An alternative proof of the non-Archimedean Montel theorem for rational dynamics

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Abstract: We will see an alternative proof of the non-Archimedean Montel theorem for rational dynamics by using the non-Archimedean Green functions.

Key words: Non-Archimedean dynamics; Green functions; Montel's theorem.

1. Introduction. In [Mont16], P. Montel proved that a family of holomorphic functions defined on an open set in the field of complex numbers is normal if the family is uniformly bounded. It can be applied to the theory of complex dynamical systems for some basic but important facts. See [Miln06] for complex dynamical systems.

Not only in complex dynamical systems, but also in *non-Archimedean dynamical systems*, which investigate the iterations of a rational map over algebraically closed, complete, and non-Archimedean fields, such a theorem is also important. In [Hs00], L.-C. Hsia originally proved the non-Archimedean Montel theorem, which is also called *Hsia's criterion*, and applied it to non-Archimedean dynamical systems. C. Favre, J. Kiwi and E. Trucco also proved several versions of Montel's theorem in a non-Archimedean setting and applied them to dynamics on the Berkovich projective and affine line in [FKT12]. The aim of this paper is to give an alternative and simple proof of Hsia's criterion for rational dynamics by using two fundamental results on non-Archimedean Green functions, namely [KS09, Theorem 6], [KS09, Proposition 3] and the idea of the author's paper [L15]. We will review the rigorous definition of the non-Archimedean Green functions and some properties of them in Section 2.

Let K be an algebraically closed field with a complete and non-Archimedean norm $|\cdot|$. The *projective line* $\mathbf{P}^1_K$ over K is defined as the quotient space $(K^2\backslash\{\mathbf{0}\})/\sim$ where $\sim$ is the equivalence relation defined by: $(z_0, z_1) \sim (w_0, w_1)$ if there exists a non-zero element c in K such that $(z_0, z_1) = (c\cdot w_0, c\cdot w_1)$. The *chordal metric* ρ on $\mathbf{P}^1_K$ is defined by

$$\rho((z_0 : z_1), (w_0 : w_1)) = \frac{|z_0\cdot w_1 - w_0\cdot z_1|}{\max\{|z_0|, |z_1|\}\cdot\max\{|w_0|, |w_1|\}}$$

for all $(z_0 : z_1)$ and $(w_0 : w_1)$ in $\mathbf{P}^1_K$. A *rational map* $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ of degree d over K is a map given by

$$f(z_0 : z_1) = (F_0(z_0, z_1) : F_1(z_0, z_1))$$

where F_0 and F_1 are two-variable homogeneous polynomials of degree d over K with no common factors. Note that we can identify $(z : 1)$ in $\mathbf{P}^1_K$ with z in K and $(1 : 0)$ in $\mathbf{P}^1_K$ with ∞. Hence a rational map naturally induces an action on $K \cup \{\infty\}$. In the rest of this paper, we will regard the projective line $\mathbf{P}^1_K$ as the union $K \cup \{\infty\}$ and a rational map as an action on $K \cup \{\infty\}$. Moreover, for a given rational map $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$, let us denote by f^k the *k-th iteration* of f.

Theorem 1.1. *Let U be a non-empty open subset in $\mathbf{P}^1_K$ and $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ a rational map of degree $d \geq 2$ over K. Suppose that there exist two distinct elements α and β in $\mathbf{P}^1_K$ such that*

$$\bigcup_{k=0}^{\infty} f^k(U) \cap \{\alpha, \beta\} = \emptyset.$$

Then $\{f^k\}_{k=0}^{\infty}$ is equicontinuous on U with respect to the chordal metric on $\mathbf{P}^1_K$.

Remark that our proof is simpler but the original statement [Hs00, Theorem 2.4] holds in a more general case than Theorem 1.1. That is, Hsia considers any family of meromorphic functions on the closed disk $\{z \in K \mid |z - a| \leq r\}$, while in Theorem 1.1 we consider the family $\{f^k\}_{k=0}^{\infty}$ given by the iterations of a rational map f.

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We end this section with some typical applications of Theorem 1.1 to non-Archimedean dynamical systems. For a rational map $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$, the *Fatou set* is defined as the largest open set of $\mathbf{P}^1_K$ on which $\{f^k\}_{k=0}^{\infty}$ is equicontinuous with respect to the chordal metric and the *Julia set* is defined as the complement of the Fatou set.

Corollary 1.2. *Let $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a rational map over K of degree ≥ 2. Then the following statements hold*:

(a) *The Julia set of f has an empty interior if the Fatou set is non-empty.*

(b) *The Julia set of f is uncountable if the Julia set is non-empty.*

(c) *The Julia set of f has no isolated points.*

(d) *The backward orbit of any points in the Julia set is dense in the Julia set.*

See [Silv07, Corollary 5.32] for the proof of Corollary 1.2. Remark that R. Benedetto proved that the Fatou set of a rational map whose degree is greater than one is non-empty in [Ben01] thus the Julia set always has empty interior if the degree of the rational map is greater than one.

2. Non-Archimedean Green functions. In this section, we will recall the definition and some properties of the non-Archimedean Green functions from [KS09]. Let us begin with the definition of a lift of a rational map.

Definition 2.1. Let $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a rational map over K. Then f can be written as

$$f = (F_0 : F_1)$$

with two-variable homogeneous polynomials F_0, F_1 over K. We call the induced map $F = (F_0, F_1) : K^2 \to K^2$ a *lift* of f.

Note that the lift is unique up to multiplication by a constant in $K\backslash\{0\}$ and the k-th iteration of a lift of a given rational map is a lift of the k-th iteration of the rational map for any natural number k. To ease notation, let us denote the k-th iteration of a lift F of a given rational map by

$$F^k = (F_0^k, F_1^k)$$

for any non-negative integer k.

Next, as in [KS09, equation (2)], we define non-Archimedean Green functions. For a given (z, w) in K^2, let $\|(z, w)\|$ denote $\max\{|z|, |w|\}$.

Proposition 2.2. *Let $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a rational map of degree d over K and $F : K^2 \to K^2$ a lift of f. The Green function $\mathcal{G}_F$ of F*

$$\mathcal{G}_F : \mathbf{P}^1_K \to \mathbf{R}$$

$$(z_0 : z_1) \mapsto \lim_{k\to\infty} \frac{1}{d^k} \cdot \log\|F^k(z_0, z_1)\| - \log\|(z_0, z_1)\|$$

is well-defined.

See [KS09, Proposition 3] for the proof of Proposition 2.2. We end this section with a one dimensional version of [KS09, Theorem 6].

Theorem 2.3. *Let $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a rational map of degree $d \geq 2$ and $F : K^2 \to K^2$ a lift of f. If the Green function $\mathcal{G}_F$ is constant on an open neighborhood of $(z_0 : z_1)$, then $\{f^k\}_{k=0}^{\infty}$ is equicontinuous on the neighborhood.*

3. The proof of Theorem 1.1. In order to prove Theorem 1.1, we will use the following lemma.

Lemma 3.1. *Let $f : K \to K$ be a non-constant polynomial map of degree d. Suppose that f has no zeros in $\{z \in K \mid |z - a| \leq r\}$ for some a in K and $r > 0$. Then for any w in $\{z \in K \mid |z - a| \leq r\}$, we have*

$$|f(w)| = |f(a)|.$$

Proof of Lemma 3.1. Since K is algebraically closed, there exist a non-zero element c and a sequence $\{\alpha_k\}_{k=1}^{d}$ in K such that

$$f(w) = c \cdot (w - \alpha_1) \cdot (w - \alpha_2) \cdot \cdots \cdot (w - \alpha_d).$$

Since f has no zeros in $\{z \in K \mid |z - a| \leq r\}$, we have $|\alpha_k - a| > r$ for any k in $\{1, 2, \cdots, d\}$ thus we have

$$\begin{aligned}|f(w)| &= |c| \cdot |w - \alpha_1| \cdot |w - \alpha_2| \cdot \cdots \cdot |w - \alpha_d| \\ &= |c| \cdot |w - a + a - \alpha_1| \cdot |w - a + a - \alpha_2| \cdot \\ &\quad \cdots \cdot |w - a + a - \alpha_d| \\ &= |c| \cdot |a - \alpha_1| \cdot |a - \alpha_2| \cdot \cdots \cdot |a - \alpha_d| = |f(a)|\end{aligned}$$

for any w in $\{z \in K \mid |z - a| \leq r\}$. □

Proof of Theorem 1.1. Let us choose a lift $F = (F_0, F_1) : K^2 \to K^2$ of f. It is sufficient to prove the case when $\{\alpha, \beta\} = \{0, \infty\}$ and $U = \{z \in K \mid |z - a| \leq r\}$ for some a in K and $r > 0$. Since

$$\bigcup_{k=0}^{\infty} f^k(U) \cap \{0, \infty\} = \emptyset,$$

the polynomial maps $F_1^k(w, 1)$ and $F_2^k(w, 1)$ have no zeros in $\{z \in K \mid |z - a| \leq r\}$ for any k in $\{0, 1, \cdots\}$. Thus it follows from Lemma 3.1 that

$$|F_i^k(w, 1)| = |F_i^k(a, 1)|$$

for any i in $\{0,1\}$, any k in $\{0,1,\cdots\}$, and any w in U. In particular, this implies

$$\begin{aligned}\mathcal{G}_F(w:1) &= \lim_{k\to\infty}\frac{1}{d^k}\cdot\log\|F^k(w,1)\| - \log\|(w,1)\| \\ &= \lim_{k\to\infty}\frac{1}{d^k}\cdot\log\|F^k(a,1)\| - \log\|(a,1)\| \\ &= \mathcal{G}_F(a:1).\end{aligned}$$

Therefore the Green function $\mathcal{G}_F$ of F is constant on U, hence, by Theorem 2.3, the family $\{f^k\}_{k=0}^{\infty}$ is equicontinuous on U. □

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