sequence $0 \to \langle\mu_L\rangle/h_*(B) \to H_1(X)/h_*(B) \to H_1(M) \to 0$ with $\langle\mu_L\rangle/h_*(B) \cong \prod_i \mathbf{Z}/e_i\mathbf{Z}$. Hence $g_h \deg(h_0) = \#(H_1(X)/h_*(B)) = \#H_1(M)\prod_i e_i$, and the assertion holds. □

Corollary 3.3. *Let $h : N \to M$ be a finite branched Galois cover of 3-manifolds. Then M is a $\mathbf{Q}HS^3$ if and only if g_h is finite.*

Proof. If M is not a $\mathbf{Q}\mathrm{HS}^3$, then $\#H_1(M) = \infty$ and so is g_h. If M is a $\mathbf{Q}\mathrm{HS}^3$, then by Theorem 3.2, $g_h < \infty$ (while N is not necessarily a $\mathbf{Q}\mathrm{HS}^3$). □

Corollary-Definition 3.4. Let $h : N \to M$ be a finite branched Galois cover of 3-manifolds. If the branch link L consists of null-homologous components, then there are a natural splitting $H_1(X) \cong H_1(M) \oplus \langle\mu_L\rangle$ ([Uek, Lemma 4.4]) and a well-defined homomorphism $\Phi : H_1(N) \to H_1(X)/h_*(B) \overset{\cong}{\to} H_1(M) \oplus \prod_i \mathbf{Z}/e_i\mathbf{Z}$ with $\Phi([c]) = ([h(c)], (\mathrm{lk}(c, K_i) \bmod e_i)_i)$ for any $c \in \mathrm{Hom}(S^1, N)$. We say that $a, b \in H_1(N)$ belong to the same *genus* over M if $\Phi(a) = \Phi(b)$. This generalizes the notion of genus over S^3 ([Mor12, Chapter 6.2]).

Remark. Let $h : N \to M$ be a finite branched Galois cover of $\mathbf{Q}\mathrm{HS}^3$. Then, since $N^{\mathrm{g}} \to N$ is unbranched and abelian, $g_h | \# H_1(N)$ holds. This fact will be used in the study of Iwasawa invariants in §5.

Remark. Let $h: N \to M$ be a finite cyclic branched cover of $\mathbf{Q}\mathrm{HS}^3$ with $G = \mathrm{Gal}(h)$, and fix finite CW-structures on them compatible with h. Then [Uek14, Proposition 16] states that $g_h = \# H_1(N)^G = \gamma(\prod_i e_i)\# H_1(M)$, $\gamma = \# \widehat{H}^0(G, Z_2(N))/\# \widehat{H}^1(G, Z_2(N))$. By the theorem above, we obtain $\gamma = 1/\deg(h)$.

Next, we study the relation between the co-invariant group and the relative genus number.

For the trivial action of a group G on $\mathbf{Z}$, we write $H_i(G) := H_i(G, \mathbf{Z})$. We have $H_1(G) = G^{\mathrm{ab}}$. If G is finite, then $H_2(G)$ is finite, and if G is cyclic in addition, then $H_2(G) = 0$. Further, for a path-connected space X, we have $H_2(\pi_1(X)) = \mathrm{Coker}(\pi_2(X) \overset{\mathrm{Hur}}{\to} H_2(X))$ (Hopf's theorem, [Bro94]). Now we have

Theorem 3.5. *Let $h: N \to M$ be a finite branched Galois cover over a $\mathbf{Q}HS^3$ with $G = \mathrm{Gal}(h)$, and put $b := (h_{0*}(B_0) : h_*(B))$ with the notation of Theorem 3.2. Then $g_h = \#(H_1(N)_G/\ \bar{\delta}(H_2(G)))/b$ for some map $\bar{\delta}$ with $1 \le b \le \deg(N \to N_0)$.*

If h is abelian, then $b = 1$. If $G = \langle \sigma \rangle$, then $g_h = \# H_1(N)_G = \# H_1(N)/(1-\sigma)H_1(N) = \# H_1(N)^G$.

Proof. By the Hochschild-Serre spectral sequence ([Bro94, VII-6]), the short exact sequence $1 \to \pi_1(Y) \to \pi_1(X) \to G \to 1$ yields an exact sequence $H_2(\pi_1(X)) \to H_2(G) \overset{\delta}{\to} H_1(\pi_1(Y))_G \to H_1(\pi_1(X)) \to H_1(G) \to 0$. By the Hurewicz isomorphism $\pi_1(X)^{\mathrm{ab}} \cong H_1(X)$, we have an exact sequence $H_2(G) \to H_1(Y)_G \overset{h_*}{\to} H_1(X) \to G^{\mathrm{ab}} \to 0$. Since $h_*(I_G H_1(Y)) = 0$, we have an exact sequence $H_2(G) \to H_1(Y)_G \overset{h_*}{\to} h_*(H_1(Y)) \to 0$.

Since $h_*(I_G B) = 0$, there is an induced surjection $h_*: B_G \twoheadrightarrow h_*(B)$. Since $(\ )_G = H_0(\ )$, an exact sequence $0 \to B \to H_1(Y) \to H_1(N) \to 0$ yields an exact sequence $\cdots \to B_G \to H_1(Y)_G \to H_1(N)_G \to 0$. Thus we have a commutative diagram

$$\begin{array}{ccccccc}
 & & H_2(G) & & & & \\
 & & \downarrow & & & & \\
 & B_G & \longrightarrow\ H_1(Y)_G & \longrightarrow & H_1(N)_G & \longrightarrow 0 & \\
 & \downarrow & \downarrow & & \downarrow & & \\
0 \longrightarrow & h_*(B) & \longrightarrow h_*(H_1(Y)) & \longrightarrow & h_*(H_1(Y))/h_*(B) & \longrightarrow 0 & \\
 & \downarrow & \downarrow & & \downarrow & & \\
 & 0 & 0 & & 0 & &
\end{array}$$

consisting of exact sequences. Let $\bar{\delta}: H_2(G) \to H_1(N)_G$ denote the natural map. Then by the snake lemma, we have $\bar{\delta}(H_2(G)) = \mathrm{Ker}(H_1(N)_G \to h_*(H_1(Y))/h_*(B))$. Hence by Theorem 3.2, we have $\# H_1(N)_G/\bar{\delta}(H_2(G)) = \# h_*(H_1(Y))/h_*(B) = g_h b$.

Now $h': N \to N_0$ satisfies $h'_*(B) < \mathrm{Ker}(B_0 \hookrightarrow H_1(Y_0) \twoheadrightarrow H_1(Y_0)/h'_*(H_1(Y)))$, and hence $b \le \# B_0/h'_*(B) \le \# H_1(Y_0)/h'_*(H_1(Y)) \le \deg(h')$. □

Finally, we give an alternative proof of Theorem 3.2, which is rather parallel to the one in [Fur67], by employing Niibo's idèle. This idèle theory was initially introduced by Niibo in [Nii14], and was modified and generalized in [NU]. We first recall definitions and results in [NU]. Let M be a 3-manifold, and let $\mathcal{K} \subset M$ be an infinite link equipped with a tubular neighborhood $V_{\mathcal{K}} = \sqcup_{K \subset \mathcal{K}} V_K$. Let $\mathcal{A}_{M,\mathcal{K}}$ denote the set of all the abelian covers of M branched over a finite sublink of $\mathcal{K}$. We call $\mathcal{K}$ a *very admissible link* of M if for any $h: N \to M \in \mathcal{A}_{M,\mathcal{K}}$, $H_1(N)$ is generated by components of the preimage $h^{-1}(\mathcal{K})$. For any link L consisting of countably many tame components in a 3-manifold M, there exists such a $\mathcal{K}$ including L ([NU, Theorem 2.3]). Let $(M, \mathcal{K})$ be such a pair. Then *the idèle group* $I_M = \coprod_{K \subset \mathcal{K}} H_1(\partial V_K)$ is defined as the restricted product with respect to the meridian subgroups $\langle \mu_K \rangle < H_1(\partial V_K)$. In other words, we put $I_M := \{(a_K) \in \prod_{K \subset \mathcal{K}} H_1(\partial V_K) \mid v_K(a_K) = 0$ for almost all $K\}$, where $v_K: H_1(\partial V_K) \to H_1(V_K)$ is a natural map. *The principle idèle group* is defined by $\mathcal{P}_M = \varinjlim_{L \subset \mathcal{K}} H_2(M, L)$ where L runs through all the finite sublinks of $\mathcal{K}$. *The unit idèle group* $U_M < I_M$ is the subgroup consisting of formal infinite sums of meridians $\sum_{K \subset \mathcal{K}} a_K \mu_K (a_K \in \mathbf{Z})$. By [NU, Lemma 5.7], we have $I_M/(P_M + U_M) \cong H_1(M)$. By [NU, Theorem 7.4] (the existence theorem), there is a natural bijective correspondence between certain subgroups $P_M < H < I_M$ and h's. For each $h: N \to M \in \mathcal{A}_{M,\mathcal{K}}$ and a very admissible link $h^{-1}(\mathcal{K})$ of N, we define I_N, P_N, U_N. By [NU, Theorem 5.5] (the global reciprocity law), there is a natural isomorphism $I_M/(P_M + h_*(I_N)) \cong \mathrm{Gal}(h)$.

An alternative proof of Theorem 3.2. We fix a very admissible link $\mathcal{K}$ of M containing $L = \sqcup_i K_i$. Since $H^* := P_M + h_*(P_N + U_N) < I_M$ corresponds to $(N^{\mathrm{g}})_0 \to M$, we have $\deg(h_0) g_h = [I_M : H^*] = [I_M : P_M + U_M] \cdot [P_M + U_M : P_M + h_*(P_N + U_N)] = \# H_1(M) \cdot [P_M + U_M : P_M + h_*(U_N)] = \# H_1(M)[U_M : h_*(U_N)]/[P_M \cap U_M : P_M \cap h_*(U_N)]$. If M is not a $\mathbf{Q}\mathrm{HS}^3$, then the formula is clear. If M is a $\mathbf{Q}\mathrm{HS}^3$, then exact sequences $0 \to \langle \mu_{L'} \rangle \to H_1(M - L') \to$

$H_1(M) \to 0$ for $L' \subset \mathcal{K}$ yield a natural injection $U_M \hookrightarrow \varprojlim_{h \in \mathcal{A}_{M,\mathcal{K}}} \mathrm{Gal}(h)$. Therefore $P_M \cap U_M = 0$ and the denominator is 1. Since $U_M/h_*(U_N) \cong \prod \mathbf{Z}/e_i\mathbf{Z}$, we have $\deg(h_0)g_h = \#H_1(M) \prod_i e_i$. $\square$

4. Iwasawa μ-invariants of $\mathbf{Z}_p$-fields. Let p be a prime number. In this section, we recall the theorems on the μ-invariants in p-extensions of $\mathbf{Z}_p$-fields given by Iwasawa in [Iwa73] with use of relative genus theory. We also refer to [Och14] for the details.

Let v_p denote the p-adic valuation. Let k_∞/k be a $\mathbf{Z}_p$-extension of a number field, and let k_n/k denote the subextension of degree p^n for each n. Then we have the Iwasawa class number formula ([Iwa59]):

$$v_p(\#\mathrm{Cl}(k_n)) = \lambda n + \mu p^n + \nu \quad \text{for } n \gg 0,$$

for some constants $\lambda = \lambda_{k_\infty/k}$, $\mu = \mu_{k_\infty/k} \in \mathbf{N} = \mathbf{N} \cup \{0\}$, $\nu = \nu_{k_\infty/k} \in \mathbf{Z}$ called the Iwasawa invariants. The value of λ and whether $\mu = 0$ or not depend only on k_∞, and are independent of the choice of k.

There is a unique $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty$ of $\mathbf{Q}$. For a number field k, the composite $k_\infty^{\mathrm{c}} = k\mathbf{Q}_\infty$ is called a *cyclotomic* $\mathbf{Z}_p$*-field*. In a cyclotomic $\mathbf{Z}_p$-extension k_∞^{c}/k, every non-p prime decomposes finitely. On the other hand, a number field k is called a *CM-field* if it is a totally imaginary quadratic extension of a totally real field k^+. Such k has a $\mathbf{Z}_p$-extension K/k which is a limit of dihedral extensions of k^+, and every prime is inert in k/k^+ decomposes completely in K/k ([Iwa73]). (If $p > 2$, then a CM-field k has *the anti-cyclotomic* $\mathbf{Z}_p^{[k:\mathbf{Q}]/2}$*-extension* k_∞^{ac}/k, whose any sub-$\mathbf{Z}_p$-extension K/k is dihedral over k^+.)

Iwasawa conjectured that $\mu = 0$ holds for every cyclotomic $\mathbf{Z}_p$-extension k_∞^c/k, and it is true by Ferrero-Washington [FW79] if k is abelian over $\mathbf{Q}$. If $\mu = 0$, then the nature of a $\mathbf{Z}_p$-field "resembles" that of a function field. For general cases, however, there exist $\mathbf{Z}_p$-extensions with arbitrary large μ:

Theorem 4.1 ([Iwa73, §1])**.** *Let $k/\mathbf{Q}$ be an extension of degree d containing primitive p-th roots of unity, and let k_∞/k be a $\mathbf{Z}_p$-extension. Suppose that there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ in k which are completely decomposed in k_∞/k, and let $k' = k(\sqrt[p]{\mathfrak{p}_1 \ldots \mathfrak{p}_t})$ and $k'_\infty = k'k_\infty$. Then, $\mu_{k'_\infty/k'} \geq t - d$ holds.*

Let k be the 4-th or $p > 2$-th cyclotomic field. Then k is a CM-field, and there is a $\mathbf{Z}_p$-extension K/k dihedral over k^+. Since there are infinitely many primes inert in k/k^+, there are infinitely many primes completely decomposed in K/k. Therefore by the previous theorem, we obtain the following

Theorem 4.2 ([Iwa73, Theorem 1])**.** *Let k be the cyclotomic field of p-th or 4-th roots of unity according as $p > 2$ or $p = 2$. Then, for any $N \in \mathbf{N}$, there exist an extension k'/k of degree p and a $\mathbf{Z}_p$-extension k'_∞/k' such that $\mu_{k'_\infty/k'} \geq N$.*

On the other hand, the following tells that there are many $\mathbf{Z}_p$-fields with $\mu = 0$.

Theorem 4.3 ([Iwa73, Theorem 2])**.** *Let k be a number field (totally imaginary if $p = 2$), k_∞/k a $\mathbf{Z}_p$-extension, k'/k a finite Galois p-extension, and put $k'_\infty = k_\infty k'$. Suppose that every prime of k which is ramified in k'/k is finitely decomposed in k_∞/k. Then $\mu_{k_\infty/k} = 0$ if and only if $\mu_{k'_\infty/k'} = 0$.*

5. Iwasawa μ-invariants of branched $\mathbf{Z}_p$-covers. In this section, we formulate analogues of Iwasawa's results recalled in the previous section.

Let $L \subset M$ be a link in $\mathbf{Q}\mathrm{HS}^3$. We call an inverse system of L-branched $\mathbf{Z}/p^n\mathbf{Z}$-covers $\widetilde{M} = \{h_n : M_n \to M\}_n$ a *branched* $\mathbf{Z}_p$*-cover*, and regard it as an analogue of a $\mathbf{Z}_p$-field. Put $X := M - L$. Then a surjective homomorphism from the pro-p completion of the fundamental group $\tau : \widehat{\pi}_1(X) \twoheadrightarrow \mathbf{Z}_p$ corresponds to such $\widetilde{M}$. Assume that M_n is a $\mathbf{Q}\mathrm{HS}^3$ for any n. Then we have an Iwasawa type formula ([HMM06], [KM08], [Uek]):

$$v_p(H_1(M_n)) = \lambda n + \mu p^n + \nu \quad \text{for } n \gg 0,$$

for some $\lambda = \lambda_{\widetilde{M}}$, $\mu = \mu_{\widetilde{M}} \in \mathbf{N}$, $\nu = \nu_{\widetilde{M}} \in \mathbf{Z}$. These constants are called the Iwasawa invariants.

Next, we review an analogous object of an extension of $\mathbf{Z}_p$-fields introduced in [Uek]. Let $\widetilde{M} = \{h_n : M_n \to M\}_n$ be an L-branched $\mathbf{Z}_p$-cover and $\widetilde{M'} = \{h'_n : M'_n \to M'\}_n$ an L'-branched $\mathbf{Z}_p$-cover. Then a *branched Galois cover* $f : \widetilde{M'} \to \widetilde{M}$ of degree r is a compatible system of branched Galois covers $\{f_n : M'_n \to M_n\}_n$ of degree r such that each induced map $\mathrm{Gal}(f_{n+1}) \to \mathrm{Gal}(f_n)$ is an isomorphism. If L and L' are properly branched in $\widetilde{M}$ and $\widetilde{M'}$, then $L' = f_0^{-1}(L)$. We can easily see that the branch links S_n of f_n satisify $S_n \subset h_n^{-1}(S_0)$. We put $S' := f_0^{-1}(S)$, $Y := M - L \cup S$, and $Y' := M' - L' \cup S'$. Then, there is a commutative diagram

$$\begin{array}{ccc} \widehat{\pi_1}(Y') & \xrightarrow{\ \tau\ } & \mathbf{Z}_p \\ {\scriptstyle f_{0*}} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \iota}\ \cong \\ \widehat{\pi_1}(Y) & \xrightarrow{\ \tau'\ } & \mathbf{Z}_p \end{array}$$

for the defining homomorphisms τ, τ' of $\widetilde{M}, \widetilde{M}'$. Conversely, if $f_0 : M' \to M$ and such a diagram are given, then $\widetilde{M}' \to \widetilde{M}$ is defined.

Let M be a $\mathbf{Q}\mathrm{HS}^3$, let $L = \sqcup K_i$ be a link which consists of null-homologous components in M, and let μ_i denote the meridian of K_i for each i. Then, the branched $\mathbf{Z}_p$-cover $\widetilde{M}$ defined by $\tau : \pi_1(M - L) \to \mathbf{Z}; \forall \mu_i \mapsto 1$ is called *the total linking number* (or *TLN* for short) *$\mathbf{Z}_p$-cover over* (M, L).

Let Σ be a Seifert surface of L, that is, a compact orientable surface Σ satisfying $\partial\Sigma = L$. Then $M - \Sigma$ gives a fundamental domain of each $\mathbf{Z}/p^n\mathbf{Z}$-cover. Let $K \subset M - L$ be a knot, and assume that K and Σ intersect transversally (perturb Σ if necessary). Then the intersection number ι satisfies $\mathrm{lk}(K, L) = \iota(K, \Sigma)$. By a standard argument similar to [Mor12, Chapter 4.1], the natural map $H_1(X) \to \mathrm{Gal}(h_n) = \langle \sigma \mid \sigma^{p^n} \rangle$ sends $[K]$ to $\sigma^{v_p(\mathrm{lk}(K,L))}$. Therefore, if $\mathrm{lk}(K, L) \neq 0$ and $n \geq v_p(\mathrm{lk}(K, L))$, then each component of $h_n^{-1}(K)$ consists of $p^{n - v_p(\mathrm{lk}(K,L))}$ copies of K, and $h_n^{-1}(K)$ is a $p^{v_p(\mathrm{lk}(K,L))}$-component link. Otherwise, $h_n^{-1}(K)$ is a p^n-component link. In particular, we have the following

Proposition 5.1. *Let $\widetilde{M} = \{h_n : M_n \to M\}_n$ be the TLN-$\mathbf{Z}_p$-cover over (M, L), and $K \subset M - L$ a knot. Then, $\mathrm{lk}(K, L) \neq 0$ holds if and only if K is finitely decomposed into $p^{v_p(\mathrm{lk}(K,L))}$ components in h_n for all $n \gg 0$, and $\mathrm{lk}(K, L) = 0$ holds if and only if K is completely decomposed in all h_n.*

If S is finitely decomposed in $\widetilde{M}$, then $f : \widetilde{M}' \to \widetilde{M}$ resembles a p-extension of cyclotomic $\mathbf{Z}_p$-field. If S is completely decomposed in $\widetilde{M}$, then $f : \widetilde{M}' \to \widetilde{M}$ resembles the case of anti-cyclotomic.

Now we present our main theorems.

Theorem 5.2 (arbitrary large μ). *Let $f : \widetilde{M}' \to \widetilde{M}$ be a branched Galois cover of degree p of $\mathbf{Z}_p$-covers of $\mathbf{Q}HS^3$. Suppose that the branch link S of $f_0 : M' \to M$ is a t-component link, and that S is completely decomposed in $\widetilde{M}$. Then, $\mu_{\widetilde{M}'} \geq t$ holds.*

Proof. Since $f_n : M'_n \to M_n$ is of degree p and the branch link S_n of f_n is a tp^n-component link $h_n^{-1}(S)$, by Theorem 3.2, we have $\#H_1(M'_n)_G = \#H_1(M_n)p^{tp^n - 1}$. Then, $\#H_1(M'_n)_G | \#H_1(M'_n)$ implies $v_p(\#H_1(M'_n)) \geq v_p(\#H_1(M_n)) + tp^n - 1$, and hence $\mu_{\widetilde{M}'} \geq t$. □

Theorem 5.3 (many $\mu = 0$). *Let $f : \widetilde{M}' \to \widetilde{M}$ be a branched Galois p-cover of branched $\mathbf{Z}_p$-covers of $\mathbf{Q}HS^3$, and suppose that any knot branched in $f_0 : M' \to M$ is finitely decomposed in $\widetilde{M}$. Then $\mu_{\widetilde{M}} = 0$ if and only if $\mu_{\widetilde{M}'} = 0$.*

Proof. Since any finite p-group has a nontrivial center, we can reduce the argument to the case of degree p.

For a finite abelian p-group A, we put $\mathrm{rank}\, A := \dim A \otimes \mathbf{F}_p$. If $B < A$, then $\mathrm{rank}\, B$, $\mathrm{rank}\, A/B \leq \mathrm{rank}\, A \leq \mathrm{rank}\, B + \mathrm{rank}\, A/B$. If a group $G = \langle \sigma \rangle \cong \mathbf{Z}/p\mathbf{Z}$ acts on A, then $(1 - \sigma)^p$ acts on $A \otimes \mathbf{F}_p$ as zero, and $\mathrm{rank}\, A \leq \sum_{i=0}^{p-1} \mathrm{rank}\, A^{(1-\sigma)^i}/A^{(1-\sigma)^{i+1}} \leq p\, \mathrm{rank}\, A/A^{1-\sigma}$ holds.

Now let $f_0 : M' \to M$ be a branched cover of degree p with $\mathrm{Gal}(f_0) = \langle \sigma \rangle$. Let A and A' denote the p-parts of $H_1(M)$ and $H_1(M')$ respectively, and put $r = r_0 := \mathrm{rank}\, A$ and $r' = r'_0 := \mathrm{rank}\, A'$. Let $s = s_0$ denote the number of components of the branch link S of f_0. Then genus theory yields $r - 1 \leq r' \leq p(r + s)$. Indeed, let $M_{\mathrm{ab}} \to M$ and $M'_{\mathrm{ab}} \to M'$ denote the maximal unbranched abelian p-covers. Then, the relative genus cover $M'^{\mathrm{g}} \to M$ of $f_0 : M' \to M$ factors through $M_{\mathrm{ab}} \to M$ by definition. Put $r_{\mathrm{g}} := \mathrm{rank}\, \mathrm{Gal}(M'^{\mathrm{g}}/M')$. Then by Theorem 3.5, $\mathrm{Gal}(M'^{\mathrm{g}}/M') \cong A'/A'^{1-\sigma}$ holds, and hence $r' \leq pr_{\mathrm{g}}$. For each i, let $T_i < \mathrm{Gal}(M'_{\mathrm{ab}}/M)$ denote the inertia group of K_i in $M'^{\mathrm{g}} \to M$. Then, since $M'^{\mathrm{g}} \to M'$ is unbranched, we have $T_i \cong \mathbf{Z}/p\mathbf{Z}$, and $\mathrm{Gal}(M'^{\mathrm{g}}/M_{\mathrm{ab}}) = T_1 \cdots T_s$. Therefore $r_{\mathrm{g}} \leq \mathrm{rank}\, \mathrm{Gal}(M'^{\mathrm{g}}/M) \leq r + \mathrm{rank}\, \mathrm{Gal}(M'^{\mathrm{g}}/M_{\mathrm{ab}}) \leq r + s$. On the other hand, we have $r \leq \mathrm{rank}\, \mathrm{Gal}(M'^{\mathrm{g}}/M) \leq 1 + \mathrm{rank}\, \mathrm{Gal}(M'^{\mathrm{g}}/M') \leq 1 + r'$.

Similarly, for each $f_n : M'_n \to M_n$, let r_n and r'_n denote the p-ranks of $H_1(M_n)$ and $H_1(M'_n)$ respectively, and let s_n denote the number of component of the branch link S_n of f_n. By a similar argument, $r_n - 1 \leq r'_n \leq p(r_n + s_n)$ holds. Since $S_n \subset h_n^{-1}(S)$ and S is finitely decomposed in $\widetilde{M}$, $\{s_n\}_n$ is bounded. By Sakuma's exact sequence ([Uek, Proposition 4.11]), there is a finitely generated torsion $\Lambda = \mathbf{Z}_p[[T]]$-module $\mathcal{H}_{\widetilde{M}}$ with $H_1(M_n, \mathbf{Z}_p)/h_n^!(H_1(M, \mathbf{Z}_p)) \cong \mathcal{H}_{\widetilde{M}}/\nu_{p^n}\mathcal{H}_{\widetilde{M}}$ for any n, where $\nu_{p^n} = ((1+T)^{p^n} - 1)/T$. By the structure theorem of finitely generated Λ-modules ([Uek, Lemma 3.1 (4)]), $\mu_{\widetilde{M}} = 0$ (resp. $\mu_{\widetilde{M}'} = 0$) is equivalent to that $\{r_n\}_n$ (resp. $\{r'_n\}_n$) is bounded. Thus the assertion holds. □

Example 5.4. Let L and S be distinct unknots in $M = S^3$ and let $f_0 : M' \to M$ be the S-branched cover of degree p. Let $\widetilde{M}$ and $\widetilde{M}'$ denote the TLN-$\mathbf{Z}_p$-covers over (M, L) and (M', L') for $L' = f_0^{-1}(L)$ respectively. Then a branched Galois cover $f : \widetilde{M}' \to \widetilde{M}$ is defined and $\mu_{\widetilde{M}} = 0$ holds. If

$\widetilde{M'}$ consists of $\mathbf{Q}$HS^3's and if $\mathrm{lk}(L,S)=p$, then $\mu_{\widetilde{M'}}=0$ by Theorem 5.3. If $p=2$, then L' is the Hopf link. If $p=3$, then L' is the Borromean ring.

If $\mathrm{lk}(L,S)=0$, then L' is a split link, its Alexander polynomial is zero, $\widetilde{M'}$ does not consist of $\mathbf{Q}$HS^3's, and $\mu_{\widetilde{M'}}$ is not defined. Instead, let $L=K_1\cup K_2$ be a Hopf link, let $L\cup S$ be 6^3_3 in Rolfsen's table ([Rol76]) and fix orientations so that $\mathrm{lk}(K_1,S)=1$ and $\mathrm{lk}(K_2,S)=-1$. Then $\mu_{\widetilde{M}}=0$ and $\mathrm{lk}(L,S)=0$. If $p=2$, then $\mu_{\widetilde{M'}}$ is defined, and $\mu_{\widetilde{M'}}\geq 1$ by Theorem 5.2. Indeed, $L'=K'_1\cup K'_2$ is 4^2_1 in Rolfsen's table and $\mu_{\widetilde{M'}}=1$ holds.

Theorems 5.2 and 5.3 give branched $\mathbf{Z}_p$-covers $\widetilde{M'}$ which are candidates for $\mu=0$ and $\mu\geq t$. We can check whether $\widetilde{M'}$ consists of $\mathbf{Q}$HS^3's or not by using the Alexander polynomials. We note that various constructions of $\widetilde{M}$ with given λ,μ,ν are studied in [KM08] and [KM13].

Remark. A $\mathbf{Z}_p$-field with $\mu=0$ resembles a function field. Especially, as an analogue of the Riemann–Hurwitz formula for a cover of Riemann surfaces, Kida's formula for a p-extension of $\mathbf{Z}_p$-fields with $\mu=0$ is known. In [Uek], following Iwasawa's second proof in [Iwa81], their analogue in the topological context was formulated. It describes the balance of Iwasawa λ-invariants, covering degree, and branching indices. We employed representation theory of finite groups, and Tate cohomology of 2-cycles $\widehat{H}^i(G,Z_2(\widetilde{N}))$. Meanwhile, Kida's formula for $\mathbf{Z}_p$-fields extension was first proved with use of genus theory ([Kid80]). We expect an alternative proof for our formula by following Kida's proof.

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References

[Bro94] K. S. Brown, *Cohomology of groups*, corrected reprint of the 1982 original, Graduate Texts in Mathematics, 87, Springer, New York, 1994.

[Fur67] Y. Furuta, The genus field and genus number in algebraic number fields, Nagoya Math. J. **29** (1967), 281–285.

[FW79] B. Ferrero and L. C. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, Ann. of Math. (2) **109** (1979), no. 2, 377–395.

[HMM06] J. Hillman, D. Matei and M. Morishita, Pro-p link groups and p-homology groups, in *Primes and knots*, Contemp. Math., 416, Amer. Math. Soc., Providence, RI, 2006, pp. 121–136.

[Iwa59] K. Iwasawa, On Γ-extensions of algebraic number fields, Bull. Amer. Math. Soc. **65** (1959), 183–226.

[Iwa73] K. Iwasawa, On the μ-invariants of Z_ℓ-extensions, in *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, Kinokuniya, Tokyo, 1973, pp. 1–11.

[Iwa81] K. Iwasawa, Riemann-Hurwitz formula and p-adic Galois representations for number fields, Tôhoku Math. J. (2) **33** (1981), no. 2, 263–288.

[Kid80] Y. Kida, l-extensions of CM-fields and cyclotomic invariants, J. Number Theory **12** (1980), no. 4, 519–528.

[KM08] T. Kadokami and Y. Mizusawa, Iwasawa type formula for covers of a link in a rational homology sphere, J. Knot Theory Ramifications **17** (2008), no. 10, 1199–1221.

[KM13] T. Kadokami and Y. Mizusawa, On the Iwasawa invariants of a link in the 3-sphere, Kyushu J. Math. **67** (2013), no. 1, 215–226.

[Mor01] M. Morishita, A theory of genera for cyclic coverings of links, Proc. Japan Acad. Ser. A Math. Sci. **77** (2001), no. 7, 115–118.

[Mor12] M. Morishita, *Knots and primes*, Universitext, Springer, London, 2012.

[Nii14] H. Niibo, Idèlic class field theory for 3-manifolds, Kyushu J. Math. **68** (2014), no. 2, 421–436.

[NU] H. Niibo and J. Ueki, Idèlic class field theory for 3-manifolds and very admissible links, arXiv:1501.03890.

[Och14] T. Ochiai, *Iwasawa theory and its perspective I*, Iwanami Studies in Advanced Mathematics, Iwanami Shoten, Tokyo, 2014.

[Rol76] D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, CA, 1976.

[Uek] J. Ueki, On the Iwasawa invariants for links and Kida's formula, arXiv:1605.09036.

[Uek14] J. Ueki, On the homology of branched coverings of 3-manifolds, Nagoya Math. J. **213** (2014), 21–39.

[Yok67] H. Yokoi, On the class number of a relatively cyclic number field, Nagoya Math. J. **29** (1967), 31–44.