The archimedean zeta integrals for $GL(3) \times GL(2)$

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Abstract: We consider here the archimedean zeta integrals for $GL(3) \times GL(2)$ and show that the zeta integral for appropriate Whittaker functions is equal to the associated L-factor.

Key words: Whittaker fucntions; automorphic forms; zeta integrals.

1. Introduction. This paper gives a new result which is a continuation of our former paper [HIM] on the archimedean Whittaker functions on $GL(3)$.

In the case of $GL(n+1) \times GL(n)$, we expect that the archimedean zeta integrals for appropriate Whittaker functions are equal to the associated L-factors. This expectation is well-grounded by Stade's result [St] for the spherical $GL(n+1, \mathbf{R}) \times GL(n, \mathbf{R})$-case and Popa's result [Po] for $GL(2) \times GL(1)$-case. We discuss here the archimedean zeta integrals for $GL(3) \times GL(2)$ and give an additional evidence for this expectation. The main result in this paper is based on explicit computation using our explicit formulas of non-spherical Whittaker functions on $GL(3)$ and $GL(2)$. Since the archimedean zeta integrals at the minimal K-types may vanish in some cases (*cf.* Lemma 4.2), the appropriate choices of K-type vectors are important.

2. Preliminaries.

2.1. Notation. Let F be $\mathbf{R}$ or $\mathbf{C}$. For $l \in F$, we define the unitary character ψ_l of F by

$$\psi_l(\xi) = \begin{cases} e^{2\pi\sqrt{-1}l\xi} & \text{if } F = \mathbf{R}, \\ e^{2\pi\sqrt{-1}(l\xi+\overline{l\xi})} & \text{if } F = \mathbf{C} \end{cases} \qquad (\xi \in F).$$

We define the norm $|\cdot|_F$ on F by $|\xi|_{\mathbf{R}} = |\xi|$ and $|\xi|_{\mathbf{C}} = |\xi|^2$ where $|\cdot|$ is the ordinary absolute value. We set $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ as usual. We denote by 1_n the unit matrix of degree n, and by E_{ij} the matrix unit of size 3×3 with 1 at (i,j)-th entry and 0 at other entries.

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2.2. Basic objects. Let G_n be the general linear group $GL(n,F)$ of degree n over F. Let N_n be the group of upper triangular matrices in G_n with diagonal entries equal to 1, and let A_n be the group of diagonal matrices with positive diagonal entries. Moreover, we fix a maximal compact subgroup K_n of G_n by

$$K_n = \begin{cases} O(n) & \text{if } F = \mathbf{R}, \\ U(n) & \text{if } F = \mathbf{C}. \end{cases}$$

Then we have an Iwasawa decomposition $G_n = N_n A_n K_n$. It is convenient to take the coordinates on N_n and A_n as follows:

$$x = (x_{ij}) \in N_n,$$
$$y = \mathrm{diag}(y_1 y_2 \cdots y_n,\ y_2 \cdots y_n,\ \cdots,\ y_n) \in A_n,$$

where $x_{ij} \in F$, $x_{ii} = 1$ $(1 \le i \le n)$, $x_{ij} = 0$ $(1 \le j < i \le n)$, and $y_k \in \mathbf{R}_{>0}$ $(1 \le k \le n)$. For $n_1, n_2, \cdots, n_m \in \mathbf{Z}_{>0}$ with $n_1 + n_2 + \cdots + n_m = n$, we associate the upper triangular parabolic subgroup $P_{n_1,n_2,\cdots,n_m}$ of G_n, whose Levi component is isomorphic to $G_{n_1} \times G_{n_2} \times \cdots \times G_{n_m}$.

2.3. Irreducible representations of K_n. Here we introduce some notations for representations of the maximal compact subgroup K_n of G_n with $n = 2, 3$. We regard K_2 as a subgroup of K_3 via the embedding

$$K_2 \ni k \mapsto \left(\begin{array}{c|c} k & \\ \hline & 1 \end{array}\right) \in K_3. \tag{2.1}$$

For $F = \mathbf{R}$, the equivalence classes of irreducible representations of $K_n = O(n)$ can be parameterized by the set

$$\Lambda_n = \begin{cases} \{(\lambda_1, 0) \mid \lambda_1 \in \mathbf{Z}_{\ge 0}\} \cup \{(0,1)\} & \text{if } n = 2, \\ \{(\lambda_1, \lambda_2) \mid \lambda_1 \in \mathbf{Z}_{\ge 0},\ \lambda_2 \in \{0,1\}\} & \text{if } n = 3. \end{cases}$$

We denote the representation of K_n associated to $\lambda = (\lambda_1, \lambda_2) \in \Lambda_n$ by $(\tau_\lambda^{(n)}, V_\lambda^{(n)})$. For $n = 2$, the dimension of the representation space $V_\lambda^{(2)}$ is 1 ($\lambda_1 = 0$) or 2 ($\lambda_1 \ne 0$), and we can take a basis $\{v_{\lambda,q}\}_{q \in S_\lambda}$,

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$S_\lambda = \{\pm\lambda_1\}$ of $V_\lambda^{(2)}$ characterized by the action

$$\tau_\lambda^{(2)}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)v_{\lambda,q} = e^{\sqrt{-1}q\theta}v_{\lambda,q} \quad (\theta \in \mathbf{R}),$$

$$\tau_\lambda^{(2)}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)v_{\lambda,q} = (-1)^{\lambda_2}v_{\lambda,-q}.$$

For $\lambda = (\lambda_1, \lambda_2) \in \Lambda_3$, let $\mathcal{P}_\lambda$ be the $\mathbf{C}$-vector space of degree λ_1 homogeneous polynomials of three variables z_1, z_2, z_3, and we define the action T_λ of K_3 on $\mathcal{P}_\lambda$ by

$$(T_\lambda(k)p)(z_1, z_2, z_3) = (\det k)^{\lambda_2}p((z_1, z_2, z_3) \cdot k)$$

for $k \in K_3$ and $p \in \mathcal{P}_\lambda$. Here $(z_1, z_2, z_3) \cdot k$ is the ordinal product of matrices. For $n = 3$, we regard $\tau_\lambda^{(3)}$ as the quotient representation of T_λ on $V_\lambda^{(3)} = \mathcal{P}_\lambda/(z_1^2 + z_2^2 + z_3^2)\mathcal{P}_{\lambda-(2,0)}$. Here we put $\mathcal{P}_{\lambda-(2,0)} = \{0\}$ if $\lambda - (2, 0) \notin \Lambda_3$. As a K_2-module, we have $V_\lambda^{(3)} \simeq \bigoplus_{\mu\in\Sigma(\lambda)} V_\mu^{(2)}$ with

$$\Sigma(\lambda) = \{(0, \lambda_2)\} \cup \{(\mu_1, 0) \mid \mu_1 \in \mathbf{Z},\ 1 \le \mu_1 \le \lambda_1\},$$

via the correspondence $v_{\mu,q}^\lambda \leftrightarrow v_{\mu,q}$ ($\mu = (\mu_1, \mu_2) \in \Sigma(\lambda)$, $q \in S_\mu$), where $v_{\mu,q}^\lambda$ is the image of

$$\begin{cases} (z_1 + \sqrt{-1}z_2)^{\mu_1} z_3^{\lambda_1-\mu_1} & \text{if } q \ge 0, \\ (-1)^{\lambda_2}(-z_1 + \sqrt{-1}z_2)^{\mu_1} z_3^{\lambda_1-\mu_1} & \text{if } q < 0 \end{cases}$$

under the natural surjection $\mathcal{P}_\lambda \to V_\lambda^{(3)}$.

For $F = \mathbf{C}$, the equivalence classes of irreducible representations of $K_n = U(n)$ can be parameterized by the set

$$\Lambda_n = \begin{cases} \{(\lambda_1, \lambda_2) \in \mathbf{Z}^2 \mid \lambda_1 \ge \lambda_2\} & \text{if } n = 2, \\ \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Z}^3 \mid \lambda_1 \ge \lambda_2 \ge \lambda_3\} & \text{if } n = 3 \end{cases}$$

and we denote the representation of K_n associated to $\lambda \in \Lambda_n$ by $(\tau_\lambda^{(n)}, V_\lambda^{(n)})$ as before. For $n = 2$, we regard $V_\lambda^{(2)}$ as the $\mathbf{C}$-vector space of degree $\lambda_1 - \lambda_2$ homogeneous polynomials in two variables z_1, z_2, on which K_2 acts by

$$(\tau_\lambda^{(2)}(k)p)(z_1, z_2) = \det(k)^{\lambda_2}p((z_1, z_2) \cdot k)$$

for $k \in K_2$ and $p \in V_\lambda^{(2)}$. We define a basis $\{v_{\lambda,q}\}_{q\in S_\lambda}$, $S_\lambda = \{q \in \mathbf{Z} \mid 0 \le q \le \lambda_1 - \lambda_2\}$ of $V_\lambda^{(2)}$ by $v_{\lambda,q}(z_1, z_2) = z_1^{\lambda_1-\lambda_2-q}z_2^q$. For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda_3$, let $\mathcal{P}_\lambda$ be the $\mathbf{C}$-vector space consisting of polynomials of six variables $z_1, z_2, z_3, z_{23}, z_{13}, z_{12}$ which are degree $\lambda_1 - \lambda_2$ homogeneous with respect to three variables z_1, z_2, z_3 and are degree $\lambda_2 - \lambda_3$ homogeneous with respect to three variables z_{23}, z_{13}, z_{12}. We define the action T_λ of K_3 on $\mathcal{P}_\lambda$ by

$$(T_\lambda(k)p)(z_1, z_2, z_3, z_{23}, z_{13}, z_{12})$$
$$= \det(k)^{\lambda_3}p((z_1, z_2, z_3) \cdot k, (z_{23}, z_{13}, z_{12}) \cdot \tilde{k})$$

for $k \in K_3$ and $p \in \mathcal{P}_\lambda$. Here $\tilde{k} = (\tilde{k}_{ij}) \in G_3$ is a matrix defined by

$$\tilde{k}_{ij} = \begin{vmatrix} k_{i_1j_1} & k_{i_1j_2} \\ k_{i_2j_1} & k_{i_2j_2} \end{vmatrix}$$

with $1 \le i_1 < i_2 \le 3$, $1 \le j_1 < j_2 \le 3$ such that $i \notin \{i_1, i_2\}$, $j \notin \{j_1, j_2\}$, for $k = (k_{ij}) \in K_3$. For $n = 3$, we regard $\tau_\lambda^{(3)}$ as the quotient representation of T_λ on $V_\lambda^{(3)} = \mathcal{P}_\lambda/(z_1z_{23} - z_2z_{13} + z_3z_{12})\mathcal{P}_{\lambda-(2,1,0)}$. Here we put $\mathcal{P}_{\lambda-(2,1,0)} = \{0\}$ if $\lambda - (2, 1, 0) \notin \Lambda_3$. As a K_2-module, we have $V_\lambda^{(3)} \simeq \bigoplus_{\mu\in\Sigma(\lambda)} V_\mu^{(2)}$ with

$$\Sigma(\lambda) = \{(\mu_1, \mu_2) \in \mathbf{Z}^2 \mid \lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \lambda_3\},$$

via the correspondence $v_{\mu,q}^\lambda \leftrightarrow v_{\mu,q}$ ($\mu = (\mu_1, \mu_2) \in \Sigma(\lambda)$, $q \in S_\mu$), where $v_{\mu,q}^\lambda$ is the image of

$$\binom{\mu_1 - \mu_2}{q}^{-1} \sum_{i=\max\{0,\mu_2-\lambda_2+q\}}^{\min\{q,\mu_1-\lambda_2\}} \binom{\mu_1 - \lambda_2}{i}\binom{\lambda_2 - \mu_2}{q - i}$$
$$\times z_1^{\mu_1-\lambda_2-i} z_{13}^{\lambda_2-\mu_2-q+i} z_2^i z_{23}^{q-i} z_3^{\lambda_1-\mu_1} z_{12}^{\mu_2-\lambda_3}$$

under the natural surjection $\mathcal{P}_\lambda \to V_\lambda^{(3)}$. Here $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the binomial coefficient.

3. Whittaker functions on G_n.

3.1. The definition of Whittaker functions. For $l \in F^\times$, let $C^\infty(N_n\backslash G_n; \psi_l)$ be the space of all smooth-functions $f: G_n \to \mathbf{C}$ satisfying

$$f(xg) = \psi_l(x_{12} + x_{23} + \cdots + x_{n-1\,n})f(g)$$

for $x = (x_{ij}) \in N_n$ and $g \in G_n$. Here G_n acts on this space by the right translation, and we equip this space with the topology of uniform convergence on compact sets of a function and its derivatives.

For an irreducible admissible Hilbert representation (Π, H_Π) of G_n, the space

$$\mathrm{Hom}_{G_n}(H_\Pi^\infty,\ C^\infty(N_n\backslash G_n; \psi_l))$$

of continuous G_n-homomorphisms is at most one dimensional ([Sha]), where H_Π^∞ is the subspace of H_Π consisting of all smooth vectors. If there is a non-zero homomorphism in this space, we denote the K_n-finite part of its image by $\mathcal{W}(\Pi, \psi_l)_{K_n}$ and say that Π is generic. Functions in $\mathcal{W}(\Pi, \psi_l)_{K_n}$ are called (K_n-finite) Whittaker functions for Π.

3.2. The archimedean L- and ϵ-factors. We recall the L- and ϵ-factors corresponding to finite dimensional semisimple representations of the Weil group W_F for F. See [HIM, §5.1 and §5.2] for details.

For $F=\mathbf{R}$, the set of equivalence classes of irreducible representations of the Weil group $W_{\mathbf{R}}$ is exhausted by characters ϕ_ν^δ ($\nu\in\mathbf{C}$, $\delta\in\{0,1\}$) and two dimensional representations $\phi_{\nu,\kappa}$ ($\nu\in\mathbf{C}$, $\kappa\in\mathbf{Z}_{>0}$), whose L- and ϵ-factors are given as follows:

$$L(s,\phi_\nu^\delta)=\Gamma_{\mathbf{R}}(s+\nu+\delta),\quad \epsilon(s,\phi_\nu^\delta,\psi_1)=(\sqrt{-1})^\delta,$$
$$L(s,\phi_{\nu,\kappa})=\Gamma_{\mathbf{C}}(s+\nu),\quad \epsilon(s,\phi_{\nu,\kappa},\psi_1)=(\sqrt{-1})^{\kappa+1}.$$

For $F=\mathbf{C}$, the set of equivalence classes of irreducible representations of the Weil group $W_{\mathbf{C}}$ is exhausted by characters ϕ_ν^d ($\nu\in\mathbf{C}$, $d\in\mathbf{Z}$), whose L- and ϵ-factors are given as follows:

$$L(s,\phi_\nu^d)=\Gamma_{\mathbf{C}}(s+\nu+|d|/2),$$
$$\epsilon(s,\phi_\nu^d,\psi_1)=(\sqrt{-1})^{|d|}.$$

For a finite dimensional semisimple representation ϕ of W_F with the irreducible decomposition $\phi\simeq\bigoplus_{i=1}^m\phi_i$, we define its L- and ϵ-factors by

$$L(s,\phi)=\prod_{i=1}^m L(s,\phi_i),\quad \epsilon(s,\phi,\psi_1)=\prod_{i=1}^m\epsilon(s,\phi_i,\psi_1).$$

By the local Langlands correspondence, an irreducible admissible representation Π of G_n corresponds to an n-dimensional semisimple representation $\phi[\Pi]$ of W_F. Let Π and π be irreducible admissible representations of G_3 and G_2, respectively. Then we define the archimedean L- and ϵ-factors for $\Pi\times\pi$ by

$$L(s,\Pi\times\pi)=L(s,\phi[\Pi]\otimes\phi[\pi]),$$
$$\epsilon(s,\Pi\times\pi,\psi_1)=\epsilon(s,\phi[\Pi]\otimes\phi[\pi],\psi_1).$$

Observe the equivalences

$$\phi_\nu^\delta\otimes\phi_{\nu'}^{\delta'}\simeq\phi_{\nu+\nu'}^{\delta-\delta'}\quad\text{if }\delta\ge\delta',$$
$$\phi_{\nu,\kappa}\otimes\phi_{\nu'}^{\delta'}\simeq\phi_{\nu+\nu',\kappa},$$
$$\phi_{\nu,\kappa}\otimes\phi_{\nu',\kappa'}\simeq\phi_{\nu+\nu',\kappa+\kappa'}\oplus\phi_{\nu+\nu'-\kappa',\kappa-\kappa'}\text{ if }\kappa>\kappa',$$
$$\phi_{\nu,\kappa}\otimes\phi_{\nu',\kappa}\simeq\phi_{\nu+\nu',2\kappa}\oplus\phi^0_{\nu+\nu'-\kappa}\oplus\phi^1_{\nu+\nu'-\kappa}$$

for $F=\mathbf{R}$ and the equivalence $\phi_\nu^d\otimes\phi_{\nu'}^{d'}\simeq\phi_{\nu+\nu'}^{d+d'}$ for $F=\mathbf{C}$, we can write the archimedean L- and ϵ-factors for $\Pi\times\pi$, explicitly.

3.3. Generic representations of G_n. It is known that any irreducible admissible generic representation of G_n is infinitesimally equivalent with an irreducible generalized principal series representation ([Ja, Lemma 2.5]). Here we recall some facts for irreducible admissible generic representations of G_n with $n=2,3$.

First, we set $F=\mathbf{R}$. We shall specify certain irreducible representations of G_1 and G_2 as follows:

(1) For $\nu\in\mathbf{C}$ and $\delta\in\{0,1\}$, let $\chi_{(\nu,\delta)}\colon G_1\to\mathbf{C}^\times$ be the character defined by
$$\chi_{(\nu,\delta)}(t)=(t/|t|)^\delta|t|_{\mathbf{R}}^\nu\quad(t\in G_1=\mathbf{R}^\times).$$

(2) For $\nu\in\mathbf{C}$ and $\kappa\in\mathbf{Z}_{\ge2}$, let $D_{(\nu,\kappa)}$ be the representation of G_2 characterized by $D_{(\nu,\kappa)}(t1_2)=t^{2\nu}$ ($t\in\mathbf{R}_{>0}$) and $D_{(\nu,\kappa)}|_{SL(2,\mathbf{R})}\simeq D_\kappa^+\oplus D_\kappa^-$, where $D_\kappa^\pm$ is the discrete series representation of $SL(2,\mathbf{R})$ with Blattner parameter $\pm\kappa$.

For $n=3$, any irreducible admissible generic representation Π of G_3 satisfies one of the following:

(1) $\Pi\simeq\mathrm{Ind}_{P_{1,1,1}}^{G_3}(\chi_{(\nu_1,\delta_1)}\boxtimes\chi_{(\nu_2,\delta_2)}\boxtimes\chi_{(\nu_3,\delta_3)})$ for some ν_i,δ_i with $\delta_1\ge\delta_2\ge\delta_3$. The minimal K_3-type is $\tau^{(3)}_{(\delta_1-\delta_3,\delta_2)}$, and $\phi[\Pi]=\phi_{\nu_1}^{\delta_1}\oplus\phi_{\nu_2}^{\delta_2}\oplus\phi_{\nu_3}^{\delta_3}$.

(2) $\Pi\simeq\mathrm{Ind}_{P_{2,1}}^{G_3}(D_{(\nu_1,\kappa)}\boxtimes\chi_{(\nu_2,\delta)})$ for some ν_i, κ, δ. The minimal K_3-type is $\tau^{(3)}_{(\kappa,\delta)}$, and $\phi[\Pi]=\phi_{\nu_1+(\kappa-1)/2,\,\kappa-1}\oplus\phi_{\nu_2}^\delta$.

For $n=2$, any irreducible admissible generic representation π of G_2 satisfies one of the following:

(1) $\pi\simeq\mathrm{Ind}_{P_{1,1}}^{G_2}(\chi_{(\nu_1',\delta_1')}\boxtimes\chi_{(\nu_2',\delta_2')})$ for some ν_i',δ_i' with $\delta_1'\ge\delta_2'$. The K_2-types are $\tau^{(2)}_{\lambda'}$ ($\lambda'\in\Lambda(\pi)$) with
$$\Lambda(\pi)=\{(\delta_1'-\delta_2',\delta_2')\}\cup\{(\lambda_1',0)\mid\lambda_1'\in\delta_1'-\delta_2'+2\mathbf{Z}_{>0}\},$$
and $\phi[\pi]=\phi_{\nu_1'}^{\delta_1'}\oplus\phi_{\nu_2'}^{\delta_2'}$.

(2) $\pi\simeq D_{(\nu',\kappa')}$ for some ν', κ'. The K_2-types are $\tau^{(2)}_{\lambda'}$ ($\lambda'\in\Lambda(\pi)$) with
$$\Lambda(\pi)=\{(\lambda_1',0)\mid\lambda_1'\in\kappa'+2\mathbf{Z}_{\ge0}\}$$
and $\phi[\pi]=\phi_{\nu'+(\kappa'-1)/2,\kappa'-1}$.

Next, we set $F=\mathbf{C}$. For $\nu\in\mathbf{C}$ and $d\in\mathbf{Z}$, let $\chi_{[\nu,d]}\colon G_1\to\mathbf{C}^\times$ be the character defined by
$$\chi_{[\nu,d]}(t)=(t/|t|)^d|t|_{\mathbf{C}}^\nu\quad(t\in G_1=\mathbf{C}^\times).$$

For $n=3$, any irreducible admissible generic representation Π of G_3 satisfies $\Pi\simeq\mathrm{Ind}_{P_{1,1,1}}^{G_3}(\chi_{[\nu_1,d_1]}\boxtimes\chi_{[\nu_2,d_2]}\boxtimes\chi_{[\nu_3,d_3]})$ for some ν_i, d_i with $d_1\ge d_2\ge d_3$. The minimal K_3-type of Π is $\tau^{(3)}_{(d_1,d_2,d_3)}$, and $\phi[\Pi]=\phi_{\nu_1}^{d_1}\oplus\phi_{\nu_2}^{d_2}\oplus\phi_{\nu_3}^{d_3}$. For $n=2$, any irreducible admissible generic representation π of G_2 satisfies $\pi\simeq\mathrm{Ind}_{P_{1,1}}^{G_2}(\chi_{[\nu_1',d_1']}\boxtimes\chi_{[\nu_2',d_2']})$ for some ν_i', d_i' with $d_1'\ge d_2'$. The K_2-types of π are $\tau^{(2)}_{\lambda'}$ ($\lambda'\in\Lambda(\pi)$) with
$$\Lambda(\pi)=\{(d_1'+m,d_2'-m)\mid m\in\mathbf{Z}_{\ge0}\},$$
and $\phi[\pi]=\phi_{\nu_1'}^{d_1'}\oplus\phi_{\nu_2'}^{d_2'}$.

3.4. Explicit formulas. Here we introduce the results for explicit formulas of the radial parts of Whittaker functions on G_n with $n=2,3$.

First, we set $n=3$. Let (Π,H_Π) be an irreducible admissible generic representation of G_3, and take $\lambda \in \Lambda_3$ such that $\tau_\lambda^{(3)}$ is the minimal K_3-type of Π. Let $\mathbf{W}_\Pi : V_\lambda^{(3)} \to \mathcal{W}(\Pi, \psi_l)_{K_3}$ be a K_3-embedding which is unique up to scalar multiple. In the former paper [HIM, Theorems 3.1 and 4.1], we give explicit formulas of $\mathbf{W}_\Pi(v)|_{A_3}$ for an image v of a monomial under the natural surjection $\mathcal{P}_\lambda \to V_\lambda^{(3)}$.

Next, we set $n = 2$. Let (π, H_π) be an irreducible admissible generic representation of G_2. For each $\lambda' \in \Lambda(\pi)$, the K_2-type $\tau_{\lambda'}^{(2)}$ occurs in $\pi|_{K_2}$ with multiplicity one. Let $\mathbf{W}_{\pi,\lambda'} : V_{\lambda'}^{(2)} \to \mathcal{W}(\pi, \psi_l)_{K_2}$ be a K_2-embedding which is unique up to scalar multiple. In the case of $F = \mathbf{R}$, for any $\lambda' \in \Lambda(\pi)$ and $q' \in S_{\lambda'}$, the explicit formula of $\mathbf{W}_{\pi,\lambda'}(v_{\lambda',q'})|_{A_2}$ is found in the standard textbooks. In the case of $F = \mathbf{C}$, the explicit formulas of Whittaker functions at the minimal K_2-type of π are found in Popa [Po, §5]. Applying the shift operator to Popa's formulas, we obtain the following:

Proposition 3.1. *We use the above notation, and assume $\pi \simeq \mathrm{Ind}_{P_{1,1}}^{G_2}(\chi_{[\nu_1',d_1']} \boxtimes \chi_{[\nu_2',d_2']})$ with $d_1' \geq d_2'$. Let $\lambda' = (d_1' + m, d_2' - m) \in \Lambda(\pi)$. There is $C \in \mathbf{C}^\times$ such that, for $y = \mathrm{diag}(y_1 y_2, y_2) \in A_2$ and $q' \in S_{\lambda'}$,*

$$\begin{aligned}\mathbf{W}_{\pi,\lambda'}(v_{\lambda',q'})(y) &= C\,(\sqrt{-1}l/|l|)^{d_1'+m-q'} y_1 y_2^{2\nu_1'+2\nu_2'} \\ &\times \sum_{i=0}^{q'} \binom{q'}{i} \frac{(-m)_i\,(-\nu_1' + \nu_2' - m - (d_1' - d_2')/2)_i}{(-d_1' + d_2' - 2m)_i\,(2\pi|l|y_1)^i} \\ &\times \frac{1}{2\pi\sqrt{-1}} \int_{\alpha-\sqrt{-1}\infty}^{\alpha+\sqrt{-1}\infty} \Gamma_{\mathbf{C}}\left(s + \nu_1' + \frac{q' + m - i}{2}\right) \\ &\times \Gamma_{\mathbf{C}}\left(s + \nu_2' + \frac{d_1' - d_2' + m - q' + i}{2}\right) (|l|y_1)^{-2s} ds.\end{aligned}$$

Here $(a)_i = \Gamma(a+i)/\Gamma(a)$ is the Pochhammer symbol, and α is a sufficiently large real number.

4. The archimedean zeta integrals.

4.1. The main result. Let Π and π be irreducible admissible generic representations of G_3 and G_2, respectively. For $W \in \mathcal{W}(\Pi, \psi_1)_{K_3}$ and $W' \in \mathcal{W}(\pi, \psi_{-1})_{K_2}$, we define the archimedean zeta integral $Z(s, W, W')$ by

$$Z(s, W, W') = \int_{N_2\backslash G_2} W\begin{pmatrix} h & \\ & 1 \end{pmatrix} W'(h) |\det(h)|_F^{s-\frac{1}{2}} d\dot{h},$$

where $d\dot{h}$ is the right G_2-invariant measure on $N_2\backslash G_2$ which is suitably normalized. Using the asymptotics of Whittaker functions, Jacquet and Shalika [JS] proved that $\dfrac{Z(s, W, W')}{L(s, \Pi \times \pi)}$ is an entire function of $s \in \mathbf{C}$, and satisfies the local functional equation:

$$\frac{Z(1-s, \tilde{W}, \tilde{W}')}{L(1-s, \tilde{\Pi} \times \tilde{\pi})} = \epsilon(s, \Pi \times \pi, \psi_1) \frac{Z(s, W, W')}{L(s, \Pi \times \pi)},$$

where tilde symbols mean the contragradients. Moreover, Jacquet [Ja] shows that there exists a finite subset $\{(W_i, W_i')\}_{1 \leq i \leq m} \subset \mathcal{W}(\Pi, \psi_1)_{K_3} \times \mathcal{W}(\pi, \psi_{-1})_{K_2}$ such that

$$\sum_{i=1}^m Z(s, W_i, W_i') = L(s, \Pi \times \pi).$$

Now we state the main theorem of this paper.

Theorem 4.1. *Let Π and π be irreducible admissible generic representations of G_3 and G_2, respectively. Then there exist $W \in \mathcal{W}(\Pi, \psi_1)_{K_3}$ and $W' \in \mathcal{W}(\pi, \psi_{-1})_{K_2}$ such that*

$$Z(s, W, W') = L(s, \Pi \times \pi).$$

This theorem is proved by the computation using the explicit formulas of Whittaker functions. In our computation, Barnes' lemma [Ba, §1.7] and appropriate choices of Whittaker functions play important roles. We introduce the appropriate choices of Whittaker functions in the next subsection.

4.2. Whittaker functions attaining the archimedean *L*-factors. For $\rho \in \Lambda_2$, we denote by $(\tilde{\tau}_\rho^{(2)}, \tilde{V}_\rho^{(2)})$ the contragradient representation of $\tau_\rho^{(2)}$, and denote by $\{\tilde{v}_{\rho,q}\}_{q \in S_\rho}$ the dual basis of $\{v_{\rho,q}\}_{q \in S_\rho}$. For $\rho = (\rho_1, \rho_2) \in \Lambda_2$ and $q \in S_\rho$, we define the symbols $\tilde{\rho}$, $\tilde{q}$ and $c(\rho, q)$ as follows:

$$\tilde{\rho} = \rho, \quad \tilde{q} = -q, \qquad c(\rho, q) = 1 \quad \text{for } F = \mathbf{R},$$
$$\tilde{\rho} = (-\rho_2, -\rho_1), \quad \tilde{q} = \rho_1 - \rho_2 - q,$$
$$c(\rho, q) = (-1)^q \binom{\rho_1 - \rho_2}{q} \qquad \text{for } F = \mathbf{C}.$$

Then we have $\tilde{V}_\rho^{(2)} \simeq V_{\tilde{\rho}}^{(2)}$ via the correspondence $\tilde{v}_{\rho,q} \leftrightarrow c(\rho, q) v_{\tilde{\rho},\tilde{q}}$ $(q \in S_\rho)$.

Let Π and π be irreducible admissible generic representations of G_3 and G_2, respectively. For $\lambda' \in \Lambda(\pi)$, let $\mathbf{W}_{\pi,\lambda'} : V_{\lambda'}^{(2)} \to \mathcal{W}(\pi, \psi_{-1})_{K_2}$ be a K_2-embedding, which is unique up to scalar multiple. We regard K_2 as a subgroup of K_3 via the embedding (2.1). Then, by Schur's orthogonality, we obtain the following lemma:

Lemma 4.2. *We use the above notation. Let $\mathbf{W} : V_\rho^{(2)} \to \mathcal{W}(\Pi, \psi_1)_{K_3}$ be a K_2-homomorphism with*

$\rho \in \Lambda_2$. *For* $q \in S_\rho$ *and* $q' \in S_{\lambda'}$, *the integral* $Z(s, \mathbf{W}(v_{\rho,q}), \mathbf{W}_{\pi,\lambda'}(v_{\lambda',q'}))$ *is equal to*

$$\frac{1}{\dim V_\rho^{(2)}} \sum_{r \in S_\rho} \frac{c(\rho,r)}{c(\rho,q)} \int_0^\infty \int_0^\infty \mathbf{W}(v_{\rho,r}) \left(\begin{array}{c|c} y & \\ \hline & 1 \end{array} \right)$$
$$\times \mathbf{W}_{\pi,\tilde{\rho}}(v_{\tilde{\rho},\tilde{r}})(y) |y_1|_F^{s-\frac{3}{2}} |y_2|_F^{2s-1} (y_1 y_2)^{-1} dy_1 dy_2$$

if $(\lambda', q') = (\tilde{\rho}, \tilde{q})$, *and is equal to* 0 *if otherwise. Here* $y = \mathrm{diag}(y_1 y_2, y_2) \in A_2$.

Let λ be the element of Λ_3 such that $\tau_\lambda^{(3)}$ is the minimal K_3-type of Π. The explicit formulas of Whittaker functions for Π are known only at $\tau_\lambda^{(3)}$. However, because of this lemma, the archimedean zeta integral $Z(s, W, W')$ vanishes for any Whittaker function W for Π at $\tau_\lambda^{(3)}$ and $W' \in \mathcal{W}(\pi, \psi_{-1})_{K_2}$ if there is no $\rho \in \Sigma(\lambda)$ such that $\tilde{\rho} \in \Lambda(\pi)$. We will construct K_2-homomorphisms $\mathbf{W}: V_\rho^{(2)} \to \mathcal{W}(\Pi, \psi_1)_{K_3}$ with $\rho \in \Lambda_2$ such that $\tilde{\rho} \in \Lambda(\pi)$, using the action of the Lie algebra of G_3.

Let $\mathfrak{g}_3$ be the complexification $\mathfrak{gl}(3, F) \otimes_{\mathbf{R}} \mathbf{C}$ of the Lie algebra of G_3, and we denote by $U(\mathfrak{g}_3)$ the universal enveloping algebra of $\mathfrak{g}_3$. We regard $U(\mathfrak{g}_3)$ as a K_2-module via the adjoint action Ad. We define a subset $\Sigma_{\mathfrak{n}}$ of Λ_2 by

$$\Sigma_{\mathfrak{n}} = \begin{cases} \{(\sigma_1, 0) \mid \sigma_1 \in \mathbf{Z}_{\geq 0}\} & \text{if } F = \mathbf{R}, \\ \{(\sigma_1, \sigma_2) \in \mathbf{Z}^2 \mid \sigma_1 \geq 0 \geq \sigma_2\} & \text{if } F = \mathbf{C}. \end{cases}$$

For $\sigma = (\sigma_1, \sigma_2) \in \Sigma_{\mathfrak{n}}$, let $\mathcal{D}_\sigma$ be a $\mathbf{C}$-vector subspace of $U(\mathfrak{g}_3)$ spanned by $\{E_{\sigma,r}\}_{r \in S_\sigma}$ with

$$E_{\sigma,r} = \begin{cases} (E_{23} \otimes 1 - E_{13} \otimes \sqrt{-1})^{\sigma_1} & \text{if } r \geq 0, \\ (E_{23} \otimes 1 + E_{13} \otimes \sqrt{-1})^{\sigma_1} & \text{if } r \leq 0 \end{cases}$$
$$\text{for } F = \mathbf{R},$$

$$E_{\sigma,r} = \sum_{i=\max\{0, r-\sigma_1\}}^{\min\{-\sigma_2, r\}} \binom{\sigma_1 - \sigma_2 - r}{-\sigma_2 - i} \binom{r}{i}$$
$$\times (E_{13} \otimes 1 - \sqrt{-1} E_{13} \otimes \sqrt{-1})^{\sigma_1 - r + i}$$
$$\times (E_{23} \otimes 1 - \sqrt{-1} E_{23} \otimes \sqrt{-1})^{r-i}$$
$$\times (E_{23} \otimes 1 + \sqrt{-1} E_{23} \otimes \sqrt{-1})^{-\sigma_2 - i}$$
$$\times (-E_{13} \otimes 1 - \sqrt{-1} E_{13} \otimes \sqrt{-1})^{i} \text{ for } F = \mathbf{C}.$$

Then $\mathcal{D}_\sigma$ is a K_2-submodule of $U(\mathfrak{g}_3)$, and $\mathcal{D}_\sigma \simeq V_\sigma^{(2)}$ via the correspondence $E_{\sigma,r} \leftrightarrow v_{\sigma,r}$ $(r \in S_\sigma)$.

Let $\mathbf{W}_\Pi: V_\lambda^{(3)} \to \mathcal{W}(\Pi, \psi_1)_{K_3}$ be a K_3-embedding, which is unique up to scalar multiple. For $\sigma = (\sigma_1, \sigma_2) \in \Sigma_{\mathfrak{n}}$ and $\mu \in \Sigma(\lambda)$, we define a K_2-homomorphism $\mathbf{W}_\Pi^{\sigma,\mu}: V_\sigma^{(2)} \otimes_{\mathbf{C}} V_\mu^{(2)} \to \mathcal{W}(\Pi, \psi_1)_{K_3}$ by $\mathbf{W}_\Pi^{\sigma,\mu}(v_{\sigma,r} \otimes v_{\mu,q}) = E_{\sigma,r} \mathbf{W}_\Pi(v_{\mu,q}^\lambda)$ for $r \in S_\sigma$ and $q \in S_\mu$. Since

$$\mathbf{W}_\Pi^{\sigma,\mu}(v_{\sigma,r} \otimes v_{\mu,q})(y) = \begin{cases} 0 & \text{if } F = \mathbf{C} \text{ and } r \neq \sigma_1, \\ (2\pi\sqrt{-1} y_2)^{\sigma_1 - \sigma_2} \mathbf{W}_\Pi(v_{\mu,q}^\lambda)(y) & \text{otherwise} \end{cases}$$

for $y = \mathrm{diag}(y_1 y_2 y_3, y_2 y_3, y_3) \in A_3$, the explicit formula of $\mathbf{W}_\Pi^{\sigma,\mu}(v_{\sigma,r} \otimes v_{\mu,q})|_{A_3}$ is obtained from the explicit formula of $\mathbf{W}_\Pi(v_{\mu,q}^\lambda)|_{A_3}$.

It is known that $V_\sigma^{(2)} \otimes_{\mathbf{C}} V_\mu^{(2)}$ is a multiplicity free K_2-module, that is, there is a subset $\Sigma(\sigma, \mu)$ of Λ_2 such that $V_\sigma^{(2)} \otimes_{\mathbf{C}} V_\mu^{(2)} \simeq \bigoplus_{\rho \in \Sigma(\sigma,\mu)} V_\rho^{(2)}$. For $\rho \in \Sigma(\sigma, \mu)$, let $I_\rho^{\sigma,\mu}: V_\rho^{(2)} \to V_\sigma^{(2)} \otimes_{\mathbf{C}} V_\mu^{(2)}$ be the K_2-embedding, which is up to scalar multiple. We define a K_2-homomorphism $\mathbf{W}_{\Pi,\rho}^{\sigma,\mu}: V_\rho^{(2)} \to \mathcal{W}(\Pi, \psi_1)_{K_3}$ by $\mathbf{W}_{\Pi,\rho}^{\sigma,\mu} = \mathbf{W}_\Pi^{\sigma,\mu} \circ I_\rho^{\sigma,\mu}$. In the rest of this paper, we consider the archimedean zeta integral of the form

$$(4.1) \quad Z(s, \mathbf{W}_{\Pi,\rho}^{\sigma,\mu}(v_{\rho,q}), \mathbf{W}_{\pi,\tilde{\rho}}(v_{\tilde{\rho},\tilde{q}})) \quad (q \in S_\rho).$$

In the case of $F = \mathbf{R}$, for $\sigma = (\sigma_1, 0) \in \Sigma_{\mathfrak{n}}$ and $\mu = (\mu_1, \mu_2) \in \Sigma(\lambda)$, we have

$$\Sigma(\sigma, \mu) = \begin{cases} \{(2\sigma_1, 0), (0,0), (0,1)\} & \text{if } \sigma_1 = \mu_1 > 0, \\ \{(0, \mu_2)\} & \text{if } \sigma_1 = \mu_1 = 0, \\ \{(\sigma_1 + \mu_1, 0)\} & \text{if } \sigma_1 \neq \mu_1 \text{ and } \sigma_1 \mu_1 = 0, \\ \{(\sigma_1 + \mu_1, 0), (|\sigma_1 - \mu_1|, 0)\} & \text{otherwise.} \end{cases}$$

Under some normalization, we obtain the following explicit expressions of $I_\rho^{\sigma,\mu}$ $(\rho \in \Sigma(\sigma, \mu))$:

- If $\sigma_1 \neq \mu_1$ or $\sigma_1 = \mu_1 = 0$, we have

  $$I_\rho^{\sigma,\mu}(v_{\rho,r+q}) = \begin{cases} v_{\sigma,r} \otimes v_{\mu,q} & \text{if } r + q \geq 0, \\ (-1)^{\mu_2} v_{\sigma,r} \otimes v_{\mu,q} & \text{if } r + q < 0 \end{cases}$$

  for $r \in S_\sigma$ and $q \in S_\mu$ such that $r + q \in S_\rho$.
- If $\sigma_1 = \mu_1 > 0$, we have

  $$I_{(2\sigma_1,0)}^{\sigma,\mu}(v_{(2\sigma_1,0),2q}) = v_{\sigma,q} \otimes v_{\mu,q} \quad (q \in S_\sigma),$$
  $$I_{(0,0)}^{\sigma,\mu}(v_{(0,0),0}) = v_{\sigma,\sigma_1} \otimes v_{\mu,-\sigma_1} + v_{\sigma,-\sigma_1} \otimes v_{\mu,\sigma_1},$$
  $$I_{(0,1)}^{\sigma,\mu}(v_{(0,1),0}) = v_{\sigma,\sigma_1} \otimes v_{\mu,-\sigma_1} - v_{\sigma,-\sigma_1} \otimes v_{\mu,\sigma_1}.$$

By these expressions, we obtain the explicit formulas of $\mathbf{W}_{\Pi,\rho}^{\sigma,\mu}(v_{\rho,q})|_{A_3}$ $(q \in S_\rho)$ for each $\rho \in \Sigma(\sigma, \mu)$. By direct computation, we know that (4.1) coincides with $L(s, \Pi \times \pi)$ up to nonzero constant multiple, if we take σ, μ and ρ as follows:

(Case 1-1) $\Pi \simeq \mathrm{Ind}_{P_{1,1,1}}^{G_3}(\chi_{(\nu_1,\delta_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)})$ with $\delta_1 \geq \delta_2 \geq \delta_3$, and $\pi \simeq \mathrm{Ind}_{P_{1,1}}^{G_2}(\chi_{(\nu_1',\delta_1')} \boxtimes \chi_{(\nu_2',\delta_2')})$ with $\delta_1' \geq \delta_2'$:

- If $\delta_1' = \delta_2' = \delta_2$, we set $\sigma = (0,0)$, $\mu = (0, \delta_2')$ and $\rho = (0, \delta_2')$.
- If $\delta_1' = \delta_2' \neq \delta_2$ and $\delta_1 = \delta_3$, we set $\sigma = (2, 0)$, $\mu = (0, \delta_2)$ and $\rho = (2, 0)$.

- If $\delta'_1 = \delta'_2 \neq \delta_2$ and $\delta_1 > \delta_3$, we set $\sigma = (1, 0)$, $\mu = (1, 0)$ and $\rho = (0, \delta'_2)$.
- If $(\delta'_1, \delta'_2) = (1, 0)$ and $\delta_1 = \delta_3$, we set $\sigma = (1, 0)$, $\mu = (0, \delta_2)$ and $\rho = (1, 0)$.
- If $(\delta'_1, \delta'_2) = (1, 0)$ and $\delta_1 > \delta_3$, we set $\sigma = (0, 0)$, $\mu = (1, 0)$ and $\rho = (1, 0)$.

(Case 1-2) $\Pi \simeq \mathrm{Ind}_{P_{1,1,1}}^{G_3}(\chi_{(\nu_1,\delta_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)})$ with $\delta_1 \geq \delta_2 \geq \delta_3$, and $\pi \simeq D_{(\nu',\kappa')}$:

- If $\delta_1 = \delta_3$, we set $\sigma = (\kappa', 0)$, $\mu = (0, \delta_2)$ and $\rho = (\kappa', 0)$.
- If $\delta_1 > \delta_3$, we set $\sigma = (\kappa' - 1, 0)$, $\mu = (1, 0)$ and $\rho = (\kappa', 0)$.

(Case 2-1) $\Pi \simeq \mathrm{Ind}_{P_{2,1}}^{G_3}(D_{(\nu_1,\kappa)} \boxtimes \chi_{(\nu_2,\delta)})$, and $\pi \simeq \mathrm{Ind}_{P_{1,1}}^{G_2}(\chi_{(\nu'_1,\delta'_1)} \boxtimes \chi_{(\nu'_2,\delta'_2)})$ with $\delta'_1 \geq \delta'_2$:

- If $\delta'_1 = \delta'_2 \neq \delta$, we set $\sigma = (1, 0)$, $\mu = (1, 0)$ and $\rho = (0, \delta'_2)$.
- If $(\delta'_1, \delta'_2) = (1, 0)$ or $\delta'_2 = \delta$, we set $\sigma = (0, 0)$, $\mu = (\delta'_1 - \delta'_2, \delta'_2)$ and $\rho = (\delta'_1 - \delta'_2, \delta'_2)$.

(Case 2-2) $\Pi \simeq \mathrm{Ind}_{P_{2,1}}^{G_3}(D_{(\nu_1,\kappa)} \boxtimes \chi_{(\nu_2,\delta)})$, and $\pi \simeq D_{(\nu',\kappa')}$:

- If $\kappa' \leq \kappa$, we set $\sigma = (0, 0)$, $\mu = (\kappa', 0)$ and $\rho = (\kappa', 0)$.
- If $\kappa' \geq \kappa$, we set $\sigma = (\kappa' - \kappa, 0)$, $\mu = (\kappa, 0)$ and $\rho = (\kappa', 0)$.

In the case of $F = \mathbf{C}$, we have

$$\Sigma(\sigma, \mu) = \{\sigma + \mu + (-i, i) \mid i \in S_\sigma \cap S_\mu\},$$

and the explicit expressions of $I_\rho^{\sigma,\mu}$ ($\rho \in \Sigma(\sigma, \mu)$) are given by Koornwinder [Ko]. Hence, we can obtain the explicit formulas of $\mathbf{W}_{\Pi,\rho}^{\sigma,\mu}(v_{\rho,q})|_{A_3}$ ($q \in S_\rho$) for each $\rho \in \Sigma(\sigma, \mu)$. Let $\Pi \simeq \mathrm{Ind}_{P_{1,1,1}}^{G_3}(\chi_{[\nu_1,d_1]} \boxtimes \chi_{[\nu_2,d_2]} \boxtimes \chi_{[\nu_3,d_3]})$ with $d_1 \geq d_2 \geq d_3$, and $\pi \simeq \mathrm{Ind}_{P_{1,1}}^{G_2}(\chi_{[\nu'_1,d'_1]} \boxtimes \chi_{[\nu'_2,d'_2]})$ with $d'_1 \geq d'_2$. By direct computation, we know that (4.1) coincides with $L(s, \Pi \times \pi)$ up to nonzero constant multiple, if we take σ, μ and ρ as follows:

- If $-d'_1 \geq d_1$, we set $\sigma = (-d'_1 - d'_2 - d_1 - d_2, 0)$, $\mu = (d_1, d_2)$ and $\rho = (-d'_1 - d'_2 - d_1, d_1)$.
- If $-d'_2 \geq d_1 \geq -d'_1 \geq d_2$, we set $\sigma = (-d'_1 - d'_2 - d_1 - d_2, 0)$, $\mu = (d_1, d_2)$ and $\rho = (-d'_2, -d'_1)$.
- If $-d'_2 \geq d_1$ and $d_2 \geq -d'_1 \geq d_3$, we set $\sigma = (-d'_2 - d_1, 0)$, $\mu = (d_1, -d'_1)$ and $\rho = (-d'_2, -d'_1)$.
- If $-d'_2 \geq d_1$ and $d_3 \geq -d'_1$, we set $\sigma = (-d'_2 - d_1, -d'_1 - d_3)$, $\mu = (d_1, d_3)$ and $\rho = (-d'_2, -d'_1)$.
- If $d_1 \geq -d'_2 \geq -d'_1 \geq d_2$, we set $\sigma = (-d'_1 - d_2, 0)$, $\mu = (-d'_2, d_2)$ and $\rho = (-d'_2, -d'_1)$.
- If $d_1 \geq -d'_2 \geq d_2 \geq -d'_1 \geq d_3$, we set $\sigma = (0, 0)$, $\mu = (-d'_2, -d'_1)$ and $\rho = (-d'_2, -d'_1)$.

Here, because of the local functional equation of the archimedean zeta integrals, we omit the following cases which are contragradient to the above cases:

- The case of $d_1 \geq -d'_2 \geq d_2$ and $d_3 \geq -d'_1$.
- The case of $d_2 \geq -d'_2 \geq -d'_1 \geq d_3$.
- The case of $d_2 \geq -d'_2 \geq d_3 \geq -d'_1$.
- The case of $d_3 \geq -d'_2$.

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